ARCHIVUM MATHEMATICUM (BRNO)
Tomus 34 (1998), 153–172

Invariant Measures for Nonlinear SPDE's: Uniqueness and Stability*

Bohdan Maslowski and Jan Seidler

Mathematical Institute, Academy of Sciences,
Žitná 25, 115 67 Praha 1, Czech Republic
Email: maslow@math.cas.cz
seidler@math.cas.cz

Abstract. The paper presents a review of some recent results on uniqueness of invariant measures for stochastic differential equations in infinite-dimensional state spaces, with particular attention paid to stochastic partial differential equations. Related results on asymptotic behaviour of solutions like ergodic theorems and convergence of probability laws of solutions in strong and weak topologies are also reviewed.

AMS Subject Classification. 60H15

Keywords. Stochastic evolution equations, invariant measures, ergodic theorems, stability

1 Introduction

The aim of the present paper is to review some recent results on uniqueness of invariant measures (that is, strictly stationary solutions) for nonlinear stochastic evolution equations (or, more generally, for stochastic differential equations in infinite-dimensional state spaces). Related asymptotic and ergodic properties of solutions like convergence of their probability laws to the invariant measure and ergodic theorems are also discussed.

The paper is divided into three parts: In Section 2, some existing results on strong and weak asymptotic stability of the invariant measure and its ergodic properties are recalled. By the strong asymptotic stability we mean convergence

* This research was supported in part by the GA ČR Grant No. 201/95/0629.

of probability laws of all solutions to the invariant measure in the norm defined by total variation of measures while the weak asymptotic stability means an analogous convergence in the weak (narrow) topology of the space of measures. It is obvious that both strong and weak asymptotic stability imply uniqueness of the invariant measure. Sections 3 and 4 contain more precise descriptions of some methods of proofs used in the papers listed in Section 2 for the respective cases of strong and weak asymptotic stabilities. In order to illustrate those methods, some typical statements and results are given. Note that the problem of existence of the invariant measure is not treated in the present paper; see for instance the monograph [20] by G. Da Prato and J. Zabczyk and the references therein.

It should be pointed out that the statements contained in Sections 3 and 4 are not always formulated in full generality. The authors' intention was to discuss some basic mathematical tools available and to avoid technical complications as much as possible. Some generalizations, improvements and applications of the presented results are referred to subsequently.

2 Review of existing results

A standard possibility to show uniqueness as well as the strong asymptotic stability (or the strong mixing property) of an invariant measure for a finite-dimensional nondegenerate stochastic differential equation is to utilize the usual correspondence between SDE's and PDE's; under suitable conditions (including, in particular, a sufficient nondegeneracy of the diffusion matrix of the SDE) the transitional densities coincide with the fundamental solution to a linear parabolic PDE (the Kolmogorov equation), which yields the strong Feller property (SFP) and the (topological) irreducibility (I) of the Markov process defined by the stochastic equation. Then the classical results of the ergodic theory of Markov processes, as developed by J. L. Doob, G. Maruyama and H. Tanaka, R. Z. Khas'minskiĭ and others (see e.g. [21], [48], [37], [22]) and later extended to more general state spaces (see the references in Section 3 and, in particular, Theorem 4), can be applied to obtain uniqueness of the invariant measure (provided it exists) as well as the the strong asymptotical stability.

For infinite-dimensional state spaces such mathematical tools are not easily available; the Lebesgue measure does not exist and equivalence of measures is in a sense a "rare" event (see, for instance, the discussion following Proposition 2 and the example at the beginning of Section 4). On the other hand, in the linear case when the transition probabilities are Gaussian measures it is possible to verify by direct computation (cf. Proposition 1) that in some important examples (typically, stochastic parabolic or parabolic-like equations) the strong asymptotic stability takes place.

There are several methods which have been used to prove similar results for nonlinear infinite-dimensional stochastic systems. At first, let us mention the approach based on verification of the strong Feller property and irreducibility of the induced Markov process which has been used in numerous papers that have appeared in recent years. We describe this method in detail in Section 3 while here we restrict ourselves to some bibliographical remarks. In early papers by B. Maslowski and R. Manthey ([50], [44]) the SFP has been proven via finite-dimensional approximations for semilinear systems under rather restrictive assumptions. Also, a controllability method to prove (I) was developed there. Those results were further extended by B. Maslowski in [52], in particular, certain smoothing properties of mild solutions to the infinite-dimensional backward Kolmogorov equation proven by G. Da Prato and J. Zabczyk ([15], see also [18]) were utilized to get the SFP for reaction-diffusion equations with additive noise. Alternatively, under different set of assumptions, the problem of equivalence of transition probabilities has been solved by means of a Girsanov type theorem in [51] and [52], cf. also [27] for an analogous but more difficult argument applied to the stochastic quantization equation. Let us mention that infinite-dimensional Kolmogorov equations have been treated very recently by many authors, their link to invariant measures of fairly general SPDE's was investigated in depth by A. Chojnowska-Michalik, B. Goldys and D. Gątarek, see [8], [28].

Another way of proving the SFP has emerged in the paper [11] by G. Da Prato, K. D. Elworthy and J. Zabczyk where a formula for directional derivatives of a Markov transition semigroup involving the L^2-derivative of the solution with respect to initial condition has been derived (cf. Proposition 9). This approach has been later extended by S. Peszat and J. Zabczyk [60] to be applicable to stochastic parabolic equations with multiplicative noise term, cf. also the already cited paper [28]. It also turned out to be useful in asymptotic analysis of various important particular systems studied in physics and chemistry, like stochastic Burgers and Navier-Stokes equations or stochastic Cahn-Hilliard equation (cf. [12], [10], [24] or [23]).

Tools from the Malliavin calculus were employed to establish the regularity of the transition semigroup (in particular the SFP) by M. Fuhrman in [26] (cf. also [14]).

Let us briefly mention some other methods of proving the strong asymptotic stability of invariant measures. S. Jacquot and G. Royer [36] used a general theory of Markov operators to prove geometric ergodicity (i.e. strong exponential stability of an invariant measure) for a particular but important stochastic parabolic equation. C. Mueller in [59] used an approach based on coupling techniques to prove strong asymptotic stability of the invariant measure for a nonlinear heat equation with multiplicative noise, defined on a circle (uniqueness of the invariant measure for this case had been proven earlier by R. Sowers in [62] by establishing suitable asymptotic stability of paths).

Very little seems to be known in the case of nonautonomous SPDE's, where the standard methods of ergodic theory are no longer available. A lower bound measure method developed in context of statistical analysis of deterministic dynamical systems has been used by B. Maslowski and I. Simão in [57] to investigate the limit behaviour (in variational norm) of Markov evolution operators corresponding to nonautonomous stochastic infinite-dimensional systems (cf. also a methodologically related paper [34]). Simulated annealing for stochastic evolution equations has been studied by S. Jacquot, see e.g. [35] where references to previous papers of the author can be found.

Results on the strong asymptotic stability, when available, usually provide us with a fairly complete description of the qualitative behaviour of solutions to the considered SPDE's. On the other hand, many stochastic equations with reasonable long-time behaviour can be never treated using the tools described above. So we shall discuss now methods for investigating the weak asymptotic stability that apply to different classes of SPDE's, including those with a degenerate noise.

