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Application of the Second Lyapunov Method to Stability Investigation of Differential Equations with Deviations

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Abstract. The paper presents overview of applications of A. M. Lyapunov's direct method to stability investigation of systems with argument delay. Methods of building Lyapunov-Krasovskiy funcionals for linear systems with constant coefficients are considered. Lyapunov quadratic forms are used to obtain applicable methods for stability investigation and estimation of solution convergence for linear stationary systems, as well as non-linear control systems and systems with quadratic and rational right hand sides.

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1 Introduction

The present paper is aimed at investigation of systems with deviating argument of delay type. The investigation is carried out using the second Lyapunov method. The following differential system with delay is considered

$$\dot{x}(t) = f(x(t), x(t-\tau)), \quad \tau > 0 \ . \tag{1}$$

Suppose that $x(t) \equiv 0$ is a solution of system (1), i.e. $f(0,0) \equiv 0$.

As opposed to ODE's, for which the Cauchy problem consists of finding a solution passing through the given point, equations with delay have an initial function. Thus, for (1) the Cauchy problem consists of finding a solution $x(t)$ that satisfies the initial condition $x(t) \equiv \varphi(t), \quad -\tau \leq t \leq 0$, where $\varphi(t)$ is a given initial function. Therefore, initial perturbations of the function $\varphi(t)$, $-\tau \leq t \leq 0$ are required to be small according to the definition of stability.

Definition 1. The solution $x(t) \equiv 0$ of system (1) is called stable according to Lyapunov if for an arbitrary $\varepsilon > 0$ there exists such $\delta(\varepsilon) > 0$ that $|x(t)| < \varepsilon$ when $t > 0$ if $\|x(0)\|_\tau < \delta(\varepsilon)$. Here $\|x(0)\|_\tau = \max_{-\tau \leq s \leq 0} \{|x(s)|\}$.

Definition 2. The solution $x(t) \equiv 0$ is called asymptotically stable if it is stable and the following condition holds

$$\lim_{t \to \infty} |x(t)| = 0 \ .$$

Definition 3. The solution $x(t) \equiv 0$ is exponentially stable if there exist such constants $N > 0$ and $\gamma > 0$ that for an arbitrary solution of the system the following estimate holds

$$|x(t)| \leq N \|x(0)\|_\tau \exp\{-\gamma t\}, \quad t \geq 0 \ .$$

System (1) cannot provide precise description of real objects. By using differential equations it is usually impossible to take into account all different factors that influence the system. Therefore, it is appropriate to consider a perturbed system in the form

$$\dot{x}(t) = f(x(t), x(t-\tau)) + q(x(t), x(t-\tau)) \ . \tag{2}$$

The following definitions of stability account for the influence of perturbation.

Definition 4. The solution $x(t) \equiv 0$ of system (1) is called stable under constantly acting perturbations when for an arbitrary $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ such that for an arbitrary solution $x_Q(t)$ of (2) the condition $|x_Q(t)| < \varepsilon$ when $t > 0$ holds if $\|x_Q(0)\|_\tau < \delta(\varepsilon)$ and $|q(x_Q(t), x_Q(t-\tau))| < \eta(\varepsilon)$.

Differential equations with delay (1) have many things in common with corresponding equations without delay. Therefore, many results from the movement stability theory for systems without delay were extended and adjusted to the equations in the form (1). One of the basic methods for investigation of system stability is the second Lyapunov method. Its application to systems with delay has been developed in two directions:

1. The first direction implies use of finite dimensional functions with an additional condition for the derivative. This is a so called B. S. Razumikhin condition [1,4].

2. The second method is a Lyapunov-Krasovskiy functional method, which has had more comprehensive theoretical ground [2,3,4].

Geometrical meaning of the Lyapunov function method involves finding the system of closed surfaces that contain the origin and are converging to it. The vector field of motion equations should be directed inside the areas limited by such surfaces. If a solution gets into such area limited by the surface, then it will never leave it again. These surfaces form level surfaces of a Lyapunov function.

For systems without argument deviation the speed vector on level surfaces is determined only by the present moment of time, i.e. by the point lying on the given surface. The speed in equations with deviating argument depends on the previous history as well; i.e. it depends on the point $x(t-\tau)$, which is usually hard to find. Therefore, it is logical to require negative definiteness of Lyapunov function derivative uniformly by the variable $x(t-\tau)$. However, this leads to an excessively sufficient character of the theorems, which in turn makes them inefficient for applications. Because of this, B. S. Razumikhin suggested to consider a previous history $x(t-\tau)$ to lie inside the level surface $v(x,t)=\alpha$ in order to be able to estimate the full derivative along system solutions. The standard technique of proving Lyapunov theorems on stability made such assumption both natural and logical. This led to an additional Razumikhin condition for the Lyapunov theorems, which included estimation of the character of Lyapunov function derivative on the curve that satisfies [1]

$$v(s,x(s)) < v(t,x(t)), \quad s<t.$$

The second approach was introduced by N. N. Krasovskiy. He suggested to consider sections $x(t+s)$, $-\tau \le s \le 0$ of the trajectory at each fixed time $t>0$ instead of functions with finite number of variables. Definitions of positive definiteness of corresponding functionals and of their derivatives on system solutions were introduced as well. Main Lyapunov theorems on stability (as well as asymptotic and exponential stability) were stated in terms of functionals and their derivatives [2].

