ARCHIVUM MATHEMATICUM (BRNO)
Tomus 34 (1998), 119–126

On Small Solutions of Second Order Differential Equations with Random Coefficients

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Abstract. We consider the equation

$$x'' + a^2(t)x = 0, \qquad a(t) := a_k \ \text{ if } t_{k-1} \le t < t_k, \ \text{ for } k = 1, 2, \dots ,$$

where $\{a_k\}$ is a given increasing sequence of positive numbers, and $\{t_k\}$ is chosen at random so that $\{t_k - t_{k-1}\}$ are totally independent random variables uniformly distributed on interval $[0, 1]$. We determine the probability of the event that all solutions of the equation tend to zero as $t \to \infty$.

AMS Subject Classification. 34F05, 34D20, 60K40

Keywords. Asymptotic stability, energy method, small solution

1 Introduction

The linear second order differential equation

$$x'' + a^2(t)x = 0 \tag{1}$$

describes the oscillation of a material point of unit mass under the action of the restoring force $-a^2(t)x$; function $a : [0, \infty) \to (0, \infty)$ is the square root of the varying elasticity coefficient a^2.

* The author was supported by the Grant FKFP 1201/1997 and Hungarian National Foundation for Scientific Research with grant number T/016367

Definition 1 (Ph. Hartman [8]). A function $t \mapsto x_0(t)$ existing and satisfying equation (1) on the interval $[0, \infty)$ is called a *small solution* of (1) if

$$\lim_{t \to \infty} x_0(t) = 0 \tag{2}$$

holds. The zero solution is called the *trivial small solution* of (1).

It is easy to see [10, p. 510] that if a is nondecreasing, then every solution of (1) is oscillatory and the successive amplitudes of the oscillation are decreasing. M. Biernacki [2] raised the question of the existence of a (nontrivial) solution whose amplitudes tend to zero, i.e., a small solution. H. Milloux answered this question by proving

Theorem A (H. Milloux [15]). *If $a : [0, \infty) \to (0, \infty)$ is differentiable, nondecreasing, and satisfies*

$$\lim_{t \to \infty} a(t) = \infty, \tag{3}$$

then equation (1) *has a non-trivial small solution.*

Milloux also provided an example of a step function a to show that one cannot conclude that *all* solutions are small.

Biernacki [2] raised also the following question: what additional conditions on a function a monotonously tending to infinity as t goes to infinity guarantee that all solutions are small? The first answer to this question was the famous Armellini-Tonelli-Sansone theorem (see, e.g., in [10]). It has been followed by many generalizations and improvements in the literature [3,9,10,13,14,16,17]. All of them require of the coefficient a to tend to infinity regularly. Roughly speaking this means that the growth of a cannot be located to a set with a small measure.

In this paper we are concerned with the case when the damping coefficient a in equation (1) is a step function. As is known such equations often serve as mathematical models in applications.

For example, let us consider the motion of the mathematical plain pendulum whose length changes by a given law $\ell = \ell(t)$. The position of the material point in the plain is described by the length $\ell(t)$ of the thread and the angle φ between the axis directed vertically downward and the thread. It is known [1,11] that the equation of the motion is

$$\varphi'' + \frac{g}{\ell(t)} \sin \varphi = 0, \tag{4}$$

where g denotes the constant of gravity. (No friction, the force of gravity acts only.) The "small oscillations" [1] are described by the linear second order differential equation

$$\varphi'' + \frac{g}{\ell(t)} \varphi = 0. \tag{5}$$

Consider the case when ℓ is a step function and $\ell(t) \to 0$ monotonously as $t \to \infty$. This is the situation when one has to lift a weight by a pulley and rope through a gape. The purpose is to guarantee $\lim_{t\to\infty} \varphi(t) = 0$.

In [12] the first author showed that the Milloux theorem can be generalized to step function coefficients, thus the existence of at least one solution with the desired property is guaranteed. However, this knowledge is useless from practical point of view. We would need a theorem guaranteeing *all* solutions to tend to zero as t goes to infinity. The Armellini-Tonelli-Sansone theorem cannot be applied because any step function can increase only irregularly: the growth of the function is located to a countable set, the function increases with jumps. Very recently the Armellini-Tonelli-Sansone theorem was generalized to impulsive systems [7] and step functions [5,6]. These theorems contain sophisticated conditions with requests of certain connections between different parameters of the step function coefficient. It is almost impossible to use these conditions for controlling the motions even if one can observe and measure the state variables during the motions, what, in general, cannot be assumed. (It is enough to mention the problem of pulling out used up graphit bars from a nuclear reactor, which can be modelled by equations similar to (5).) For this reason the first author [12] formulated the following practical problem: How many solutions are small if we do not require any additional condition on $\ell(t)$ beyond $\lim_{t\to\infty} \ell(t) = 0$? In other words, how often does it happen that $\lim_{t\to\infty} \varphi(t) = 0$?