As is known from the finite-dimensional case, uniqueness of an invariant measure may be obtained as a consequence of pathwise stability of the process, which, in turn, is often investigated by means of well developed Lyapunov techniques (see e.g. [38]). The Lyapunov functions methods were extended to semilinear SPDE's by A. Ichikawa in [32] (see also [31] for slight modifications), who found sufficient conditions for uniqueness, and further strengthened in [49] to yield stability as well. Later, these methods proved themselves applicable to nonhomogeneous boundary value problems for stochastic parabolic equations ([53], [54]). G. Leha and G. Ritter developed a rather general Lyapunov approach for establishing existence, uniqueness and attractiveness of invariant measures for Markov processes in topological spaces, that covers also some classes of stochastic infinite-dimensional differential equations (see [42], [43]). A recent paper [4] on uniqueness of an invariant measure for a stochastic parabolic variational inequality is virtually based on the same technique. We discuss the Lyapunov method in some detail in Section 4.

A special attention must be paid to the dissipativity method (sometimes also called “the remote start method”) since most of recent results on invariant measures for SPDE's (both abstract theorems and results about important particular equations) seem to have been obtained using this procedure. The method was developed by G. Da Prato and J. Zabczyk in [16], [17], [19] for equations with additive noise and by them together with D. Gątarek in [13] for the multiplicative noise case; see the monographs [18], Chapter 11.5, [20] for a systematic account. (We list here only papers dealing with uniqueness and weak asymptotic stability, not the copious articles concerning applications of the dissipativity method to existence of invariant measures.) More factual description of the method is provided in Section 4.

Finally, we are going to list briefly other papers containing related results. R. Marcus in the early papers [45], [46], [47] considered stochastic parabolic equations with an additive noise under rather restrictive hypotheses and sketched a proof of the weak asymptotic stability of an invariant measure (using a procedure that can be viewed as a variant of the remote start trick). In particular, he investigated the case of the drift term having a potential, when the invariant measure may be given explicitly, see also [40] and [25] for uniqueness results in this direction. (These results are now partly covered by those based on the equivalence of transition kernels.) I. D. Chueshov and T. V. Girya proved existence and weak asymptotic stability of an invariant measure as a consequence of their results on inertial manifolds for parabolic SPDE's driven by additive noise ([9], [30]). Uniqueness and stability theorems on invariant measures for semilinear stochastic parabolic equations, proved in the framework of the variational approach to SPDE's, can be found in [33] and [29], see also the book [65], §XII.7 and the references therein.

An analytic approach to invariant measures for infinite-dimensional stochastic systems, using logarithmic Sobolev inequalities (see the surveys [64] or [67] for references) or Dirichlet forms techniques (see [6], [5], [1], [7]), has found many applications to lattice systems (cf. e.g. [2], [3]). Applications to stochastic partial differential equations are up to now less frequent, see, however, the papers [58] and [39] in which ergodic properties of invariant measures for SPDE's are dealt with by means of Dirichlet forms.

3 Strong asymptotic stability

In the present section, some basic results on uniqueness, ergodicity and strong asymptotic stability of an invariant measure for stochastic evolution equations are listed and basic methods of their proofs are explained. By the strong asymptotic stability we understand convergence of probability laws of all solutions to a given stochastic evolution equation to the corresponding invariant measure in norm defined by the total variation of measures. In what follows, we denote by $|||\varrho|||$ the total variation of a signed measure ϱ and by $\mathscr{N}(m, U)$ the Gaussian measure with mean m and covariance operator U.

We start with the linear equation in which case the problem of strong asymptotic stability is in a sense much simpler than for the nonlinear equation. However, some "typical" difficulties (as well as differences between finite- and infinite-dimensional stochastic equations) can be seen already in that case.

Consider a linear stochastic equation of the form

$$\mathrm{d}Z_t = AZ_t \,\mathrm{d}t + \mathrm{d}W_t, \tag{1}$$

in a real separable Hilbert space $H = (H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ where $A : \mathrm{Dom}(A) \subseteq H \longrightarrow H$ is an infinitesimal generator of a strongly continuous semigroup $(e^{At},\, t \geq 0)$ on H, W_t is a Wiener process on H defined on a probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$ with an incremental covariance operator $Q \in \mathscr{L}(H)$. The operator Q is not necessarily nuclear (which means that W_t may be just cylindrical, not really H-valued, Wiener process). In the sequel we shall assume

$$\int_0^T \left\| e^{At} Q^{1/2} \right\|_{\mathrm{HS}}^2 \mathrm{d}t < \infty \tag{2}$$

for some $T > 0$, where $\|\cdot\|_{\mathrm{HS}}$ stands for the Hilbert-Schmidt norm of an operator on H. It is well known that under the condition (2) the equation (1) has for any initial datum $Z_0 = x \in H$ a unique mild solution defined as a continuous H-valued process satisfying the variation of constants formula

$$Z_t = e^{At}x + \int_0^t e^{A(t-r)} \,\mathrm{d}W_r, \quad t \geq 0, \tag{3}$$

whose transition probabilities $P_t(x,\cdot)$ are Gaussian measures $\mathscr{N}(e^{At}x, Q_t)$ for $t \ge 0$, $x \in H$, where

$$Q_t = \int_0^t e^{Ar} Q e^{A^* r} \, \mathrm{d}r$$

is a nuclear operator (cf. [18] for basic results on the semigroup theory of stochastic evolution equations). An invariant measure μ^* for the Markov process induced by the equation (1) exists if and only if

$$\sup_{t \ge 0} \operatorname{Tr} Q_t < \infty \tag{4}$$

in which case $\mu^* = \mathscr{N}(0, Q_\infty)$, where $Q_\infty = \lim_{t\to\infty} Q_t$. In general, it can happen that μ^* is not the only invariant measure; the problem of uniqueness and characterization of all invariant measures has been treated in [66] (see also [18] and the references therein). As far as the strong asymptotic stability is concerned we expose the following result the proof of which can be found in [49] (for simplicity, we consider only the case $Q > 0$):

Proposition 1. *Assume* (2) *and let K be a linear subspace of H such that*

$$e^{At}x \in \operatorname{Im}\bigl(Q_t^{1/2}\bigr), \quad t > t_0(x) \ge 0 \tag{5}$$

and

$$\|Q_t^{-1/2} e^{At} x\| \longrightarrow 0, \quad t \to \infty, \tag{6}$$

for $x \in K$. Then

$$|||P_t(x,\cdot) - P_t(y,\cdot)||| \longrightarrow 0, \quad t \to \infty \tag{7}$$

for each $x, y \in K$. If, moreover, (4) *holds true and $\mu^*(K) = 1$ then*

$$|||P_t(x,\cdot) - \mu^*||| \longrightarrow \infty, \quad t \to \infty \tag{8}$$

for any $x \in K$. In particular, μ^* is the only invariant probability measure concentrated on K.

In particular examples, the condition (5) can be usually verified for $t_0 = 0$ (or for some t_0 independent of x) and for $K = H$. In this case the assumptions of Proposition 1 can be simplified as follows:

Proposition 2. *Assume* (2) *and* (4) *and let*

$$\operatorname{Im}\bigl(e^{At}\bigr) \subseteq \operatorname{Im}\bigl(Q_t^{1/2}\bigr) \tag{9}$$

be satisfied for $t > 0$. Then (8) *holds true and, in particular, μ^* is the only invariant probability measure for the problem* (1).