Both methods are thought to have certain advantages and disadvantages. However, both methods have capacity for existence and further development according to opinions of many scientists.

2 Lyapunov-Krasovskiy Functional Method

Let us consider the basic idea of Lyapunov-Krasovskiy functional method. Denote vector-function defined on the interval $-\tau \le s \le 0$ for each fixed $t>0$ by $x(t+s)$. The functional $V[x(t),t]$ is determined on the vector-functions $x(t+s)$, $-\tau \le s \le 0$. Using introduced functionals N. N. Krasovskiy obtained theorems on stability and asymptotic stability of zero solution of system (1) with delay, which was analogous to the well known Lyapunov theorems.

In the theorems on stability (asymptotic stability, unstability) stated in terms of Lyapunov-Krasovskiy functional the following value (called right upper derivative number)

$$\bar{D}_+V = \lim_{\Delta t \to +0} \sup \frac{1}{\Delta t}\left\{V[x(t+\Delta t), t+\Delta t] - V[x(t),t]\right\}$$

played role of a function derivative dv/dt along solutions $x(t)$ of a system with delay.

We should draw our attention to the two steps in development of the Lyapunov-Krasovskiy functional method. The first step included development of a theoretical ground for the method. The second step used theoretical results to make theorems more applicable to construction of the functionals. Let us consider these two stages in more details.

The first step was to formulate theorems on stability and asymptotic stability, and invert them. All conditions of the theorems were formulated in terms of a uniform norm

$$\|x(t)\|_\tau = \sup_{-\tau \le s \le 0} \{|x(t+s)|\}$$

for a zero solution of system (1) with delay, which was similar to the well known Lyapunov's theorem.

The main results are as follows

Theorem 5 (Stability by Lyapunov). *Let differential equations of system* (1) *be such that there exists a functional* $V[x(t), t]$ *satisfying the following conditions:*

1. $a(\|x(t)\|_\tau) \le V[x(t), t]$,
2. $\bar{D}_+ V[x(t), t] \le 0$.

Here $a(r)$ *is a continuous non-decreasing function positive for all* $r > 0$ *and* $a(0) = 0$. *Then the zero solution* $x(t) \equiv 0$ *of system* (1) *is stable according to Lyapunov's definition.*

Theorem 6 (Asymptotic stability). *Let differential equations of system* (1) *be such that there exists a functional* $V[x(t), t]$ *satisfying the following conditions:*

1. $a(\|x(t)\|_\tau) \le V[x(t), t] \le b(\|x(t)\|_\tau)$,
2. $\bar{D}_+ V[x(t), t] \le -c(\|x(t)\|_\tau)$.

Here $a(r), b(r), c(r)$ *are continuous non-decreasing functions positive for all* $r > 0$ *and equal to zero at* $r = 0$. *Then the zero solution* $x(t) \equiv 0$ *of system* (1) *is asymptotically stable.*

It should be noted that the conditions of the above formulated theorems use the uniform metric, which essentially limits the number of differential systems for which functionals can be constructed in an explicit form. For example, for a linear stationary system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) \tag{3}$$

with constant matrices A and B and a functional in a quadratic form

$$V[x(t)] = x^T(t)Hx(t) + \int_{-\tau}^{0} x^T(t+s)Gx(t+s)ds,$$

where H, G are constant positive definite matrices it is impossible to find functions $a(r)$ and $c(r)$ that would satisfy theorem's conditions.

Therefore, the second step formulated stability theorems in terms of such norms, that are more convenient for constructing the functionals.

Theorem 7 (Asymptotic stability). *Let differential equations of system* (1) *be such that there exists a functional* $V[x(t),t]$ *satisfying the following conditions:*

1. $a(|x(t)|) \leq V[x(t),t] \leq b(\|x(t)\|_\tau)$,
2. $\bar{D}_+V[x(t),t] \leq -c(|x(t)|)$.

Then the zero solution $x(t) \equiv 0$ *of system* (1) *is asymptotically stable.*

2.1 Quadratic Functionals in a General Form

Let us consider constructive methods for construction of Lyapunov-Krasovskiy functionals for linear stationary systems with delay (3). It is obvious that the natural form of a functional is a quadratic one, the same as for systems without delay. Yu. M. Repin constructed quadratic functionals in the following general form [5]

$$\begin{aligned} V[x(t)] = {} & x^T(t)Hx(t) + \int_{-\tau}^{0} x^T(t+s)K(s)x(t)dt \\ & + \int_{-\tau}^{0} x^T(t+s)G(s)x(t+s)ds \\ & + \int_{-\tau}^{0}\int_{-\tau}^{0} x^T(t+s_1)M(s_1,s_2)x(t+s_2)ds_1ds_2 \ . \end{aligned} \tag{4}$$

