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Quadratic Functionals: Positivity, Oscillation, Rayleigh's Principle

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Abstract. In this paper we give a survey on the theory of quadratic functionals. Particularly the relationships between positive definiteness and the asymptotic behaviour of Riccati matrix differential equations, and between the oscillation properties of linear Hamiltonian systems and Rayleigh's principle are demonstrated. Moreover, the main tools form control theory (as e.g. characterization of strong observability), from the calculus of variations (as e.g. field theory and Picone's identity), and from matrix analysis (as e.g. l'Hospital's rule for matrices) are discussed.

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1 Introduction

This article presents a survey on the theory of quadratic functionals as described in a recent book by W. Kratz [13]. This theory is based mainly on the work by M. Morse and W. T. Reid (see [16] and [18]). We introduce the necessary notions and formulate the central results, but without any proofs. The setup of the paper is as follows.

In the next section we introduce the necessary notation and basic concepts, namely: We consider *quadratic functionals* for state and control functions, which satisfy a linear differential system (called the *equations of motion*), and for which the state function satisfies additionally some linear and homogeneous boundary condition. Classical methods of the calculus of variations lead to a *self-adjoint eigenvalue problem* consisting of a *linear Hamiltonian system* and boundary conditions (including the corresponding Euler equation and the natural boundary conditions). These eigenvalue problems contain e.g. the well-known Sturm-Liouville problems. The study of oscillation properties of the Hamiltonian system requires the concept of *conjoined bases* and their *focal points*, which includes in a certain sense the basic notion of disconjugacy. Moreover, the central notions of *controllability* and *strong observability* (or observability with unknown inputs) from control theory play a key role in the theory as well as *Riccati matrix differential equations* corresponding to linear Hamiltonian systems.

In Section 3 we formulate the main results concerning the positivity of quadratic functionals. Theorem 4 states a *Reid Roundabout Theorem*, which includes e.g. the well-known Jacobi condition from the calculus of variations as a special case. Theorem 5 concerns the positivity of a quadratic functional depending on a parameter. It states essentially that the positivity of the functional for small values of the parameter is equivalent to a certain asymptotic behaviour of the corresponding Riccati equation, and that it is also equivalent to strong observability of the underlying linear system.

In Section 4 we present in Theorem 7 the central oscillation theorem for Hamiltonian systems. The next Theorem 9 states the basic properties of the corresponding eigenvalue problem, i.e., existence of eigenvalues, Rayleigh's principle, and the expansion theorem.

Finally, we describe in Section 5 the main tools for the proofs. These tools include results from the calculus of variations (as e.g. Picone's identity), from matrix analysis (as e.g. properties of monotone matrix-valued functions), from linear control theory (as e.g. a canonical form for controllable systems), and from functional analysis (Ehrling's lemma). We formulate explicitly two basic results, namely a substitute of l'Hospital's rule for matrices in Theorem 10 and a characterization of strong observability for time-dependent systems in Theorem 11.

2 Notation and basic concepts

First we introduce *quadratic functionals*

$$\mathcal{F}(x) := \int_a^b \{x^T Cx + u^T Bu\}(t)dt + \begin{pmatrix} -x(a) \\ x(b) \end{pmatrix}^T S_1 \begin{pmatrix} -x(a) \\ x(b) \end{pmatrix}, \tag{1}$$

and *bilinear forms*

$$\langle x, y\rangle_0 := \int_a^b \{x^T C_0 y\}(t)dt\ ,\ \langle x, y\rangle := \langle x, y\rangle_0 + \begin{pmatrix} -x(a) \\ x(b) \end{pmatrix}^T S_0 \begin{pmatrix} -y(a) \\ y(b) \end{pmatrix}, \tag{2}$$

where x (or (x,u)) is (A,B)-*admissible* , i.e., the so-called *equations of motion* $\dot{x} = Ax + Bu$ hold on $\mathcal{I} := [a,b]$ for some *control* u with $Bu \in C_s(\mathcal{I})$ (i.e., Bu is piecewise continuous on $\mathcal{I}$), and where it satisfies boundary conditions of the form $\begin{pmatrix} -x(a) \\ x(b) \end{pmatrix} \in \mathcal{V}$, we write $x \in \tilde{\mathcal{R}}$, and where the same holds for y. Throughout we impose the following assumptions on the given data:

