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A New Finite Element Approach for Problems Containing Small Geometric Details

W. Hackbusch and S. Sauter

Lehrstuhl Praktische Mathematik, Universität Kiel, 24098 Kiel, Germany
Email: wh@numerik.uni-kiel.de and sas@numerik.uni-kiel.de
WWW: http://www.numerik.uni-kiel.de

Abstract. In this paper a new finite element approach is presented which allows the discretization of PDEs on domains containing small micro-structures with extremely few degrees of freedom. The applications of these so-called *Composite Finite Elements* are two-fold. They allow the efficient use of multi-grid methods to problems on complicated domains where, otherwise, it is not possible to obtain very coarse discretizations with standard finite elements. Furthermore, they provide a tool for discrete homogenization of PDEs without requiring periodicity of the data.

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1 Introduction

Before the mid-sixties the finite difference method was the standard discretization method for differential equations. The following two severe drawbacks of finite differences lead to the development of the finite element method. First, the use of Cartesian difference quotients made the treatment of complicated and curved boundaries difficult and many technical tricks have to be employed to overcome this problem. Furthermore, it turned out that only the variational setting of the continuous problem leads to satisfactory existence and uniqueness results in appropriate function spaces, usually the convergence results of FDM require too much smoothness. Nowadays, we know that the question whether a discretization method is a FDM or a FEM is often only a matter of interpretation. In numerical linear algebra where one is interested in the algebraic properties of the linear system as, e.g., the M-matrix property, it is, in many case, very useful to interpret the discretization as a discrete, FD-like method while for the estimates of the discretization error the powerful apparatus of finite elements is employed.

An advantage, however, of FDM is the easy regular structure of the grid. Hence, the matrix pattern has a very regular structure, too. We know that this is very essential in the performance of iterative solvers as, e.g. ILU-like methods, while in an a priori unstructured FE mesh, sometimes, big effort has to be spent to find an advantageous numbering of the grid points. Furthermore, the simple structure of the matrix pattern makes the implementation of FDM much easier compared to FEM. Additionally, the efficient use of high performance computers as, e.g., vector computers, favorites such simple data structures.

On the other hand, the FEM has big advantages compared to FDM, namely, it provides a powerful apparatus for convergence analysis and is very flexible with respect to an appropriate geometric discretization of the domain allowing adaptive refinement strategies and proper resolution of the boundary.

However, the latter mentioned feature is true, only, if the grid size is small enough resolving essentially all micro-structures of the domain and differential equation. Very coarse discretizations (step size much larger than the geometric details) are not possible. In the context of homogenization and in order to apply multi-grid methods where the efficiency depends on how coarse the coarsest grid can be chosen this is a severe drawback. The Shortley-Weller FDM [14] which is in the literature since 1938 allows that the Cartesian grid overlap the boundary and appropriate weights are introduced in the difference quotients. The first multi-grid computations [3] use this discretization method in order to get very coarse coarse-grid approximations.

Since recently, various approaches have been presented in the literature concerning coarsening strategies for finite element spaces or, more general, discretizations with only few degrees of freedom which have already the asymptotic accuracy. In [1], [2], and [9], approaches are presented which can be used in the context of BPX-multigrid methods and hierarchical basis multigrid methods.

An approach which is based on pure algebraic considerations is the so-called *Algebraic Multigrid Method (AMG)* where only the information of the system matrix is used to obtain matrices of lower dimension. For details see [11]. A further related paper in this context is [10].

Composite Finite Elements were first presented by the authors in [8] and [6] where the aim was to define finite element spaces which have the asymptotic approximation property and the possibly low number of unknowns is independent of the shape of the domain. They can be used for both pure Galerkin discretization and in combination with standard multigrid methods and are not necessarily linked to a special solver.

In the present paper, we will, in the light of the Shortley-Weller discretization, define a new class of finite elements which is appropriate to resolve complicated geometries with very few degrees of freedom.

The paper is organized as follows. First, we recapitulate the principle of the Shortley-Weller method within an elementary setting. Then, we will introduce the new class of finite elements called *Composite Finite Elements* which resolves complicated boundaries with a very small number of degrees of freedom satisfying the usual asymptotic approximation property. We will show that the implementation of this method is very easy and the application to 3-d problems does not involve further difficulties compared to the 2-d version.

2 Shortley-Weller Finite Difference Discretization

In this section we recall the principles of the Shortley-Weller method for finite difference discretization of partial differential equations (PDEs) on domains having complicated boundary. The basic principles of this method will be used for the design of the new class of *Composite Finite Elements*.