Here H is a constant quadratic $n \times n$ positive definite matrix; $K(s), G(s)$, $M(s_1,s_2)$ are continuous matrices, and H and $M(s_1,s_2)$ are symmetric matrices. Functionals are chosen in such a way that

$$\frac{d}{dt}V[x(t)] = W[x(t)],$$

where

$$\begin{aligned} W[x(t)] = {} & x^T(t)Qx(t) + x^T(t-\tau)Rx(t) + x^T(t-\tau)Sx(t-\tau) \\ & + \int_{-\tau}^{0} x^T(t+s)D(s)x(t)ds + \int_{-\tau}^{0} x^T(t+s)E(s)x(t+s)ds \\ & + \int_{-\tau}^{0}\int_{-\tau}^{0} x^T(t+s_1)F(s_1,s_2)x(t+s_2)ds_1ds_2 \end{aligned} \tag{5}$$

for given matrices $Q, R, S, D(s), E(s), F(s_1,s_2)$. These matrices satisfy conditions ensuring negative definiteness of $W[x(s)]$ on system's solutions.

By taking a derivative of the functional (4) we obtain a system of algebraic equations that consists of ordinary matrix differential equations and partial differential equations

$$HA + A^TH + \frac{1}{2}[K(0) + K^T(0)] + G(0) = Q\,,$$

$$A^T K(s) - \frac{d}{ds}K(s) + M(s,0) = D(s), \quad -\frac{d}{ds}G(s) = E(s),$$

$$\frac{\partial M(s_1,s_2)}{\partial s_1} + \frac{\partial M(s_1,s_2)}{\partial s_2} = -F(s_1,s_2), \tag{6}$$

$$2HB - K(-\tau) = R, \quad B^T K(s) - M(-\tau,s) = 0 \ .$$

In some cases solutions of system (3) can be found, however in a general case the question of existence of a solution for such system cannot be addressed.

Simplified quadratic functional was proposed in the form [6]

$$V[x(t)] = x^T(t)H(0)x(t) + 2x^T(t)\int_{t-\tau}^{t} H(s-t+\tau)Bx(s)ds$$
$$+\int_{t-\tau}^{t}\int_{t-\tau}^{t} x^T(s_1)B^T H(s_2-s_1)Bx(s_2)ds_1 ds_2 \ .$$

Theorem 8. *Let there exist a matrix function $H(t)$, a solution of the matrix differential equation*

$$\ddot{H}(t) = A^T\dot{H}(t) - \dot{H}(t)A + A^T H(t)A - B^T H(t)B, \quad t \geq 0,$$

and let it satisfy

1. $\dot{H}(t) = A^T H(t) + B^T H(t-\tau), \quad t \geq 0,$
2. $H(t) = H^T(-t), \quad H(0) = H^T(0),$
3. $A^T H(0) + H(0)A + B^T H^T(\tau) + H(\tau)B = -C\,,$

where C is a positive definite matrix. If $H(t)$ is such that the functional $V[x(t)]$ satisfies bilateral estimates

$$a(|x(t)|) \leq V[x(t)] \leq b_1(|x(t)|) + b_2(\|x(t)\|_\tau),$$

then the system is asymptotically stable.

The important fact about this theorem is that the theorem can be reversed.

Theorem 9. *Let a linear system with a delay be asymptotically stable. Then there exists a quadratic functional $V[x(t)]$. Let a matrix function $H(t)$ be a solution of the ordinary differential equation*

$$\ddot{H}(t) = A^T\dot{H}(t) - \dot{H}(t)A + A^T H(t)A - B^T H(t)B, \quad t \geq 0,$$

and let it satisfy

1. $\dot{H}(t) = A^T H(t) + B^T H(t-\tau), \quad t \geq 0,$
2. $H(t) = H^T(-t), \quad H(0) = H^T(0),$

3. $A^T H(0) + H(0)A + B^T H^T(\tau) + H(\tau)B = -C\,,$

where C is a positive definite matrix. Then on solutions $x(t)$ of the system the functional $V[x(t)]$ satisfies bilateral estimates

$$a(|x(t)|) \leq V[x(t)] \leq b_1(|x(t)|) + b_2(\|x(t)\|_\tau),$$

and its full derivative satisfies

$$\dot{V}[x(t)] \leq -\lambda_{\min}(C)|x(t)|^2 \ .$$

If we consider a functional in the form

$$V[x(t)] = x^T(t)Hx(t) + \int_{-\tau}^{0} x^T(t+s)Gx(t+s)ds,$$

then for an asymptotic stability of system (3) it is sufficient that such positive matrices H and G exist that the matrix

$$C[G,H] = \begin{bmatrix} -A^T H - HA - G & -HB \\ -B^T H & G \end{bmatrix}$$

is also positive definite.

Let us transform the problem of finding matrices H and G into an optimizational problem [7,8]

$$(G_0, H_0) = \arg \inf_{(G,H)\in \bar{L}^1_G \times \bar{L}^1_H} \{\varphi_0(G,H)\},$$

where $\varphi_0(G,H) = -\lambda_{\min}[C(G,H)]$, $\lambda_{\min}(\bullet)$ is minimal eigenvalue of the matrix $C[G,H]$; $\bar{L}^1_G, \bar{L}^1_H$ are sets of positive definite matrices G and H that lie within a unit circle.