(A) $A(t), B(t), C(t), C_0(t)$ are real $n \times n$-matrix-valued functions, which are piecewise continuous on $\mathbb{R}$, $B(t), C(t), C_0(t)$ are symmetric, and $B(t), C_0(t)$ are non-negative definite (we write $B(t) \geq 0$, $C_0(t) \geq 0$) for $t \in \mathbb{R}$. $\mathcal{V} \subset \mathbb{R}^{2n}$ is a subspace of $\mathbb{R}^{2n}$, S_0 and S_1 are real and symmetric $2n \times 2n$-matrices, and $S_0 \geq 0$.

Let R_2, S_2 be $2n \times 2n$-matrices, such that $\mathcal{V} = \operatorname{Im} R_2^T$ and $S_2 R_2^T = 0$, $\operatorname{rank}(R_2, S_2) = 2n$ (see [13, Corollary 3.1.3]). We put

$$R_1(\lambda) := R_2(S_1 - \lambda S_0) + S_2 \quad , \quad R_1 := R_1(0) \tag{3}$$

for $\lambda \in \mathbb{R}$. By Im, rank, ker we denote the *image*, *rank*, *kernel* of a matrix, and I denotes the *identity matrix* of corresponding size.

The pair (x,u) is stationary for the functional $\mathcal{F}$ if it satisfies the natural boundary conditions and the *Euler equations* $\dot{u} = Cx - A^T u$. These Euler equations lead together with the equations of motion to the linear *Hamiltonian system*

$$\dot{x} = Ax + Bu \quad , \quad \dot{u} = Cx - A^T u \, , \tag{H}$$

and the natural boundary conditions together with the given boundary conditions $\tilde{\mathcal{R}}$ lead to the self-adjoint boundary conditions (B_λ) below with $\lambda = 0$.

We need the following basic notions (see [13]).

Definition 1.

(i) (X,U) is called a *conjoined basis* of (H), if $X(t), U(t)$ are real $n \times n$-matrix-valued solutions of (H) with

$$\operatorname{rank}(X^T(t), U^T(t)) \equiv n \quad , \quad X^T(t)U(t) - U^T(t)X(t) \equiv 0 \ \text{ on } \mathbb{R} \, .$$

(ii) Two conjoined bases (X_1, U_1) and (X_2, U_2) are called *normalized conjoined bases* of (H) if $X_1^T(t)U_2(t) - U_1^T(t)X_2(t) \equiv I$ on $\mathbb{R}$; and $(\tilde{X}_1, \tilde{U}_1), (\tilde{X}_2, \tilde{U}_2)$ denote the special normalized bases of (H), which satisfy the initial conditions

$$\tilde{X}_1(a) = \tilde{U}_2(a) = 0 \quad , \quad \tilde{U}_1(a) = -\tilde{X}_2(a) = I \, ,$$

and then $(\tilde{X}_1, \tilde{U}_1)$ is called the *principal solution at a* .

(iii) A point $t_0 \in \mathbb{R}$ is called a *focal point* of X (or (X,U)) for a conjoined basis (X,U), if $X(t_0)$ is non-invertible, and the dimension of the kernel of $X(t_0)$ is called its multiplicity.

(iv) The pair (A, B) is called *controllable* on $\mathcal{J}$, if $\dot{v} = -A^T(t)v$, $B^T(t)v(t) \equiv 0$ on some non-degenerate interval $\tilde{\mathcal{J}} \subset \mathcal{J}$ always implies that $v(t) \equiv 0$ on $\tilde{\mathcal{J}}$ (see also [6, Definition 1.2.4] for "uniformly controllable").