Note that Proposition 1 has been proven in [49] by direct computation using the Cameron-Martin formula for density of a Gaussian measure while Proposition 2 is a corollary of a more general statement given below (Theorem 4). The assumption (9) is an if and only if condition on the Gaussian transition probabilities $P_t(x,\cdot)$ to be equivalent (i.e., mutually absolutely continuous) for $t>0$, as can be seen easily by the Hájek-Feldman theorem. In some cases, (9) can be shown to be a necessary condition for the strong asymptotic stability (8) (see the example at the beginning of Section 4) and since it is rather restrictive (for example, the semigroup e^{At} satisfying (9) is necessarily Hilbert-Schmidt) it can be expected that the cases when the strong asymptotic stability takes place in the infinite-dimensional space H are rather "rare". However, it turns out that parabolic and parabolic-like stochastic equations with enough nondegenerate diffusion term are natural field for applications of Propositions 1 and 2 as may be seen from the simple example below.

Example 3. Assume that A is self-adjoint, negative, and has compact resolvent and denote by $\{e_j\}_{j\geq 1}$ the orthonormal basis of H such that

$$Ae_j = -\alpha_j e_j, \tag{10}$$

where $0<\alpha_j \longrightarrow \infty$, $j\in\mathbb{N}$. Assume that A, Q are such that for some $0<\lambda_j\leq \lambda_0<\infty$, we have

$$Qe_j = \lambda_j e_j, \quad j\in\mathbb{N}. \tag{11}$$

Then it is easy to check that the conditions (2) and (4) are satisfied if

$$\sum_{i=1}^{\infty}\frac{\lambda_i}{\alpha_i}<\infty, \tag{12}$$

which is sufficient for the mild solution Z_t of the equation (1) and the corresponding invariant measure μ^* to exist. The condition (9) verifying the strong asymptotic stability is now equivalent to the requirement that the sequence

$$\left\{\frac{\alpha_i}{\lambda_i}\exp(-2\alpha_i t)\right\}_{i\in\mathbb{N}} \tag{13}$$

is bounded for each $t>0$.

In particular, the process Z_t in the present example can represent a solution to a linear stochastic parabolic equation like, for instance, the equation

$$\frac{\partial u}{\partial t}(t,\xi) = \frac{\partial^2 u}{\partial \xi^2}(t,\xi)+\eta(t,\xi), \quad (t,\xi)\in\mathbb{R}_+\times(0,1), \tag{14}$$

with an initial condition $u(0,\xi)=x(\xi)$, $\xi\in(0,1)$, and the Dirichlet boundary conditions $u(t,0)=u(t,1)=0$, $t\in\mathbb{R}_+$, where η is a space-dependent noise, white in time. This can be achieved by the particular choice $H=L^2(0,1)$, and $A=\frac{\partial^2}{\partial\xi^2}$

with $\mathrm{Dom}(A) = H_0^1(0,1) \cap H^2(0,1)$. Now (13) can be viewed as a condition on the noise term (the covariance operator Q) for the strong asymptotic stability to hold. For example, if η represents a noise white in both space and time then we can take for Q the identity I and (13) is satisfied.

Our next aim is to describe a method based on the general ergodic theory of Markov processes that allows to prove the strong asymptotic stability (and, also, ergodic theorems) for nonlinear stochastic evolution equations. We shall utilize an abstract result stated in Theorem 4 below which was obtained independently by Stettner [63] and Seidler [61].

Theorem 4. *Let $\big((X_t)_{t\geq 0}, (\boldsymbol{P}_x)_{x\in H}\big)$ be a Markov process in a Polish space H with a transition probability function $P_t(x,\cdot)$, $t \geq 0$, $x \in H$, having an invariant probability measure μ^*. Assume that all the measures $P_t(x,\cdot)$, $t > 0$, $x \in H$, are equivalent. Then*

(i) for each bounded Borel function $\phi : H \longrightarrow \mathbb{R}$ and every $x \in H$ we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \phi(X_t)\,\mathrm{d}t = \int_H \phi\,\mathrm{d}\mu^* \quad \boldsymbol{P}_x\text{-a.s.} \tag{15}$$

(ii) for every $x \in H$ we have

$$|\!|\!| P_t(x,\cdot) - \mu^* |\!|\!| \longrightarrow 0, \quad t \to \infty. \tag{16}$$

In particular, both (i) and (ii) imply that the invariant measure μ^* is unique.

The assertion (15) is known as the pointwise ergodic theorem (or the strong law of large numbers). As mentioned above, the condition (9) is an if and only if condition for the equivalence of transition probability functions for the linear equation (1). The simplest nonlinear case into which Theorem 4 can be applied is the one allowing reduction of the nonlinear problem to a linear one by means of a Girsanov type theorem. We shall present a simple result of this type now. Consider a stochastic semilinear equation

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + f(X_t)\,\mathrm{d}t + \mathrm{d}W_t, \tag{17}$$

in the Hilbert space H, where A and W have the same meaning as in the equation (1) and $f : H \longrightarrow \mathrm{Im}\big(Q^{1/2}\big)$ satisfies

$$\|Q^{-1/2}\big(f(x) - f(y)\big)\| \leq K\|x - y\| \tag{18}$$

for a $K < \infty$ and all $x, y \in H$.

Theorem 5. *Assume* (2), (9) *and* (18). *Then $P_t(x,\cdot)$ and $\tilde{P}_t(x,\cdot)$ are equivalent measures for every $t > 0$, $x \in H$, where P and $\tilde{P}$ denote the transition probability functions for the solutions of the equations* (17) *and* (1), *respectively. If, moreover, there exists an invariant measure μ^* for the equation* (17) *then the strong asymptotic stability* (16) *holds true.*

Theorem 5 can be proven as a corollary of Theorem 4; note that (9) yields the equivalence of measures $\tilde{P}_t(x,\cdot)$ for $t > 0$, $x \in H$, and the equivalence $P_t(x,\cdot) \sim \tilde{P}_t(x,\cdot)$ follows from the Girsanov theorem. The condition (18) appears here because the Girsanov factor has the form

$$\exp\Big\{ \int_0^T \langle Q^{-\frac{1}{2}} f(Z_t), \mathrm{d}W_t \rangle - \frac{1}{2} \int_0^T \big\| Q^{-\frac{1}{2}} f(Z_t) \big\|^2 \, \mathrm{d}t \Big\} \tag{19}$$

for $T > 0$ (cf. [18]). Theorem 5 can be generalized to cover also nonlinear terms which are not Lipschitz continuous or even only densely defined in H (cf. [52], [18]). The main disadvantage of this approach is that the inclusion $\mathrm{Im}(f) \subseteq \mathrm{Im}\big(Q^{1/2}\big)$ is required which makes the abstract results easily applicable only if Q is boundedly invertible, that is, just for a cylindrical Wiener process W (typically it can represent a space-time white noise; see however [56] for examples of stochastic parabolic equations in which a nonlinear term of the form $Q^{1/2} f$ occurs in a natural way in the drift part of the equation).

For equations of the form (17) where the covariance Q is "too degenerate" for the Girsanov theorem to be applied, it is sometimes possible to verify the equivalence of transition probability functions by means of Lemma 6 below which holds true even if the state space H is an arbitrary Polish space. Recall that a Markov process is called strongly Feller if its transition probability function $P_t(x,\Gamma)$ is continuous in the variable x for each fixed $t > 0$ and every Borel set Γ in H. Furthermore, the Markov process is called irreducible if $P_t(x,U) > 0$ holds for each $t > 0$, $x \in H$ and $U \neq \emptyset$, U open in H.