To be more precise, let us suppose that the length $\ell(t)$ is of the form

$$\ell(t) := \ell_k, \qquad \text{if} \qquad t_{k-1} \le t < t_k,\ k = 1, 2, \dots,$$

where $\{\ell_k\}_{k=1}^{\infty}$ is given, $\lim_{k\to\infty} \ell_k = 0$, and the sequence $\{t_k\}_{k=0}^{\infty}$ of the moments of pulling the rope is chosen "at random" such that $\lim_{k\to\infty} t_k = \infty$. *For an arbitrarily fixed pair of initial data φ_0, φ_0', what is the probability, that $\lim_{t\to\infty} \varphi(t) = 0$?*

In this paper we give an answer to this problem in the case when the differences $t_k - t_{k-1}$ $(k = 1, 2, \dots)$ are independent random variables uniformly distributed on interval $[0, 1]$. Namely, we prove that in this case $\lim_{t\to\infty} \varphi(t) = 0$ is almost sure (it is an event of probability 1).

2 Preliminaries and Results

Let $\{t_k\}_{k=1}^{\infty}$ be an increasing sequence of positive numbers tending to infinity as k goes to infinity, and define $t_0 := 0$. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive numbers such that

$$0 < a_0 \le a_1 \le \dots \le a_k \le a_{k+1} \le \dots,$$

and consider the equation

$$x'' + a^2(t)x = 0, \qquad a(t) := a_k \ \text{ if } t_{k-1} \le t < t_k, \ \text{ for } k = 1, 2, \dots. \tag{6}$$

A function $x : [0, \infty) \to (-\infty, \infty)$ is a *solution of* (6) if it is continuously differentiable on $[0, \infty)$ and it solves the equation on every (t_{k-1}, t_k) for $k = 1, 2, \dots$.

Write (6) as a system of first order differential equations for a 2-dimensional vector (x, y), where $y := x'/a_k$. The resulting system is

$$x' = a_k y, \quad y' = -a_k x \qquad (t_{k-1} \le t < t_k; \quad k = 1, 2, \dots). \tag{7}$$

One has to be careful defining what it means that a function $t \mapsto (x(t), y(t))$ is a solution of (7) on the interval $[0, \infty)$. The function $t \mapsto x'(t) = a_k y(t)$ has to be continuous, so we require that the function $t \mapsto y(t)$ is continuous to the right for all $t \ge 0$ and satisfies $a_k y(t_k - 0) = a_{k+1} y(t_k)$ for $k = 1, 2, \dots$, where $y(t_k - 0)$ denotes the left-hand side limit of y at t_k. Accordingly, the system of first order differential equations for (x, y) equivalent with (6) is

$$\begin{aligned} &x' = a_k y, \quad y' = -a_k x \qquad (t_{k-1} \le t < t_k) \\ &y(t_k) = \frac{a_k}{a_{k+1}} y(t_k - 0), \qquad k = 1, 2, \dots . \end{aligned} \tag{8}$$

It is easy to see that introducing the polar coordinates (r, φ) by the equations $x = r\cos\varphi,\ y = r\sin\varphi$, we can rewrite system (7) into the form

$$r' = 0, \quad \varphi' = -a_k \qquad (t_{k-1} \le t < t_k, \quad k = 1, 2, \dots).$$

So, system (8) turns the plane uniformly around the origin for $t \in [t_{k-1}, t_k)$, and then contracts it along the y-axis by a_k/a_{k+1} at $t = t_k$. Introduce the notations

$$\tau_k := t_k - t_{k-1}, \quad \varphi_k := a_k \tau_k, \quad \alpha_k := \frac{a_k}{a_{k+1}},$$

$$T_k := \begin{pmatrix} 1 & 0 \\ 0 & \alpha_k \end{pmatrix} \begin{pmatrix} \cos\varphi_k & \sin\varphi_k \\ -\sin\varphi_k & \cos\varphi_k \end{pmatrix}, \quad k = 1, 2, \dots; \qquad T_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then from (8) we obtain

$$\xi_k := \begin{pmatrix} x(t_k) \\ y(t_k) \end{pmatrix} = T_k T_{k-1} \dots T_2 T_1 \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \in \mathbb{R}^2, \quad k = 0, 1, 2, \dots . \tag{9}$$

Since $\alpha_k \le 1$, $k = 1, 2 \dots$, for every solution $t \mapsto (x(t), y(t))$ the limit

$$\omega := \lim_{t\to\infty} (x^2(t) + y^2(t)) = \lim_{k\to\infty} \|\xi_k\|^2 \tag{10}$$

exists and is finite, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$.