(v) The triple (A, B, C_0) (or the *linear system*

$$\dot{x} \;=\; Ax + Bu \quad , \quad y \;=\; C_0\, x \tag{LS}$$

with state x, input u, and output y) is called *strongly observable* on $\mathcal{J}$ if $\dot{x} = A(t)x + B(t)u$, $C_0(t)x(t) \equiv 0$ on some non-degenerate interval $\tilde{\mathcal{J}} \subset \mathcal{J}$ for some function u with $Bu \in C_s(\tilde{\mathcal{J}})$ always implies that $x(t) \equiv 0$ on $\tilde{J}$.

Given any conjoined basis (X, U) of (H) there exists another conjoined basis (X_2, U_2) such that $(X_1 = X, U_1 = U)$, (X_2, U_2) are normalized conjoined bases (see [13, Proposition 4.1.1]). By [13, Theorem 4.1.3] controllability of (A, B) is the same as saying that the focal points of every conjoined basis of (H) are isolated. Obviously, strong observability of (A, B, C_0) means that the bilinear form $\langle \cdot, \cdot \rangle_0$ is an inner product on the space of all (A, B)-admissible functions. Moreover, it is well-known (see [19] or [13]) that, for any conjoined basis (X, U) of (H), the quotient $Q(t) := U(t)X^{-1}(t)$ satisfies the *Riccati matrix differential equation*

$$\dot{Q} + A^T Q + QA + QBQ - C \;=\; 0\;, \tag{R}$$

whenever $X(t)$ is invertible.

The investigation of extremal values if the so-called *Rayleigh quotient* $R(x) := \mathcal{F}(x)/\langle x, x\rangle$ leads to functions x and reals λ, where the functional

$$\mathcal{F}(x, \lambda) \;:=\; \mathcal{F}(x) - \lambda \langle x, x\rangle \tag{4}$$

is stationary. Hence, these values λ are the eigenvalues of the *eigenvalue problem* $(\mathcal{E})$, which consists of the Hamiltonian system

$$\dot{x} \;=\; Ax + Bu\;,\; \dot{u} \;=\; (C - \lambda\, C_0)x - A^T u\;, \tag{H$_\lambda$}$$

and of the $2n$ linear and homogeneous boundary conditions

$$R_1(\lambda)\begin{pmatrix} -x(a) \\ x(b) \end{pmatrix} + R_2 \begin{pmatrix} u(a) \\ u(b) \end{pmatrix} \;=\; 0\;,\ \text{i.e.}\;, (x, u) \in \mathcal{R}(\lambda)\;. \tag{B$_\lambda$}$$

Note that (H)=(H$_0$), and that $(x, u) \in \mathcal{R}(\lambda)$ implies that $x \in \tilde{\mathcal{R}}$. Moreover, as above, there corresponds to (H$_\lambda$) a Riccati equation, namely:

$$\dot{Q} + A^T Q + QA + QBQ - C + \lambda\, C_0 \;=\; 0\;. \tag{R$_\lambda$}$$

Remark 2. If the matrix $B(t)$ is *positive definite* for $t \in \mathcal{I}$, then the functional $\mathcal{F}$ and the equation of motion, i.e., $u = B^{-1}(\dot{x} - Ax)$, reduce to a quadratic functional occurring as *second variation* in the classical calculus of variations, which satisfies the strengthened Legendre condition. Moreover, our eigenvalue problems $(\mathcal{E})$ include the self-adjoint *Sturm-Liouville problems* of even order as a special case, where e.g. rank $B(t) =$ rank $C_0(t) \equiv 1$.

3 Positivity

In this section we derive criteria for the positive definiteness of the functionals $\mathcal{F}(\cdot)$ and $\mathcal{F}(\cdot,\lambda)$.

Definition 3. The functional $\mathcal{F}$ is called *positive definite*, we write $\mathcal{F} > 0$, if $\mathcal{F}(x) > 0$ for all (A,B)-admissible x with $x \in \tilde{\mathcal{R}}$ and $x(t) \not\equiv 0$ on $\mathcal{I}$.

Our first result includes the classical *Jacobi condition* from the calculus of variations, and it is often called "Reid Roundabout Theorem" (see [1], [2], [4], [5], [18]).