The Lagrange function is constructed in the form

$$\begin{aligned} \mathcal{L}(G,H,u) =& \varphi_0(G,H) + u_1\varphi_1(G) + u_2\varphi_2(G) + u_3\varphi_3(H) \\ &+ u_4\varphi_4(H), \quad u_i \geq 0, i = \overline{1,4}; \\ \varphi_1(G) =& \lambda_{\max}(G) - 1, \quad \varphi_2(G) = -\lambda_{\min}(G), \\ \varphi_3(H) =& \lambda_{\max}(H) - 1, \quad \varphi_4(H) = -\lambda_{\min}(H) \ . \end{aligned}$$

Theorem 10. *For a function $\varphi_0(G,H)$ to reach its minimal value, it is necessary and sufficient for the point (G_0, H_0, u_0), $u_0^T = (u_1^0, u_2^0, u_3^0, u_4^0)$ to be a saddle point of the Lagrange function.*

The following theorem provides constructive conditions for finding matrices G_0 and H_0 such that the Lyapunov-Krasovskiy functional from a given class resolves a stability question.

Theorem 11. *The Lyapunov-Krasovskiy functional with matrices G_0, H_0 resolves a problem of stability within a given class of functionals (i.e. it is the optimal functional in a given class) if and only if the vector $u_0^T = (u_1^0, u_2^0, u_3^0, u_4^0)$ exists such that*

1. *A gradient set R_L^0 of the Lagrange function $\mathcal{L}(G, H, u)$ on variables (G, H) at the point G_0, H_0, u_0 contains a pair of zero matrices, i.e. $(\theta, \theta) \in R_L^0$.*
2. *Conditions of additional non-stiffness hold:*

$$u_1^0 \varphi_1(G_0) = 0, \quad u_2^0 \varphi_2(G_0) = 0, \quad u_3^0 \varphi_3(H_0) = 0, \quad u_4^0 \varphi_4(H_0) = 0 \ .$$

3 Lyapunov Function Method with Razumikhin Condition

Proofs of main Lyapunov's theorems are based on estimate of a speed vector direction at the moment $x(t)$ on level surfaces $v(x, t) = \alpha$ of the Lyapunov function $v(x, t)$. In other words, the sign of $\dot{v}(x, t)$ is studied, where

$$\frac{dv(x(t), t)}{dt} = \frac{\partial v(x(t), t)}{\partial t} + \mathrm{grad}_x^T v(x(t), t) f(x(t), x(t - \tau)) \ . \tag{7}$$

For systems with argument deviation this expression is a functional that depends on the previous history $x(t - \tau)$. On the basis of the stability definition we can assume that points lie inside the area limited by level surfaces before points of the previous history leave the level surfaces. In other words, the condition $v(x(t - \tau), t - \tau) < v(x(t), t)$ holds.

B. S. Razumikhin proposed to find the estimate of functional (7) not for all curves that correspond to solutions $x(t)$ of the system, but only for those that leave areas limited by level surfaces, i.e. $v(x(s), s) < v(x(t), t), \quad s < t$.

Theorem 12. *Let for system* (1) *a continuously differentiable function $v(x, t)$ exist and satisfy the conditions:*

1. $a(|x|) \le v(x, t)$,
2. $\frac{dv(x(t))}{dt} \le 0$ *for curves $x(t)$ that satisfy $v(x(s), s) < v(x(t), t), \quad s < t$.*

Here $a(r)$ is a continuous non-decreasing function positive for all $r > 0$ and $a(0) = 0$. Then the zero solution $x(t) \equiv 0$ of the system (1) *is stable according to Lyapunov.*

Theorem 13. *Let for the system* (1) *a continuously differentiable function $v(x, t)$ exist and satisfy the conditions:*

1. $a(|x|) \le v(x, t) \le b(|x|)$,
2. $\frac{dv(x(t))}{dt} \le -c(|x(t)|)$ *for curves $x(t)$ that satisfy $v(x(s), s) < v(x(t), t), \quad s < t$.*

Here $a(r)$, $b(r)$, $c(r)$ are continuous non-decreasing functions positive for all $r > 0$ and equal to zero at $r = 0$. Then the zero solution $x(t) \equiv 0$ of the system (1) *is asymptotically stable.*

3.1 Asymptotic Stability of Systems with One Delay

Suppose that the system without deviation (3)

$$\dot{x}(t) = (A + B)x(t) \tag{8}$$

is asymptotically stable. Stability investigation is performed using Lyapunov function in the form $v(x) = x^T Hx$, where H is a solution of the equation

$$(A + B)^T H + H(A + B) = -C \ . \tag{9}$$

Here C is an arbitrary positive definite matrix.

Denote $\varphi(H) = \lambda_{\max}(H)/\lambda_{\min}(H)$, where $\lambda_{\max}(\bullet), \lambda_{\min}(\bullet)$ are maximal and minimal eigenvalues of the matrix H [9,10].