Lemma 6. *Assume that a Markov process is strongly Feller and irreducible. Then the measures $P_t(x,\cdot)$ are equivalent for $t > 0$, $x \in H$.*

Note that both the strong Feller property and irreducibility are of independent interest (for example, to investigate recurrence of the process, cf. [55] and [61]).

In the rest of the section we shall illustrate some methods which allow to verify irreducibility or the strong Feller property for stochastic evolution equations. At first we describe a method based on an argument of approximate controllability for a deterministic evolution equation, which yields the irreducibility property for the corresponding stochastic evolution equation. We again shall illustrate the method in the simple case (17) where A and W are as above and $f : H \longrightarrow H$ is assumed to be Lipschitz continuous. Note that the mild solution to the equation (17) with initial condition $X_0 = x \in H$ (which exists and is unique in this case) can be written $X_t = u(t,x;\tilde{Z})$, $t \geq 0$, where $\tilde{Z}$ solves the linear equation (1) with initial condition $\tilde{Z}_0 = 0$ and $u(t,x;\phi)$ is the solution of the integral equation

$$u(t,x;\phi) = e^{At}x + \int_0^t e^{A(t-r)} f(u(r,x;\phi)) \, \mathrm{d}r + \phi(t), \quad t \in [0,T], \tag{20}$$

with $\phi \in \mathscr{C}_0([0,T];H) := \{ g \in \mathscr{C}([0,T];H);\ g(0) = 0 \}$. The method consists in finding a suitable space $\mathscr{X}$ of trajectories such that the paths of the Gaussian process $\tilde{Z}$ belong a.s. to $\mathscr{X}$, $\tilde{Z}$ induces a full Gaussian measure (that is, a measure whose closed support is the whole space) in $\mathscr{X}$, and $u(\cdot,x;\phi_n) \to u(\cdot,x;\phi)$ in $\mathscr{C}([0,T];H)$ as $\phi_n \to \phi$ in $\mathscr{X}$. The irreducibility of the transition probability function $P_t(x,\cdot)$ for the equation (17) follows easily. If the nonlinear term f is (globally) Lipschitz continuous on H it is natural to take $\mathscr{X} = \mathscr{C}_0([0,T];H)$ and it only remains to find conditions under which the measure induced by $\tilde{Z}$ in $\mathscr{X}$ is full. Since the closed support of a Gaussian measure is just the closure of its reproducing kernel we have a following result:

Theorem 7. *Define a mapping* $\mathscr{K} : L^2(0,T;H) \longrightarrow \mathscr{C}_0([0,T];H)$ *by*

$$\mathscr{K}\psi(t) := \int_0^t e^{A(t-r)} Q^{1/2} \psi(r)\, \mathrm{d}r, \quad t \in [0,T].$$

If f is Lipschitz continuous and $\mathrm{Im}(\mathscr{K})$ *is dense in* $\mathscr{C}_0([0,T];H)$ *then the transition probability function* $P_t(x,\cdot)$ *corresponding to the equation* (17) *is irreducible.*

Theorem 7 is a particular case of a result proven in [52] for the case of non-Lipschitz and densely defined nonlinear terms f, which is applicable to stochastic reaction-diffusion equations. More sophisticated versions of this method have been applied, for example, to stochastic Burgers equation [12], stochastic Cahn-Hilliard equation [10] and stochastic Navier-Stokes equation [24], [23].

Now we focus our attention on the strong Feller property of solutions to stochastic evolution equations. The usual procedure of verification of the strong Feller property in the finite-dimensional case utilizes the smoothing properties of the Kolmogorov equation. A similar theory for Kolmogorov backward equation in infinite dimensions is being developed in recent years ([18], [8], and others). The main tool to prove both existence and uniqueness of solutions and the required smoothing properties is the concept of mild solutions to the backward Kolmogorov equation, which we shall recall now. Basically, we follow the paper [8]. Assume (2), (4) and let $\mu = \mathscr{N}(0, Q_\infty)$ be the invariant measure for the linear equation (1) and T_t its Markov transition semigroup considered on the space $L^2(H,\mu)$, i.e., $T_t\phi(x) = \boldsymbol{E}_x \phi(Z_t)$, $t \geq 0$, $x \in H$, $\phi \in L^2(H,\mu)$. Further, denote by P_t the Markov transition semigroup defined by the nonlinear equation (17). Analogously to the finite-dimensional case it can be expected that, under suitable conditions, the semigroup P_t corresponds to solutions of the mild backward Kolmogorov equation

$$u(t,\cdot) = T_t\phi + \int_0^t T_{t-s}\langle f, Du(s,\cdot)\rangle\, \mathrm{d}s, \quad t > 0, \tag{21}$$

where D denotes the Fréchet derivative. A precise statement is formulated now ($\mathscr{C}_b(H)$ and $\mathscr{C}_b^1(H)$ denote the space of bounded continuous functions on H and its subspace of functions having bounded and continuous Fréchet derivative on H, respectively).

Theorem 8. *Let f be bounded and continuous, assume* (2), (4), (9) *and*

$$\int_0^T \|Q_t^{-1/2} e^{At}\|_{\mathscr{L}(H)} \, \mathrm{d}t < \infty \tag{22}$$

for some $T > 0$. Then for every bounded Borel function ϕ on H there exists a unique solution u to (21) *and $u(t,\cdot) \in \mathscr{C}_b^1(H)$, $t > 0$. Moreover, $u(t,x) = P_t\phi(x) \equiv \boldsymbol{E}_x\phi(X_t)$, $t > 0$, $x \in H$, provided $\phi \in \mathscr{C}_b(H)$. In particular, we have*

$$\|P_t(x,\cdot) - P_t(y,\cdot)\| \le \gamma(t)\|x-y\|, \quad t>0,\ x,y \in H, \tag{23}$$

where $\gamma(t) := \sup\{\|DP_t\phi(z)\|;\ z \in H,\ \phi \in \mathscr{C}_b(H),\ |\phi| \le 1\} < \infty$, hence the strong Feller property holds true.

For the proof see [8] or (in certain earlier version) [18], [15]. The assumption of boundedness of f is not always essential for the strong Feller property and can be weakened by suitable truncation procedures (see [52], [28]) so that stochastic parabolic equations with polynomial-type nonlinearities could be included. The important assumption is (22) which is further strengthening of (9) and means certain "nondegeneracy of the noise" which, of course, is needed (even in finite-dimensional state space) for the strong Feller property to hold. It can be shown ([15]) that if the covariance Q is boundedly invertible then (22) is satisfied.

Theorem 8 is applicable only to equations with additive noise (if the diffusion term is a constant operator). Now we shall mention another method of establishing the strong Feller property, which is useful also in the case of multiplicative noise. The method was developed in [11] and is based on the so-called Elworthy formula which we present in the simple case of equation (17) where Q is assumed to be boundedly invertible and f is Lipschitz continuous and Gateaux differentiable on H with the Gateaux derivative continuous as a mapping from H into the space $\mathscr{L}(H)$ endowed with the strong operator topology.

Proposition 9. *Under the above hypotheses, we have that $P_t\phi \in \mathscr{C}^1(H)$ for each $t > 0$, ϕ bounded Borel, and*

$$\langle DP_t\phi(x), h\rangle = \frac{1}{t}\boldsymbol{E}_x\Big(\phi(X_t)\int_0^t \langle Q^{-1/2}X_s^h, \mathrm{d}W_s\rangle\Big) \tag{24}$$

holds for $x, h \in H$, where X_t^h denotes the directional derivative in the L^2-sense of the solution X_t to (17) *in the direction $h \in H$.*

For the proof see [11]. The usefulness of the formula (24) lies with the fact that it allows to estimate the value of $\|DP_t\phi(x)\|$ for a fixed $t > 0$, independently of $\phi \in \mathscr{C}_b(H)$, $|\phi| \le 1$ and the strong Feller property follows in the same way as in (23).