Theorem 4. *Assume* (A)*, and suppose that the pair* (A,B) *is controllable on* $\mathcal{I}$*. Then* $\mathcal{F} > 0$ *if and only if the following two assertions hold:*

(i) $\tilde{X}_1(t)$ *possesses no focal point in* $(a,b]$*.*

(ii) *The matrix* $M := R_2\{S_1 + \tilde{M}\}\, R_2^T$ *is positive definite on* $\mathrm{Im}\, R_2$*, where the matrix* $\tilde{M}$ *is defined by*

$$\tilde{M} \;:=\; \begin{pmatrix} -\tilde{X}_1^{-1}\tilde{X}_2 & \tilde{X}_1^{-1} \\ \left(\tilde{X}_1^{-1}\right)^T & \tilde{U}_1\tilde{X}_1^{-1} \end{pmatrix}(b)\ . \tag{5}$$

This result is [13, Theorem 2.4.1]. Note that the matrices $\tilde{M}$ and M are symmetric and that assertion (ii) is empty (i.e., always satisfied), if $R_2 = 0$. The connection between the Hamiltonian system (H) and the Riccati equation (R) yields quite easily that the assertion (i) is equivalent with:

(i') The Riccati equation (R) possesses a symmetric solution $Q(t)$ on $\mathcal{I}$.

Moreover, in the case of so-called "separated boundary conditions" there is another result [13, Theorem 2.4.2], which uses only one conjoined basis (rather than $(\tilde{X}_1, \tilde{U}_1)$ and $(\tilde{X}_2, \tilde{U}_2)$ as above) depending on the boundary conditions (see also [5]).

Our next result concerns the positivity of $\mathcal{F}(\cdot,\lambda)$ for sufficiently small values of λ, and it is contained in the recent paper [15, Theorem 2]. It requires additional smoothness assumptions on the given data, i.e., for $n \geq 2$,

$$A \in C_s^{2n-3}(\mathbb{R})\ ,\ B \in C_s^{2n-3}(\mathbb{R})\ ,\ C_0 \in C_s^{2n-2}(\mathbb{R})\ . \tag{A1}$$

Theorem 5. *Assume* (A), (A1)*, and suppose that the pair* (A,B) *is controllable on* $\mathbb{R}$*. Then the following statements are equivalent:*

(i) *The linear system* (LS) *is strongly observable on* $\mathbb{R}$*.*

(ii) *For all non-degenerate intervals* $\mathcal{I} = [a, b]$ *and symmetric matrices* Q_0 *the solution* $Q(t; \lambda)$ *of* (R_λ) *with the initial condition* $Q(a; \lambda) \equiv Q_0$ *exists on* $\mathcal{I}$*, if* λ *is sufficiently small, and*

$$\lim_{\lambda \to -\infty} Q(b; \lambda) = \infty$$

(i.e., all eigenvalues of the symmetric matrix $Q(b; \lambda)$ *tend to infinity as* $\lambda \to -\infty$*).*

(iii) *For all non-degenerate intervals* $\mathcal{I} = [a, b]$*, subspaces* $\mathcal{V} \subset \mathbb{R}^{2n}$*, symmetric matrices* S_1 *and* S_0 *with* $S_0 \geq 0$*, there exists* $\lambda_0 \in \mathbb{R}$ *such that (see* (4)*)*

$$\mathcal{F}(\cdot, \lambda) > 0 \quad \text{for all} \quad \lambda \leq \lambda_0 \ ,$$

and then, moreover,

$$\min\{R(x) = \mathcal{F}(x)/\langle x, x\rangle : x \text{ is } (A, B)\text{-admissible} \ , \ x \in \tilde{\mathcal{R}} \ , \ x \not\equiv 0\}$$

exists.

Remark 6. The assertion (iii) has the following interpretation in terms of the “optimal linear regulator problem” in control theory (see [10]), namely:
For given data the following LQ-problem with output energy constraints possesses a minimum. Minimize the quadratic functional $\mathcal{F}(x)$ for (A, B)-admissible $x \in \tilde{\mathcal{R}}$ under the additional “output energy” constraint $\langle x, x\rangle = 1$.