Theorem 14. *Let the system* (8) *be asymptotically stable. If there exists a positive definite matrix* H*, which is a solution of* (9)*, and if the inequality*

$$\lambda_{\min}(C) - 2|HB|(1 + \sqrt{\varphi(H)}\,) > 0 \tag{10}$$

is satisfied, then the system (3) *is asymptotically stable for an arbitrary* $\tau > 0$. *Moreover, for an arbitrary solution* $x(t)$ *of the system* (3) *the condition* $|x(t)| < \varepsilon$, $t > 0$ *holds only if* $\|x(0)\|_\tau < \delta(\varepsilon)$, *where* $\delta(\varepsilon) = \varepsilon/\sqrt{\varphi(H)}$.

Conditions of the Theorem 14 provide exponential decay of solutions of the system (3).

Theorem 15. *Let the system* (8) *be asymptotically stable. If a positive definite matrix* H*, which is a solution of the equation* (9)*, exists and if an inequality* (10) *holds, then for solutions* $x(t)$ *of the system* (3) *the following inequality holds*

$$|x(t)| < \sqrt{\varphi(H)}\,\|x(0)\|_\tau \exp\{-\gamma t/2\}, \quad t > 0,$$

where

$$\gamma = \left\{ \frac{2}{\tau} \ln^{-1}\left[\frac{\lambda_{\min}(C) - 2|HB|}{2|HB|\sqrt{\varphi(H)}}\right] + \frac{\lambda_{\max}(H)}{\lambda_{\min}(C) - 2|HB|(1 + \sqrt{\varphi(H)}\,)} \right\}^{-1} .$$

Let the system (8) be asymptotically stable, but there is no such H that satisfies the inequality (10).

Theorem 16. *Let the system* (8) *be asymptotically stable. If* $\tau < \tau_0$*, where*

$$\tau_0 = \frac{\lambda_{\min}(C)}{2(|A| + |B|)|HB|\sqrt{\varphi(H)}}, \tag{11}$$

then the system (3) *is also asymptotically stable. Also* $|x(t)| < \varepsilon, \quad t > 0$, *only if* $\|x(0)\|_\tau < \delta(\varepsilon, \tau)$, *where*

$$\delta(\varepsilon, \tau) = (1 + |B|\tau)^{-1} \exp\{-|A|\tau\}\varepsilon/\sqrt{\varphi(H)} \ .$$

Theorem 17. *Let the system* (8) *be asymptotically stable. If $\tau < \tau_0$, where τ_0 is defined in* (11)*, then the following inequality holds*

$$|x(t)| < \begin{cases} \sqrt{\varphi(H)}\,(1+|B|\tau)\|x(0)\|_\tau \exp\{|A|\tau\}, & 0 \le t \le \tau, \\ \sqrt{\varphi(H)}\,(1+|B|\tau)\|x(0)\|_\tau \exp\{|A|\tau - \gamma t/2\}, & t > \tau\,, \end{cases}$$

where

$$\gamma = \left(1 - \frac{\tau}{\tau_0}\right)\left[\frac{\lambda_{\max}(H)}{\lambda_{\min}(C)} - \frac{(1-\tau/\tau_0)\tau}{\ln(\tau/\tau_0)}\right]^{-1} \quad .$$

3.2 Estimation of Delay Influence on System Solution

A system in the form

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + Q(x(t), x(t-\tau)) \tag{12}$$

is called "perturbed" to (3) [11].

Theorem 18. *Let the system* (8) *be asymptotically stable and let there exist a positive definite matrix H such that it is a solution of the equation* (9) *and the inequality* (10) *holds. Then for an arbitrary solution $x_Q(t)$ of the system* (12) *the following holds: $|x_Q(t)| < \varepsilon, \quad t > 0$, if $\|x_Q(0)\|_\tau < \delta(\varepsilon)$ and $|Q(x_Q(t), x_Q(t-\tau))| < \eta(\varepsilon)$, where*

$$\delta(\varepsilon) = \varepsilon/\sqrt{\varphi(H)}\,, \quad \eta(\varepsilon) = \frac{\lambda_{\min}(C) - 2|HB|(1+\sqrt{\varphi(H)}\,)}{2|H|\sqrt{\varphi(H)}}\varepsilon \quad .$$

Let there be no such matrix H that satisfies the inequality (10).

Theorem 19. *Let the system* (8) *be asymptotically stable. Then if $\tau < \tau_0$, where τ_0 is defined in* (11)*, the following holds for a solution $x_Q(t)$ of the system* (12)*: $|x_Q(t)| < \varepsilon, \quad t > 0$, only if $\|x_Q(0)\| < \delta(\varepsilon,\tau)$, and $|Q(x_Q(t), x_Q(t-\tau)| < \eta(\varepsilon,\tau)$, where*

$$\delta(\varepsilon,\tau) = (1-\zeta)(1+|B|\tau)^{-1}\exp\{-|A|\tau\}\varepsilon/\sqrt{\varphi(H)}\,,$$
$$\eta(\varepsilon,\tau) = \min\left\{\frac{\zeta}{\tau}e^{-|A|\tau}, \frac{\lambda_{\min}(C)(1-\tau/\tau_0)}{2(|HB|\tau + |H|)}\right\}\frac{\varepsilon}{\sqrt{\varphi(H)}},$$

where $0 < \zeta < 1$ is an arbitrary fixed constant.