Suppose that $\tau_1, \tau_2, \dots, \tau_k, \dots$ *are totally independent random variables uniformly distributed on interval* $[0, 1]$. Limit ω is a function of the sequqnce $\{\tau_k\}_{k=1}^{\infty}$, so it is also random. Now we introduce the probability space where ω can be defined as a random variable.

For every natural number n, let $\mathcal{P}_n = (\Omega_n, \mathcal{A}_n, \mu_n)$ be the probability space with $\Omega_n := \prod_{k=1}^{n} [0, 1]$, the class $\mathcal{A}_n$ of Lebesgue measurable subsets of Ω_n, and the Lebesgue measure μ_n in Ω_n. By the Fundamental Theorem of Kolmogorov [4] there exists the infinite product probability space $\mathcal{P} = (\Omega := \prod_{k=1}^{\infty}[0,1], \mathcal{A}, \mu)$, having the following property:

$$\mu\Bigg(H \times \prod_{k=n+1}^{\infty} [0,1]\Bigg) = \mu_n(H) \quad \text{for every } H \in \mathcal{A}_n. \tag{11}$$

Limit ω defined by (10) is a random variable on probability space $\mathcal{P}$. Our purpose is to determine the probability

$$\mathbf{P}(\omega = 0 \text{ for all } \xi_0 \in \mathbb{R}^2).$$

Obviously, the event ($\omega = 0$ for all $\xi_0 \in \mathbb{R}^2$) is independent of the choices $\{\tau_k\}_{k=1}^{n}$ for every finite n. By Kolmogorov's Zero-Or-One Law, the probability of such an event equals either zero or one. The following theorems are in accordance with this law.

Theorem 2. *If* $\lim_{k\to\infty} a_k = \infty$*, then it is almost sure (i.e., it is an event of probability 1 in probability space* $\mathcal{P}$*) that*

$$\lim_{t\to\infty}\left(x^2(t) + \frac{(x'(t))^2}{a^2(t)}\right) = 0$$

for all solutions of equation (6)*.*

Corollary 3. *If* $\lim_{k\to\infty} a_k = \infty$*, then it is almost sure (i.e., it is an event of probability 1 in probability space* $\mathcal{P}$*) that*

$$\lim_{t\to\infty} x(t) = 0$$

for all solutions of equation (6)*.*

Theorem 4. *If* $\lim_{k\to\infty} a_k < \infty$*, then*

$$\lim_{t\to\infty}\left(x^2(t) + \frac{(x'(t))^2}{a^2(t)}\right) > 0$$

for every non-trivial solution x *of equation* (6)*.*

Corollary 5. *If* $\lim_{k\to\infty} a_k < \infty$*, then it is an impossible event in probability space* $\mathcal{P}$ *that there exists a non-trivial solution* x *of equation* (6) *with*

$$\lim_{t\to\infty} x(t) = 0.$$

3 Proofs

3.1 Proof of Theorem 2

Let $(x(0), y(0)) \in \mathbb{R}^2$ be fixed, and consider the solution of equation (8) starting from this point. If $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^2$, then for a fixed $k \geq 1$ we have

$$\|\xi_k\|^2 = \langle \xi_k, \xi_k \rangle = \langle T_k \xi_{k-1}, T_k \xi_{k-1} \rangle = \langle T_k^* T_k \xi_{k-1}, \xi_{k-1} \rangle \leq \Lambda_k \|\xi_{k-1}\|^2,$$

where T_k^* denotes the transposed of matrix T_k, and Λ_k denotes the greater eigenvalue of the symmetric matrix $T_k^* T_k$. The random variables $\xi_1, \xi_2, \ldots, \xi_k$ are independent; consequently, for the expected values we obtain the inequality

$$\mathbf{E}(\|\xi_k\|^2) \leq \mathbf{E}(\Lambda_k)\mathbf{E}(\|\xi_{k-1}\|^2). \tag{12}$$

Now we compute $\mathbf{E}(\Lambda_k)$. First we determine the expected value of matrix $T_k^* T_k$:

$$\begin{aligned}
\mathbf{E}(T_k^* T_k) &= \int_0^1 \begin{pmatrix} \cos a_k\tau & -\sin a_k\tau \\ \sin a_k\tau & \cos a_k\tau \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_k^2 \end{pmatrix} \begin{pmatrix} \cos a_k\tau & \sin a_k\tau \\ -\sin a_k\tau & \cos a_k\tau \end{pmatrix} d\tau \\
&= \int_0^1 \cos^2 a_k\tau d\tau \begin{pmatrix} 1 & 0 \\ 0 & \alpha_k^2 \end{pmatrix} + \int_0^1 \sin^2 a_k\tau d\tau \begin{pmatrix} \alpha_k^2 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + \int_0^1 \sin a_k\tau \cos a_k\tau d\tau \begin{pmatrix} 0 & \alpha_k^2 - 1 \\ \alpha_k^2 - 1 & 0 \end{pmatrix} \\
&= \frac{1 + \alpha_k^2}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin 2a_k}{4a_k}(1 - \alpha_k^2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\sin^2 a_k}{2a_k}(\alpha_k^2 - 1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