4 Oscillation and Rayleigh's principle

In this section we formulate the main results on the oscillation of solutions of the Hamiltonian system (H) and on the eigenvalue problem ($\mathcal{E}$). The oscillation theorem follows immediately from [13, Theorem 7.2.2] by using assertion (ii) or (iii) of Theorem 5.

Theorem 7 (Oscillation). *Assume* (A), (A1)*, and suppose that the pair* (A, B) *is controllable on* $\mathcal{I}$ *and that the linear system* (LS) *is strongly observable on* $\mathcal{I}$*. Let* (X, U) *be any conjoined basis of* (H)*, such that* $X(a)$ *and* $X(b)$ *are invertible. Then*

$$n_1 + n_2 = n_3 + n \ ,$$

where n_1 *denotes the number of focal points of* X *(including multiplicities) in* (a, b) *;*

n_3 *denotes the number of eigenvalues* ($\mathcal{E}$) *(including multiplicities), which are less than zero; and where*

n_2 *denotes the number of negative eigenvalues of the symmetric* $3n \times 3n$*-matrix* $\mathcal{M}$*, which is defined by*

$$\mathcal{M} := \mathcal{R}_2 \begin{pmatrix} \Delta & -X^{-1}(a) & -X^{-1}(b) \\ -(X^{-1})^T(a) & -U(a)X^{-1}(a) & 0 \\ -(X^{-1})^T(b) & 0 & U(b)X^{-1}(b) \end{pmatrix} \mathcal{R}_2^T + \mathcal{R}_1 \mathcal{R}_2^T \ ,$$

where $\mathcal{R}_1 := \begin{pmatrix} 0 & 0 \\ 0 & R_1 \end{pmatrix}$, $\mathcal{R}_2 := \begin{pmatrix} I & 0 \\ 0 & R_2 \end{pmatrix}$, $\Delta := X^{-1}(a)X_2(a) - X^{-1}(b)X_2(b)$ *with conjoined basis* (X_2, U_2) *such that* (X, U) *and* (X_2, U_2) *constitute normalized conjoined bases of* (H).

Remark 8. If the matrix $X(a)$ or $X(b)$ is not invertible, then the same result holds but with a more complicated matrix $\mathcal{M}$, the definition of which needs more notation (see [13, Theorem 7.2.2]).

By using a generalized "Picone identity" [13, Theorem 1.2.1] this oscillation theorem is the main tool to derive Rayleigh's principle for our eigenvalue $(\mathcal{E})$ (see [13, Theorem 7.7.1 and Theorem 7.7.6]), namely:

Theorem 9 (Rayleigh's principle). *Assume* (A), (A1)*, and suppose that the pair* (A, B) *is controllable on* $\mathcal{I}$ *and that the linear system* (LS) *is strongly observable on* $\mathcal{I}$*. Then the following statements hold:*

(i) *There **exist** infinitely many eigenvalues* λ_k *of the eigenvalue problem* $(\mathcal{E})$ *with* $\lambda_k \to \infty$ *(let* $-\infty < \lambda_1 \le \lambda_2 \le \cdots$ *denote these eigenvalues including multiplicities with corresponding orthonormal eigenfunctions* (x_1, u_1), $(x_2, u_2), \dots$ *, so that* $\langle x_k, x_\ell \rangle = \delta_{k\ell}$*).*

(ii) ***Rayleigh's principle*** *holds, i.e. for* $k = 0, 1, 2, \dots$,

$$\lambda_{k+1} \ = \ \min\Big\{ R(x) \ = \ \frac{\mathcal{F}(x)}{\langle x, x \rangle} \ : \ x \ \textit{is} \ (A,B)\textit{-admissible} \ , x \in \tilde{\mathcal{R}} \ , \ x \not\equiv 0 \ , \\ \textit{and} \ \langle x, x_\nu \rangle = 0 \ \textit{for} \ \nu = 1, \dots, k \Big\}.$$

(iii) *The **expansion theorem** holds, i.e.*

$$x \ = \ \sum_{\nu=1}^{\infty} \langle x_\nu, x \rangle x_\nu \ , \quad \textit{i.e.,} \quad \lim_{k \to \infty} \Big\| x - \sum_{\nu=1}^{k} \langle x_\nu, x \rangle x_\nu \Big\| \ = \ 0 \ ,$$

for all (A, B)*-admissible* x *with* $x \in \tilde{\mathcal{R}}$*, where* $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$*.*

5 Tools

In this section we discuss the main tools for the proof of our theorems cited above.