In fact, the method is applicable to more general cases as well as to some special equations which are rather difficult to handle (usually it is possible to use suitable approximations of the equation, which can be typically finite-dimensional approximations or approximations by smooth nonlinearities). Thus, in [60] the strong Feller property has been proven for stochastic semilinear equations with multiplicative noise (with boundedly invertible diffusion coefficients). In [12] and [10] the stochastic Burgers and Cahn-Hilliard, respectively, equations are treated. The 2-dimensional stochastic Navier-Stokes equation is dealt with in [24] and [23]. In all those cases the limit and ergodic properties of solutions listed in Theorem 4 are proved in respective state spaces.

4 Weak asymptotic stability

However efficient are the methods of investigating the long time behaviour of Markov processes based on the strong Feller property, they are relevant for a rather limited class of equations that are, roughly speaking, subject to a sufficiently non-degenerated noise. But such a nondegeneracy is necessary neither for the existence, nor for uniqueness and attractiveness of invariant measures. To indicate what may happen, let us consider a simple linear equation

$$\mathrm{d}Z = AZ\,\mathrm{d}t + \mathrm{d}W \tag{25}$$

in a separable Hilbert space H, where W is a Wiener process in H with a covariance operator Q and $A : \mathrm{Dom}(A) \longrightarrow H$ is a self-adjoint operator. Assume that the hypotheses (10)–(12) of Example 3 are satisfied. Denote by $P = P_t(x,\cdot)$ the transition function of the Markov process defined by (25). As above we set

$$Q_t = \int_0^t e^{Ar} Q e^{Ar}\,\mathrm{d}r, \quad 0 \le t \le \infty.$$

If (9) holds, that is

$$\mathrm{Im}\big(e^{At}\big) \subseteq \mathrm{Im}\big(Q_t^{1/2}\big) \quad \text{for each } t > t_0 \tag{26}$$

for a $t_0 \ge 0$, then the kernels $P_t(x,\cdot)$ are strong Feller and the theory discussed in Section 3 applies, so let us assume that (26) is violated. (Note that this is possible only in the "degenerate" case when Q is noninvertible, cf. [18], Remark B.9.) Then we can always find an $x_0 \in H$ satisfying

$$e^{At}x_0 \notin \mathrm{Im}\big(Q_t^{1/2}\big) \quad \text{for every } t > 0. \tag{27}$$

The semigroup (e^{At}) is exponentially stable, so there exists a unique invariant measure μ^* for (25), namely $\mu^* = \mathscr{N}(0, Q_\infty)$, see e.g. [18], Theorem 11.11(ii). At the same time, $P_t(x_0,\cdot) = \mathscr{N}(e^{At}x_0, Q_t)$, hence the measures $P_t(x_0,\cdot)$ and μ^* are mutually singular according to (27) and the Hájek-Feldman theorem (cf. e.g. [41], Theorems II.3.1 and II.3.4). This implies $\|P_t(x_0,\cdot) - \mu^*\| = 2$ and the measures $P_t(x_0,\cdot)$ cannot converge to the invariant measure in the total variation norm. Moreover, we see that nor the weaker assertion

$$\lim_{t\to\infty} P_t(x_0, B) = \mu^*(B) \quad \text{for any } B \subseteq H \text{ Borel} \tag{28}$$

holds true. Indeed, we know that there are Borel sets A_n, $n \geq 1$, such that $\mu^*(A_n) = 0$, $P_n(x_0, A_n) = 1$, so setting $B = \bigcup_{n\geq 1} A_n$ we obtain a counterexample to (28).

On the other hand we have

$$P_t(y,\cdot) \xrightarrow[t\to\infty]{\mathrm{w}^*} \mu^* \quad \text{for any } y \in H$$

by [49], Proposition 3.1, or [18], Theorem 11.11(i), therefore the invariant measure is globally asymptotically stable with respect to the narrow convergence. Hereafter, we denote by $\xrightarrow{\mathrm{w}^*}$ the narrow (or weak) convergence of finite (signed) Borel measures on H, that is,

$$\mu_\alpha \xrightarrow{\mathrm{w}^*} \mu \quad \text{if and only if} \quad \int_H f \,\mathrm{d}\mu_\alpha \longrightarrow \int_H f\,\mathrm{d}\mu \quad \forall f \in \mathscr{C}_b(H).$$

In finite-dimensional spaces, Lyapunov functions techniques are the basic tool for investigating stability properties of solutions to SDE's. A. Ichikawa [32] employed such an argument to establish uniqueness of an invariant measure for stochastic evolution equations, and later the procedure was extended to yield attractiveness as well, see the discussion in Section 2 above. The proofs based on Lyapunov functions have usually a lucid structure and lead, in a straightforward manner, to sufficient conditions for stability in terms of the coefficients of the equation. The known sufficient conditions, however, may be often too restrictive to cover interesting models. Furthermore, Itô's formula is not directly applicable to *mild* solutions of stochastic partial differential equations, nontrivial approximations are needed, and the class of admissible Lyapunov functions may be too narrow for useful applications, in particular if the Wiener process is cylindrical. Hence we content ourselves with stating a single typical result.

Let us consider a stochastic evolution equation

$$\mathrm{d}X_t = \bigl\{AX_t + f(X_t)\bigr\}\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t \tag{29}$$

in a separable Hilbert space H, where $A : \mathrm{Dom}(A) \longrightarrow H$ is an infinitesimal generator of a C_0-semigroup on H, W is a Wiener process in another (real, separable) Hilbert space U, with the covariance operator Q nuclear, and the mappings $f : H \longrightarrow H$, $\sigma : H \longrightarrow \mathscr{L}(U,H)$ are globally Lipschitz continuous. Denote by $\mathscr{C}^2(H)$ the set of all real valued functions on H having continuous the first and second Fréchet derivatives.

Theorem 10 ([49], Corollary 2.3). *Let there exist a function $V \in \mathscr{C}^2(H)$ satisfying:*

i) $V(0) = 0$ and

$$\inf_{\|y\|\geq r} V(y) > 0 \quad \textit{for any } r > 0;$$

ii) for some $k < \infty$, $p > 0$ and any $y \in H$ we have

$$V(y) + \|DV(y)\| + \|D^2V(y)\| \leq k\bigl(1 + \|y\|^p\bigr);$$

iii) there exists $a > 0$ such that

$$\big\langle DV(x-y), Ax - Ay + f(x) - f(y)\big\rangle \\ + \frac{1}{2}\operatorname{Tr}\big\{\big(\sigma(x)-\sigma(y)\big)^* D^2V(x-y)\big(\sigma(x)-\sigma(y)\big)Q\big\} \le -aV(x-y)$$

for all $x, y \in \operatorname{Dom}(A)$.

Then

$$\big(P_t(x,\cdot) - P_t(y,\cdot)\big) \xrightarrow[t\to\infty]{\mathrm{w}^*} 0$$

for any $x, y \in H$.

In particular, if there exists an invariant measure for (29) then it is globally asymptotically stable for the narrow convergence and, *a fortiori*, unique.