Let us estimate the maximum deviation $\tau = \tau_{\max}$, such that the divergence $|x(t) - x_0(t)| < \varepsilon, \quad t > 0$ holds. Denote $x_0(t)$ to be a solution (8), and

$$q = |B(A+B)||x_0(0)| \quad .$$

Theorem 20. *Let the system* (8) *be asymptotically stable, and let there exist* H *— a solution of* (9) *— satisfying* (10)*. Then for an arbitrary* $\varepsilon > 0$, $\delta < \varepsilon/\sqrt{\varphi(H)}$ *the following is true:* $|x(t) - x_0(t)| < \varepsilon$, $t > 0$ *only when* $\|x(0) - x_0(0)\|_\tau < \delta$, *and* $\tau \le \tau_{\max}$, *where*

$$\tau_{\max} = \frac{\lambda_{\min}(C) - 2|HB|(1 + \sqrt{\varphi(H)}\,)}{2|H|q\varphi(H)}\varepsilon \ .$$

Let us introduce the following notations

$$M_1 = 1 - \delta\sqrt{\varphi(H)}\,/\varepsilon, \quad M_2 = |A| + |B|\sqrt{\varphi(H)}\,\delta/\varepsilon,$$
$$N_1 = \varepsilon\lambda_{\min}(C)/\varphi(H)q, \quad N_2 = |H| + \varepsilon\lambda_{\min}(C)/2\varphi(H)q\tau_0.$$

Theorem 21. *Let the system* (8) *be asymptotically stable. Then for any* $\varepsilon > 0$ *and* $\delta < \varepsilon/\sqrt{\varphi(H)}$ *we have* $|x(t) - x_0(t)| < \varepsilon, \quad t > 0$ *only if* $\|x(0) - x_0(0)\|_\tau < \delta$ *and* $\tau \le \tau_{\max}$, *where*

$$\tau_{\max} = \min\left\{ 2M_1\left[\sqrt{M_2^2 + 4M_1\varphi(H)q/\varepsilon} + M_2\right]^{-1}, \right.$$
$$\left. N_1\left[\sqrt{N_2^2 + 2N_1|HB|} + N_2\right]^{-1} \right\} \ .$$

3.3 Absolute Stability of "Direct" Control Systems with Delay

Consider the following system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau) + b_0 f(\sigma[t]) + b_1 f(\sigma[t-\tau]), \\ \sigma[t] = c_0^T x(t) + c_1^T x(t-\tau) \ . \end{cases} \tag{13}$$

Function $f(\sigma)$ satisfies the Lipshitz condition with a constant L and a sector $(0, k)$; i.e.

$$f(\sigma)(K\sigma - f(\sigma)) > 0 \ . \tag{14}$$

Lyapunov function is used in the form

$$v(x) = x^T Hx + \beta\int_0^{\sigma(x)} f(\xi)d\xi, \quad \sigma(x) = c^T x, \quad c = c_0 + c_1 \ .$$

Matrix H is found from the equation (9). For the function $v(x)$ the following condition holds:

$$\lambda_{\min}(\tilde{H})|x|^2 \le v(x) \le \lambda_{\min}(\tilde{H})|x|^2,$$

where

$$\lambda_{\min}(\tilde{H}) = \begin{cases} \lambda_{\min}(H), & \beta \geq 0, \\ \lambda_{\min}(H + \beta k c c^T/2), & \beta < 0\,; \end{cases}$$

$$\lambda_{\max}(\tilde{H}) = \begin{cases} \lambda_{\max}(H + \beta k c c^T/2), & \beta \geq 0, \\ \lambda_{\max}(H), & \beta < 0\ . \end{cases}$$

Definition 22. The system (13) is absolutely stable if the solution $x(t) \equiv 0$ is stable for an arbitrary function $f(\sigma)$ that satisfies (14).

Denote

$$\varphi(\tilde{H}) = \lambda_{\max}(\tilde{H})/\lambda_{\min}(\tilde{H}), \quad p_1 = 2(|HB| + L|Hb_0||c_1| + L|Hb_1||c_0|),$$
$$p_2 = |\beta c^T B| + |\beta c^T b_0|L|c_1| + |\beta c^T b_1|L|c_0|, \quad b = b_0 + b_1,$$
$$q_1 = 2L|Hb_1||c_1|, \quad q_2 = |\beta c^T b_1|L_1|c_1|, \quad c = c_0 + c_1,$$

$$\tilde{C}_1 = \begin{bmatrix} -[(A+B)^T H + H(A+B)] & \vdots & -[Hb + (\beta(A+B)^T + E)c/2] \\ -(p_1 + q_1 + (p_2+q_2)/2\xi^2) & \vdots & \\ \times(1 + \sqrt{\varphi(\tilde{H})})E & \vdots & \\ \cdots & \cdots & \cdots \\ -[Hb + (\beta(A+B)^T + E)c/2]^T & \vdots & 1/k - \beta b^T c - (p_2+q_2)\xi^2 \\ & \vdots & \times(1 + \sqrt{\varphi(\tilde{H})})/2 \end{bmatrix}.$$

Theorem 23. *Let matrix H and a parameter β be such that $\lambda_{\min}(\tilde{H}) > 0$, and let there exist such ξ that $\tilde{C}_1$ is positive definite. Then the system* (13) *is absolutely stable for any $\tau > 0$. In such case $|x(t)| < \varepsilon, \quad t > 0$ only when $\|x(0)\|_\tau < \delta(\varepsilon)$, where $\delta(\varepsilon) = \varepsilon/\sqrt{\varphi(\tilde{H})}$.*

When conditions of the theorem hold, solutions of the system decay.