As already mentioned in the previous section a generalization of an identity due to *Picone* [17] (see also [21]) is the basis of the proof of Rayleigh's principle, i.e., Theorem 9, given in [13]. This extended version of Picone's formula can be derived from *field theory* (see e.g. [18]) as discussed in [13, Section 1.3].

The basic tools for the proof of the oscillation theorem, i.e., Theorem 7, come from *matrix analysis* (see [13, Chapter 3]). These results concern in particular limit and rank theorems for monotone matrix-valued functions (see [8], [11]). The basis for the limit theorem is a *substitute of l'Hospital's rule for matrices* [7, Theorem 1], namely:

Theorem 10. *Suppose that X, U are real $n \times n$-matrices such that it is fulfilled* $\operatorname{rank}(X^T, U^T) = n$ *and $X^T U = U^T X$. Then*

$$\lim_{S \to 0+} X(X+SU)^{-1} \;=\; 0\,,$$

where $S \to 0+$ stands for $S \to 0$ and $S > 0$.

This theorem together with monotonicity properties of the Riccati matrix differential equation (R) [13, Section 5.1] leads to the asymptotic behaviour of solutions of (R) (see [9] or [13, Chapter 6]).

Moreover, there are needed results from *linear control theory.* The asymptotics of Riccati equations requires, besides the results from matrix analysis above, in particular a certain *canonical form* of controllable pairs. While the Reid Roundabout Theorem, i.e., Theorem 4, may be proven by using mainly Picone's identity, the proof of Theorem 5 given in [15] depends essentially on two results. The first result is the following *characterization of strong observability* for time-dependent systems [14, Theorem 2].

Theorem 11. *Assume* (A1). *Then the linear system* (LS) *is strongly observable on some interval $\mathcal{I}$ if and only if*

$$\operatorname{rank} S(t) \;=\; n + \operatorname{rank} T(t)$$

for $t \in \mathcal{I}$ except on a nowhere dense subset of $\mathcal{I}$, where the matrix-valued functions $S : \mathbb{R} \to \mathbb{R}^{n^2 \times n^2}$, $T : \mathbb{R} \to \mathbb{R}^{n^2 \times n(n-1)}$ are defined as follows: First denote recursively $C_k = C_k(t)$, $B_{\mu\nu} = B_{\mu\nu}(t)$ by

$$\begin{aligned}
C_1 &:= C_0\,, C_{\mu+1} := \dot{C}_\mu + C_\mu A && \textit{for} \quad \mu = 1, \ldots, n-1, \\
B_{\mu+1,\mu} &:= C_0 B && \textit{for} \quad 0 \le \mu \le n-1\,, \\
B_{\mu+1,0} &:= C_{\mu+1} B + \dot{B}_{\mu 0} && \textit{for} \quad 1 \le \mu \le n-1\,, \\
B_{\mu+1,\nu} &:= B_{\mu,\nu-1} + \dot{B}_{\mu\nu} && \textit{for} \quad 1 \le \nu < \mu \le n-1\,;
\end{aligned}$$

and (in block form) $S := [Q, T]$ with

$$Q \;:=\; \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix}, \qquad T \;:=\; \begin{bmatrix} 0 & 0 & \ldots & 0 \\ B_{10} & 0 & \ldots & 0 \\ B_{20} & B_{21} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{n-1,0} & B_{n-1,1} & \ldots & B_{n-1,n-2} \end{bmatrix}.$$

This result reduces to [12, Theorem 2] or [13, Theorem 3.5.7] for time-invariant systems. In case $B = 0$, the theorem gives a characterization of controllability/observability (see [14, Theorem 1] and [6, Theorem 1.3.2 and Theorem 1.4.4]).

The second tool for the proof of Theorem 5 is a result from *functional analysis*, namely an application of the so-called Ehrling lemma (see [20, Lemma 8] or [3, 8.3]).

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