According to Corollary 2.8 in [49], the hypotheses of Theorem 10 are fulfilled with the natural choice $V = \|\cdot\|^p$ (for a suitable $p > 0$), provided

$$\langle Ax, x\rangle \le \beta\|x\|^2 \quad \text{for a } \beta \in \mathbb{R} \text{ and every } x \in \operatorname{Dom}(A),$$
$$\big\|Q^{1/2}\big(\sigma(x)-\sigma(y)\big)^*(x-y)\big\| \ge \alpha\|x-y\|^2 \quad \text{for an } \alpha \ge 0,$$

and

$$\beta + \operatorname{Lip}(f) + \frac{1}{2}\operatorname{Lip}(\sigma)^2 \operatorname{Tr} Q < \alpha^2,$$

$\operatorname{Lip}(\Upsilon)$ denoting the Lipschitz constant of a mapping Υ.

As we have explained in Section 2, most of the recent results on the weak stability have been obtained by the "dissipativity method" of G. Da Prato and J. Zabczyk. To show the core of the method, we sketch here a proof of one of their results concerning a stochastic partial differential equation

$$\mathrm{d}X = \big(AX + f(X)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W \tag{30}$$

with an additive noise in a separable Hilbert space H. We assume that W is a standard cylindrical Wiener process in a Hilbert space U, $\sigma \in \mathscr{L}(U,H)$, and $A : \operatorname{Dom}(A) \longrightarrow H$ is a closed linear operator. To state the other hypotheses, we need a few additional definitions. If E is a Banach space, we denote by $\partial\|x\|_E$ the subdifferential of the norm $\|\cdot\|_E$ at the point $x \in E$. We say that a mapping $\gamma : \operatorname{Dom}(\gamma) \subseteq E \longrightarrow E$ is dissipative, provided for any $x, y \in \operatorname{Dom}(\gamma)$ there exists $z^* \in \partial\|x-y\|_E$ such that

$$z^*\big(\gamma(x) - \gamma(y)\big) \le 0.$$

A dissipative mapping γ is called m-dissipative, if $\operatorname{Im}(\lambda I - \gamma) = E$ for a $\lambda > 0$. Let $G \subseteq E$ be a subspace, a part γ_G of the mapping γ on G is defined by

$$\operatorname{Dom}(\gamma_G) = \{x \in \operatorname{Dom}(\gamma) \cap G;\ \gamma(x) \in G\}, \quad \gamma_G = \gamma \text{ on } \operatorname{Dom}(\gamma_G).$$

For completeness, we list here assumptions under which there exists a unique (generalized) mild solution of the equation (30) for any initial condition $X(0) = x \in H$, and (30) defines a Feller Markov process in H. We suppose:

1) There exists $\eta \in \mathbb{R}$ such that the mappings $A - \eta I$ and $f - \eta I$ are m-dissipative on H.

2) There exists a reflexive Banach space K densely and continuously imbedded in H, and $(A - \eta I)_K$, $(f - \eta I)_K$ are m-dissipative on K.

3) $\mathrm{Dom}(f) \supseteq K$ *and f maps bounded set in K into bounded sets in H.*

4) The process

$$W_A(t) = \int_0^t e^{A(t-r)}\sigma\,\mathrm{d}W(r), \quad t \geq 0,$$

is $\mathrm{Dom}(f_K)$*-valued, with paths continuous in H, and*

$$\sup_{t\in[0,T]} \big\{\|W_A(t)\|_K + \|f(W_A(t))\|_K\big\} < \infty \quad \textit{almost surely}$$

for every $T > 0$.

Let us note that the introduction of an auxiliary space K is inevitable as interesting nonlinearities f are not defined (or do not behave well) on the basic state space H (compare Example 12 below).

Now we are prepared to state a theorem on existence and stability of an invariant measure (see Theorem 2.3 in [19], cf. also [20], Theorem 6.3.3).

Theorem 11. *Let there exist $\omega_1, \omega_2 \in \mathbb{R}$ such that $\omega \equiv \omega_1 + \omega_2 > 0$ and the mappings $A + \omega_1 I$, $f + \omega_2 I$ are dissipative on H. Suppose that*

$$\sup_{t\geq 0} \boldsymbol{E}\big\{\|W_A(t)\|_H + \|f(W_A(t))\|_H\big\} < \infty.$$

Then there exists a unique invariant measure μ for (30) *and for any $y \in H$ we have*

$$P_t(y,\cdot) \xrightarrow[t\to\infty]{\mathrm{w}^*} \mu.$$

Moreover, there exists a constant $L < \infty$ such that

$$\left|\int_H g\,\mathrm{d}P_t(y,\cdot) - \int_H g\,\mathrm{d}\mu\right| \leq L\big(1 + \|y\|\big)e^{-\omega t/2}\mathrm{Lip}(g) \tag{31}$$

for any $y \in H$, $t > 0$ and any bounded Lipschitz function $g : H \longrightarrow \mathbb{R}$.

The procedure used in the proof, that is known as the "remote start method", yields in the present case existence and uniqueness of the invariant measure at the same time. We shall consider the equation (30) on the whole real line $\mathbb{R}$, that is, we shall work with solutions to

$$\mathrm{d}X_t = \big(AX_t + f(X_t)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}\widetilde{W}_t, \tag{32}$$

where

$$\widetilde{W}(t) = \begin{cases} W(t), & t \geq 0, \\ Y(-t), & t < 0, \end{cases}$$

Y being a standard cylindrical Wiener process independent of W. Denote by $X(t; s, y)$, $t \geq s$, the unique solution of (32) with the initial datum $X(s; s, y) = y$. First, we derive an a priori estimate

$$\boldsymbol{E}\|X(t; s, y)\| \leq c + \|y\| \tag{33}$$

valid for all $s < 0$, $t \geq s$, $y \in H$. Setting

$$\Psi(t) = X(t; s, y) - \int_s^t e^{A(t-r)} \sigma \, \mathrm{d}\widetilde{W}(r)$$

we see that Ψ pathwise solves the equation

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = A\Psi + f\left(\Psi + \int_s^t e^{A(t-r)} \sigma \, \mathrm{d}\widetilde{W}(r)\right), \quad \Psi(s) = y.$$

Using the dissipativity hypothesis of Theorem 11 one easily finds that

$$\frac{\mathrm{d}^-}{\mathrm{d}t}\|\Psi(t)\| \leq -\omega\|\Psi(t)\| + \left\| f\left(\int_s^t e^{A(t-r)} \sigma \, \mathrm{d}\widetilde{W}(r)\right)\right\|,$$

which yields the desired estimate (33).

Analogously, for $v < s < 0$ one arrives at an estimate

$$\boldsymbol{E}\|X(t; s, y) - X(t; v, y)\| \leq e^{-\omega(t-s)}\big(2\|y\| + c\big), \quad t \geq s, \tag{34}$$

and it follows that the net $\{X(0; s, y),\ s \leq 0\}$ is Cauchy in $L^1(\Omega; H)$ as $s \to -\infty$. Let $p \in L^1(\Omega; H)$ be its limit, then the law μ of p is an invariant measure for (30): The L^1-convergence obviously implies the narrow convergence, therefore (P_t^* denoting the adjoint Markov semigroup)

$$P_t^* \delta_y = P_t(y, \cdot) = \mathrm{Law}(X(t; 0, y)) = \mathrm{Law}(X(0; -t, y)) \xrightarrow[t \to +\infty]{\mathrm{w}^*} \mathrm{Law}(p) = \mu,$$

and, since the Markov process solving (30) is Feller, we obtain

$$P_s^* \mu = P_s^*\Big(\lim_{t\to\infty} P_t^* \delta_y\Big) = \lim_{t\to\infty} P_{t+s}^* \delta_y = \mu$$

for any $s \geq 0$. The estimate (31) on the speed of convergence now follows from (34) in a straightforward way.