It is easy to check that the greater eigenvalue of a symmetric matrix $(d_{ik})_{i,k=1}^2$ is determined by the formula

$$\frac{d_{11} + d_{22} + \sqrt{(d_{11} - d_{22})^2 + (2d_{12})^2}}{2}.$$

Λ_k is the greater eigenvalue of matrix $\mathbf{E}(T_k^* T_k)$; therefore,

$$\Lambda_k = \frac{1}{2}\left(1 + \alpha_k^2 + (1 - \alpha_k^2)\left|\frac{\sin a_k}{a_k}\right|\right). \tag{13}$$

Applying inequality (12) for $k = 1, 2, \ldots$ we obtain the estimate

$$\mathbf{E}(\|\xi_n\|^2) \leq \left(\prod_{k=1}^n \Lambda_k\right) \|\xi_0\|^2. \tag{14}$$

Now we prove

$$\prod_{k=1}^{\infty} \Lambda_k = 0. \tag{15}$$

This assertion is equivalent with

$$\sum_{k=1}^{\infty} \ln \left[1 - \frac{1-\alpha_k^2}{2}\left(1 - \left|\frac{\sin a_k}{a_k}\right|\right)\right] = -\infty.$$

This is obviously satisfied if $\liminf_{k\to\infty} \alpha_k < 1$. If $\lim_{k\to\infty} \alpha_k = 1$, then it is enough to show that

$$\sum_{k=1}^{\infty}(1-\alpha_k^2)\left(1 - \left|\frac{\sin a_k}{a_k}\right|\right) = \infty,$$

i.e., $\sum_{k=1}^{\infty}(1-\alpha_k) = \infty$. But this is equivalent with $\sum_{k=1}^{\infty} \ln \alpha_k = -\infty$, i.e.,

$$\lim_{n\to\infty} \prod_{k=1}^{n} \alpha_k = \lim_{n\to\infty} \frac{a_0}{a_{n+1}} = 0,$$

which was assumed.

From (14) and (15) it follows that $\lim_{n\to\infty} \mathbf{E}(\|\xi_n\|^2) = 0$. Then by Fatou's Lemma [4] and property (11) we have

$$\begin{aligned}\mathbf{E}(\omega) = \mathbf{E}(\lim_{n\to\infty}(\|\xi_n\|^2) &= \int_{\Omega} \lim_{n\to\infty}(\|\xi_n\|^2\, d\mu \le \lim_{n\to\infty} \int_{\Omega} \|\xi_n\|^2\, d\mu \\ &= \lim_{n\to\infty} \int_{\Omega_n} \|\xi_n\|^2\, d\mu_n = \lim_{n\to\infty} \mathbf{E}(\|\xi_n\|^2) = 0.\end{aligned}$$

We have proved that for every fixed individual solution of (6) there holds $\mathbf{P}(\omega = 0)$. Since all solutions of the linear equation (6) can be represented as linear combinations of two fixed linearly independent solutions of the equation, this implies that

$$\mathbf{P}(\omega = 0 \text{ for all solutions of } (6)) = 0,$$

which completes the proof of Theorem 2.

3.2 Proof of Theorem 4

Suppose that $\lim_{n\to\infty} a_n =: a_\infty < \infty$. From the representation (9) and the definition of T_k we have

$$\|\xi_k\|^2 = \langle \xi_k, \xi_k\rangle = \langle T_k\xi_{k-1}, T_k\xi_{k-1}\rangle = \langle T_k^* T_k \xi_{k-1}, \xi_{k-1}\rangle \ge \alpha_k^2 \|\xi_k\|^2.$$

Iterating this estimate we obtain the inequality

$$\omega = \lim_{n\to\infty} \|\xi_n\|^2 \ge \left(\lim_{n\to\infty} \prod_{k=1}^{n} \alpha_k^2\right)\|\xi_0\|^2 = \left(\lim_{n\to\infty} \frac{a_0^2}{a_{n+1}^2}\right)\|\xi_0\|^2 = \frac{a_0^2}{a_\infty^2}\|\xi_0\|^2 > 0,$$

whenever $\|\xi_0\|^2 > 0$. This completes the proof.

The proofs of Corollaries 3 and 5 are trivial, so they are omitted.

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