Theorem 24. *Let matrix H and a parameter β be such that $\lambda_{\min}(\tilde{H}) > 0$ and $\tilde{C}_1$ exists and is positive definite. Then for solutions $x(t)$ of the system* (13) *the following holds*

$$|x(t)| < \sqrt{\varphi(\tilde{H})}\|x(0)\|_{2\tau} \exp\{-\gamma t/2\}, \quad t > 0,$$

where

$$\gamma = \min\left\{\frac{\gamma_1\lambda_{\min}(\tilde{C}_1)}{\gamma_1\lambda_{\max}(\tilde{H}) + \lambda_{\min}(\tilde{C}_1)}, \gamma_2\right\},$$

$$\gamma_1 = \frac{2}{\tau}\ln\left\{\left[\sqrt{[(p_1+p_2)+2(q_1+q_2)]^2 + 4\lambda_{\min}(\tilde{C}_1)(q_1+q_2)/\sqrt{\varphi(\tilde{H})}} - (p_1+p_2)\right]/2(q_1+q_2)\right\},$$

Introduce the following designations

$$M(0) = |A| + |B|\,|K|(|b_0| + |b_1|)(|c_0| + |c_1|),$$
$$N(0) = p_1 + 2q_1 + \sqrt{(p_1+2q_1)^2 + (p_2+2q_2)^2},$$

$$\tilde{C}_2 = \begin{bmatrix} -[(A+B)^T H + H(A+B)] & \vdots & -[Hb + (\beta(A+B)^T + E)c/2] \\ \cdots & \cdots & \cdots \\ -[Hb + (\beta(A+B)^T + E)c/2]^T & \vdots & 1/k - \beta b^T c \end{bmatrix}.$$

Theorem 25 ([13]). *Let matrix H and a parameter β be such that $\lambda_{\max}(\tilde{H}) > 0$, and let $\tilde{C}_2$ be positive definite. Then, when $\tau < \tau_0$, where*

$$\tau_0 = \frac{2\lambda_{\max}(\tilde{C}_2)}{M(0)N(0)\sqrt{\varphi(\tilde{H})}}, \tag{15}$$

the system (13) *is absolutely stable. Moreover, $|x(t)| < \varepsilon, \quad t > 0$ if $\|x(0)\|_{2\tau} < \delta(\varepsilon,\tau)$, where*

$$\delta(\varepsilon,\tau) = [(1+\bar{R}\tau)e^{\bar{L}\tau}]^{-2}\varepsilon/\sqrt{\varphi(\tilde{H})}$$

Theorem 26. *Let matrix H and a parameter β be such that $\lambda_{\min}(\tilde{H}) > 0$ and $\tilde{C}_2$ is positive definite. Then if $\tau < \tau_0$, where τ_0 is defined in* (15)*, the following inequality holds for solutions $x(t)$ of the system* (13)

$$|x(t)| < \begin{cases} \sqrt{\varphi(\tilde{H})}\|x(0)\|_{2\tau}(1+R\tau)^2\exp\{2\bar{L}\tau\}, & 0 \le t \le 2\tau, \\ \sqrt{\varphi(\tilde{H})}\|x(0)\|_{2\tau}(1+R\tau)^2\exp\{2\bar{L}\tau - \gamma t/2\}, & \tau > 2\tau, \end{cases}$$

where

$$\gamma = \frac{\gamma_1\lambda_{\min}(\tilde{C}_2)(1-\tau/\tau_0)}{\gamma_1\lambda_{\max}(\tilde{H}) + \lambda_{\min}(\tilde{C}_2)(1-\tau/\tau_0)},$$

and γ_1 *is a root of the equation*

$$1-[M(\gamma)N(\gamma)][M(0)N(0)]^{-1}e^{\gamma\tau/2}=0,$$

$$\begin{aligned} M(\gamma) &= |A|+|B|e^{\gamma\tau/2}+K(|b_0|+|b_1|e^{\gamma\tau/2})(|c_0|+|c_1|e^{\gamma\tau/2}),\\ N(\gamma) &= (p_1+2q_1e^{\gamma\tau/2})+\sqrt{(p_1+2q_1e^{\gamma\tau/2})^2+(p_2+2q_2e^{\gamma\tau/2})^2},\\ \bar{R} &= |B|+K(|b_0||c_1|+|b_1||c_0|+|b_1||c_1|),\\ \bar{L} &= |A|+K|b_0||c_0| \ . \end{aligned}$$