A similar theorem holds for equations with multiplicative noise, that is, for equations of the form (29), where W is now assumed to be a standard cylindrical Wiener process, see [13], Theorem 1, and [20], Theorem 6.3.2. We shall not cite the result precisely, let us only note that in this case the dissipativity assumption includes also the Yosida approximations $A_n = nA(nI - A)^{-1}$ of the operator A and reads as follows:

$$\big\langle A_n(x-y) + f(x) - f(y), x-y \big\rangle + \|\sigma(x) - \sigma(y)\|_{\mathrm{HS}}^2 \le -\varpi \|x-y\|^2$$

for a $\varpi > 0$ and any $x, y \in H$ and $n \in \mathbb{N}$.

We finish this section with an example which is very particular case of the example discussed in [19], Section 4, and in [20], §11.4, this example being based on Theorem 11.

Example 12. Let us consider a stochastic parabolic equation

$$\mathrm{d}X(t,\xi) = \big\{(\Delta - \alpha)X(t,\xi) + f(X(t,\xi))\big\}\,\mathrm{d}t + \mathrm{d}W(t,\xi), \quad \xi \in \mathbb{R},\ t \ge 0, \tag{35}$$

where $\alpha > 0$ and W is a standard cylindrical Wiener process in $L^2(\mathbb{R})$. Assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f = f_0 + f_1$, f_0 being (globally) Lipschitz continuous, $\xi \mapsto f_1(\xi) + b\xi$ is a continuous decreasing function for a $b \in \mathbb{R}$, and

$$|f_1(\xi)| \le c(1 + |\xi|^p)$$

for some $p \ge 1$, $c < \infty$ and every $\xi \in \mathbb{R}$. (For example, if f_1 is an odd degree polynomial with a negative leading coefficient,

$$f_1(\xi) = -\xi^{2k+1} + \sum_{j=0}^{2k} a_j \xi^j,$$

then the assumptions are satisfied.) Under the above hypotheses, there exists a unique (generalized) mild solution of (35) in the weighted space $L^2(\mathbb{R}; e^{-\varkappa|\xi|}\,\mathrm{d}\xi)$, for any $\varkappa > 0$. Moreover, suppose that f_1 is decreasing and

$$\alpha - \mathrm{Lip}(f_0) > 0.$$

Then there exists $\varkappa_0 > 0$ such that for any $\varkappa \in \,]0, \varkappa_0[$ the Markov process defined by (35) in the space $L^2(\mathbb{R}; e^{-\varkappa|\xi|}\,\mathrm{d}\xi)$ has a unique invariant measure, and an estimate of the type (31) holds for any $\omega \in \,]0, 2(\alpha - \mathrm{Lip}(f_0))[$.

References

1. S. Albeverio, V. Bogachev, M. Röckner: On uniqueness of invariant measures for finite and infinite dimensional diffusions, Universität Bielefeld, SFB 343, Preprint 97–057
2. S. Albeverio, Yu. G. Kondratiev, M. Röckner: Ergodicity of L^2-semigroups and extremality of Gibbs states, *J. Funct. Anal.* **144** (1997), 394–423
3. S. Albeverio, Yu. G. Kondratiev, M. Röckner: Ergodicity for the stochastic dynamics of quasi-invariant measures with applications to Gibbs states, *J. Funct. Anal.* **149** (1997), 415–469

4. A. Bensoussan, A. Răşcanu: Large time behaviour for parabolic stochastic variational inequalities, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat.* **42** (1996), 149–173
5. V. I. Bogachev, N. Krylov, M. Röckner: Regularity of invariant measures: the case of non-constant diffusion part, *J. Funct. Anal.* **138** (1996), 223–242
6. V. I. Bogachev, M. Röckner: Regularity of invariant measures in finite and infinite dimensional spaces and applications, *J. Funct. Anal.* **133** (1995), 168–223
7. V. Bogachev, M. Röckner, T. S. Zhang: Existence and uniqueness of invariant measures: an approach via sectorial forms, Universität Bielefeld, SFB 343, Preprint 97–072
8. A. Chojnowska-Michalik, B. Goldys: Existence, uniqueness and invariant measures for stochastic semilinear equations in Hilbert spaces, *Probab. Theory Related Fields* **102** (1995), 331-356
9. I. D. Chueshov, T. V. Girya: Inertial manifolds and forms for semilinear parabolic equations subjected to additive noise, *Lett. Math. Phys.* **34** (1995), 69–76
10. G. Da Prato, A. Debussche: Stochastic Cahn-Hilliard equation, *Nonlinear Anal.* **26** (1996), 241–263
11. G. Da Prato, K. D. Elworthy, J. Zabczyk: Strong Feller property for stochastic semilinear equations, *Stochastic Anal. Appl.* **13** (1995), 35–45
12. G. Da Prato, D. Gątarek: Stochastic Burgers equation with correlated noise, *Stochastics Stochastics Rep.* **52** (1995), 29–41
13. G. Da Prato, D. Gątarek, J. Zabczyk: Invariant measures for semilinear stochastic equations, *Stochastic Anal. Appl.* **10** (1992), 387–408
14. G. Da Prato, D. Nualart, J. Zabczyk: Strong Feller property for infinite-dimensional stochastic equations, Scuola Normale Superiore Pisa, Preprints di Matematica n. 33/1994
15. G. Da Prato, J. Zabczyk: Smoothing properties of transition semigroups in Hilbert spaces, *Stochastics Stochastics Rep.* **35** (1991), 63–77
16. G. Da Prato, J. Zabczyk: Non-explosion, boundedness and ergodicity for stochastic semilinear equations, *J. Differential Equations* **98** (1992), 181–195
17. G. Da Prato, J. Zabczyk: On invariant measure for semilinear equations with dissipative nonlinearities, *Stochastic partial differential equations and their applications* (Charlotte, 1991), 38–42, Lecture Notes in Control Inform. Sci. 176, Springer 1992
18. G. Da Prato, J. Zabczyk: *Stochastic equations in infinite dimensions*, Cambridge University Press, Cambridge 1992
19. G. Da Prato, J. Zabczyk: Convergence to equilibrium for classical and quantum spin systems, *Probab. Theory Related Fields* **103** (1995), 529–552
20. G. Da Prato, J. Zabczyk: *Ergodicity for infinite dimensional systems*, Cambridge University Press, Cambridge 1996
21. J. L. Doob: Asymptotic properties of Markoff transition probabilities, *Trans. Amer. Math. Soc.* **63** (1948), 393–421
22. M. Duflo, D. Revuz: Propriétés asymptotiques de probabilités de transition des processus de Markov récurrents, *Ann. Inst. H. Poincaré Probab. Statist.* **5** (1969), 233–244
23. B. Ferrario: Ergodic results for stochastic Navier-Stokes equation, *Stochastics Stochastics Rep.* **60** (1997), 271–288
24. F. Flandoli, B. Maslowski: Ergodicity of the 2–D Navier-Stokes equation under random perturbations, *Comm. Math. Phys.* **171** (1995), 119–141
25. M. I. Freidlin: Random perturbations of reaction-diffusion equations: the quasideterministic approximation, *Trans. Amer. Math. Soc.* **305** (1988), 665–697