3.4 Differential Systems with a Quadratic Right-Hand Side

Difference-differential equation with a quadratic right-hand side

$$\begin{aligned} \dot{x}(t) = {} & Ax(t)+Bx(t-\tau)+X^T(t)D_1x(t)+X^T(t)D_2x(t-\tau)\\ & +X(t-\tau)D_3x(t-\tau) \end{aligned} \tag{16}$$

recently became very popular. Here $X(t), D_i,\ i=\overline{1,3}$ are rectangular $n^2\times n$ matrices in the form

$$\begin{aligned} X(t) &= \{X_1(t),X_2(t),\dots,X_n(t)\}\\ D_j^T &= \{D_{1j},D_{2j},\dots,D_{nj}\} \ . \end{aligned}$$

Here $X_k(t)$, where $k=\overline{1,n}$, are quadratic matrices that have a vector $x(t)=(x_1(t),x_2(t),\dots,x_n(t))^T$ in place of a kth column, and other elements are zero. D_{ij} are symmetric matrices that define quadratic ith rows.

Theorem 27. *Let there exist such a matrix* H *that* (10) *holds. Then the solution* $x(t)\equiv 0$ *of the system* (3) *is asymptotically stable at any* $\tau>0$. *The sphere* U_R *that lies in the area of asymptotic stability has the radius*

$$R=\frac{\lambda_{\min}(C)-2|HB|(1+\sqrt{\varphi(H)}\,)}{2\lambda_{\max}(H)\sum_{i=1}^{3}|D_i|(\sqrt{\varphi(H)}\,)^i} \ .$$

For solutions $x(t)$ *from the sphere* U_R *the following convergence estimate holds:*

$$\begin{aligned} |x(t)| &< \frac{R\sqrt{\varphi(H)}\,\|x(0)\|_\tau\exp\{-\gamma t/2\}}{R-\|x(0)\|_\tau[1-\exp\{-\gamma t/2\}]}, \quad t>0,\\ \gamma &= [\lambda_{\min}(C)-2|HB|(1+\sqrt{\varphi(H)}\,)]/\lambda_{\max}(H) \ . \end{aligned}$$

Theorem 28. *Let the system* (8) *be asymptotically stable. Then for* $\tau<\tau_0$, *where* τ_0 *is denoted in* (11), *solution* $x(t)\equiv 0$ *of the system* (16) *will also be asymptotically stable. For such solutions* $x(t)$ *that satisfy the condition* $\|x(\tau)\|_\tau<\bar{R}\zeta$, $0<\zeta<1$ *the following convergence estimate holds:*

$$|x(t)|\le\begin{cases}\|x(t)\|_\tau, & 0\le t\le\tau,\\ \frac{\bar{R}\zeta\sqrt{\varphi(H)}\,\|x(\tau)\|_\tau\exp\{-\gamma t/2\}}{\bar{R}\zeta-\|x(\tau)\|_\tau[1-\exp\{-\gamma t/2\}]}, & t>\tau \ .\end{cases}$$

Here

$$\bar{R} = \frac{\lambda_{\min}(C)(1-\tau/\tau_0)}{2\sum_{i=1}^{3}\left[|HB||D_i|(\sqrt{\varphi(H)}\,)^3 + \lambda_{\max}(H)|D_i|(\sqrt{\varphi(H)}\,)^i\right]},$$

γ *is a solution of a special equation.*

3.5 Differential Systems with Rational Right-Hand Sides

Recently developed mathematical models of ordinary differential equations with rational right-hand sides were found adequate for description of various models in biology and medicine. The systems have the form [15,16]

$$\dot{x}(t) = [E + X(t)D_1 + X(t-\tau)D_2]^{-1}[Ax(t) + Bx(t-\tau)]. \tag{17}$$

Theorem 29. *Let there exist a symmetric positive definite matrix H that satisfies* (10)*. Then the solution $x(t) \equiv 0$ of the system* (17) *is asymptotically stable for an arbitrary delay $\tau > 0$. The asymptotic stability region contains the ball $U_R = \{x : |x| \le R\}$, where*

$$R = \frac{[\lambda_{\min}(C) - 2|HB|(1+\sqrt{\varphi(H)}\,)]/\sqrt{\varphi(H)}}{(|D_1| + |D_2|\sqrt{\varphi(H)}\,)[[\lambda_{\min}(C) - 2|HB|(1+\sqrt{\varphi(H)}\,)] + 2|H|(|A| + |B|\sqrt{\varphi(H)}\,)]}.$$

Theorem 30. *Let the system* (8) *be asymptotically stable. Then for all $\tau < \tau_0$, where*

$$\tau_0 = \frac{\lambda_{\min}(C)(1-\zeta)^3}{2(|A|+|B|)|H|\sqrt{\varphi(H)}\,[|B| + (|D_2||A_1| - |D_1||B|)R\zeta]}.$$

Then the solution $x(t) \equiv 0$ of the system (17) *is asymptotically stable. The asymptotic stability region contains a ball with the radius*

$$R = \min\left\{\frac{1}{(|D_1|+|D_2|)\sqrt{\varphi(H)}}, \frac{\lambda_{\min}(C)/\sqrt{\varphi(H)}}{[2|HB| + \lambda_{\min}(C)]|D_1+D_2|}\right\}.$$

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