26. M. Fuhrman: Smoothing properties of nonlinear stochastic equations in Hilbert spaces, *NODEA Nonlinear Differential Equations Appl.* **3** (1996), 445–464
27. D. Gątarek, B. Gołdys: Existence, uniqueness and ergodicity for the stochastic quantization equation, *Studia Math.* **119** (1996), 179–193
28. D. Gątarek, B. Goldys: On invariant measures for diffusions on Banach spaces, *Potential Anal.* **7** (1997), 539–553
29. T. V. Girya: On stabilization of solutions to nonlinear stochastic parabolic equations, *Ukrain. Mat. Zh.* **41** (1989), 1630–1636 (in Russian)
30. T. V. Girya, I. D. Khueshov: Inertial manifolds and stationary measures for dissipative dynamical systems with a random perturbation, *Mat. Sb.* **186** (1995), 29–46 (in Russian)
31. Hu Xuanda: Boundedness and invariant measures of semilinear stochastic evolution equations, *Nanjing Daxue Xuebao Shuxue Bannian Kan* **4** (1987), 1–14
32. A. Ichikawa: Semilinear stochastic evolution equations: Boundedness, stability and invariant measures, *Stochastics* **12** (1984), 1–39
33. Ya. Sh. Il'yasov, A. I. Komech: The Girsanov theorem and ergodic properties of statistical solutions to nonlinear parabolic equations, *Trudy Sem. Petrovskogo* **12** (1987), 88–117 (in Russian)
34. S. Jacquot: Strong ergodicity results on Wiener space, *Stochastics Stochastics Rep.* **51** (1994), 133–154
35. S. Jacquot: Simulated annealing for stochastic semilinear equations in Hilbert spaces, *Stochastic Process. Appl.* **64** (1996), 73–91
36. S. Jacquot, G. Royer: Ergodicité d'une classe d'équations aux dérivées partielles stochastiques, *C. R. Acad. Sci. Paris Sér. Math.* **320** (1995), 231–236
37. R. Z. Khas'minskiĭ: Ergodic properties of recurrent diffusion processes and stabilization of solutions to the Cauchy problem for parabolic equations, *Teor. Veroyatnost. i Primenen.* **5** (1960), 196–214 (in Russian)
38. R. Z. Khas'minskiĭ: *Stability of systems of differential equations under random perturbations of their parameters*, Nauka, Moskva 1969 (in Russian); English translation: *Stochastic stability of differential equations*, Sijthoff & Noordhoff, Alphen aan den Rijn 1980
39. Yu. G. Kondratiev, S. Roelly, H. Zessin: Stochastic dynamics for an infinite system of random closed strings: A Gibbsian point of view, *Stochastic Process. Appl.* **61** (1996), 223–248
40. S. M. Kozlov: Some problems concerning stochastic partial differential equations, *Trudy Sem. Petrovskogo* **4** (1978), 147–172 (in Russian)
41. H.-H. Kuo: *Gaussian measures in Banach spaces*, Lecture Notes in Math. 463, Berlin 1975
42. G. Leha, G. Ritter: Lyapunov-type conditions for stationary distributions of diffusion processes on Hilbert spaces, *Stochastics Stochastics Rep.* **48** (1994), 195–225
43. G. Leha, G. Ritter: Stationary distributions of diffusion processes with singular drift on Hilbert spaces, in preparation
44. R. Manthey, B. Maslowski: Qualitative behaviour of solutions of stochastic reaction-diffusion equations, *Stochastic Process. Appl.* **43** (1992), 265–289
45. R. Marcus: Parabolic Itô equations, *Trans. Amer. Math. Soc.* **198** (1974), 177–190
46. R. Marcus: Parabolic Itô equations with monotone nonlinearities, *J. Funct. Anal.* **29** (1978), 275–286
47. R. Marcus: Stochastic diffusion on an unbounded domain, *Pacific J. Math.* **84** (1979), 143–153

48. G. Maruyama, H. Tanaka: Ergodic property of N-dimensional recurrent Markov processes, *Mem. Fac. Sci. Kyushu Univ. Ser. A* **13** (1959), 157–172
49. B. Maslowski: Uniqueness and stability of invariant measures for stochastic differential equations in Hilbert spaces, *Stochastics Stochastics Rep.* **28** (1989), 85–114
50. B. Maslowski: Strong Feller property for semilinear stochastic evolution equations and applications, *Stochastic systems and optimization* (Warsaw, 1988), 210–224, Lecture Notes in Control Inform. Sci. 136, Springer-Verlag, Berlin 1989
51. B. Maslowski: On ergodic behaviour of solutions to systems of stochastic reaction-diffusion equations with correlated noise, *Stochastic processes and related topics* (Georgenthal, 1990), 93–102, Akademie-Verlag, Berlin 1991
52. B. Maslowski: On probability distributions of solutions of semilinear stochastic evolution equations, *Stochastics Stochastics Rep.* **45** (1993), 17–44
53. B. Maslowski: Stability of semilinear equations with boundary and pointwise noise, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **22** (1995), 55–93
54. B. Maslowski: Asymptotic properties of stochastic equations with boundary and pointwise noise, *Stochastic processes and related topics* (Siegmundsburg, 1994), 67–76, Gordon and Breach, Amsterdam 1996
55. B. Maslowski, J. Seidler: Ergodic properties of recurrent solutions of stochastic evolution equations, *Osaka J. Math.* **31** (1994), 965–1003
56. B. Maslowski, J. Seidler: Probabilistic approach to the strong Feller property, in preparation
57. B. Maslowski, I. Simão: Asymptotic properties of stochastic semilinear equations by method of lower measures, *Colloq. Math.* **79** (1997), 147–171
58. S. Mück: Semilinear stochastic equations for symmetric diffusions, Stochastics Stochastics Rep., to appear (1998)
59. C. Mueller: Coupling and invariant measures for the heat equation with noise, *Ann. Probab.* **21** (1993), 2189–2199
60. S. Peszat, J. Zabczyk: Strong Feller property and irreducibility for diffusions on Hilbert spaces, *Ann. Probab.* **23** (1995), 157–172
61. J. Seidler: Ergodic behaviour of stochastic parabolic equations, *Czechoslovak Math. J.* **47** (122) (1997), 277–316
62. R. Sowers: Large deviations for the invariant measure of a reaction-diffusion equation with non-Gaussian perturbations, *Probab. Theory Related Fields* **92** (1994), 393–421
63. Ł. Stettner: Remarks on ergodic conditions for Markov processes on Polish spaces, *Bull. Polish Acad. Sci. Math.* **42** (1994), 103–114
64. D. W. Stroock: Logarithmic Sobolev inequality for Gibbs states, *Dirichlet forms* (Varenna, 1992), 194–228, Lecture Notes in Math. 1563, Springer-Verlag, Berlin 1993
65. M. J. Vishik, A. V. Fursikov: *Mathematical problems of stochastic hydromechanics*, Kluwer Academic Publishers, Dordrecht 1988
66. J. Zabczyk: Structural properties and limit behaviour of linear stochastic systems in Hilbert spaces, *Mathematical control theory*, 591–609, Banach Center Publications Vol. 14, PWN, Warsaw 1985
67. B. Zegarlinski: Ergodicity of Markov semigroups, *Stochastic partial differential equations* (Edinburgh, 1994), 312–337, Cambridge University Press, Cambridge 1995