In order to approximate the second derivative of a function u at a point $x \in \mathbf{R}$ using a non-uniformly spaced grid, Newton's divided second differences are employed

$$\begin{aligned}-u''(x) &\approx \frac{2}{h_1+h_2}\left(\frac{u(x)-u(x-h_1)}{h_1} - \frac{u(x+h_2)-u(x)}{h_2}\right)\\ &= -\frac{2}{(h_1+h_2)\,h_2}u(x+h_2) + \frac{2}{h_1h_2}u(x) - \frac{2}{(h_1+h_2)\,h_1}u(x-h_1)\end{aligned}$$

Symbolically, the *matrix stencil* is given by

$$L_h = \left[-\frac{2}{(h_1+h_2)\,h_2}, \frac{2}{h_1h_2}, -\frac{2}{(h_1+h_2)\,h_1}\right]. \tag{1}$$

The use of non-uniform spaced Cartesian grids for finite difference approximation is necessary if non-rectangular geometries as depicted in Figure 1 occur. A coarse

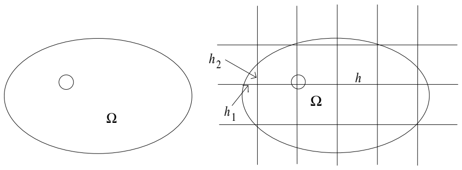

Fig. 1. Domain Ω with curved boundary and a small hole. The Cartesian grid does not fit in the domain and defines local stepsizes h_j near the boundary.

Cartesian grid will overlap the domain substantially. Instead of deforming the

Cartesian grid we use 2-d analogues of (1). The arising system matrix $\mathbf{L}_h$ has favorable properties. $\mathbf{L}_h$ is an M-matrix and has special stability properties (see [5, Theorem 4.8.4]) which can be expressed by

$$\left\|\mathbf{L}_h^{-1}\right\|_\infty \leq \frac{d^2}{8}.$$

However, difficulties arise if the micro-structures of the grid are not visible on the coarse grid. This would arise if, e.g., a hole lies in the interior of a grid cell and no Cartesian line of the grid intersects the hole. To overcome this problem we consider a hierarchy of Cartesian grids τ_ℓ of step size h_ℓ satisfying

$$\begin{aligned} h_0 &= O(1) \approx \operatorname{diam}(\Omega), \\ h_\ell &= 2^{-\ell} h_0. \end{aligned}$$

We assume that $\ell_{\max}$ is such that $\tau_{\ell_{\max}}$ resolves all necessary details of the domain. Hence, the matrix $\mathbf{L}_{\ell_{\max}}$ can be generated by using the Shortley-Weller scheme. Matrices corresponding to coarser grids are then extracted from the fine grid matrix by introducing prolongations $p_{\ell\leftarrow\ell-1}$ and restrictions $r_{\ell-1\leftarrow\ell}$ linking grid functions on different grids $\tau_{\ell-1}$ and τ_ℓ with each other. Having defined these operators the coarse grid matrices are given recursively by the Galerkin product

$$\mathbf{L}_{\ell-1} = r_{\ell-1\leftarrow\ell} \mathbf{L}_\ell p_{\ell\leftarrow\ell-1}.$$

In standard cases, the prolongation and restriction can be defined, e.g., via interpolation in the following way. First, we consider the one-dimensional case which is illustrated in Figure 2.

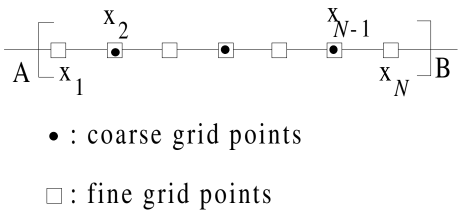

Fig. 2. Domain $\Omega = [A, B]$ with non-fitting fine and coarse grids.

The prolongation in the case of homogeneous Dirichlet boundary conditions is given for all fine grid points x_i by interpolating the neighbouring coarse grid values.

$$[p_{\ell\leftarrow\ell-1} u](x_i) = \begin{cases} u(x_i) & \text{if } x_i \text{ is also a coarse grid point,} \\ \frac{1}{2}\left(u(x_{i-1}) + u(x_{i+1})\right) & \text{otherwise and } i \neq 1, N, \\ \frac{\|x_1 - A\|}{\|x_2 - A\|} u(x_2) & i = 1, \\ \frac{\|x_{N-1} - B\|}{\|x_N - B\|} u(x_{N-1}) & i = N. \end{cases}$$

In more than one-dimension one has to interpolate sequentially in all directions. We state that in regular situation, i.e., in the case of domain-fitting grids, the prolongation is the bilinear interpolation. In any case the restriction $r_{\ell-1\leftarrow\ell}$ is defined as the adjoint of $p_{\ell\leftarrow\ell-1}$ with respect to the weighted Euclidean scalar product:

$$\langle u, v\rangle = \frac{1}{N}\sum_{i=1}^{N} u\left(x_i\right)\bar{v}\left(x_i\right).$$

An important feature of the prolongation and restriction above is that the sparsity of the system matrix is preserved and the regular distribution of the non-zero entries as well. If $\mathbf{L}_\ell$ is given by a 9-point stencil, i.e., 9 non-vanishing entries per matrix line, then, the same is true for $\mathbf{L}_{\ell-1}$.

Using these system matrices $\{\mathbf{L}_\ell\}_{0\leq\ell\leq\ell_{\max}}$ in a multi-grid method one observes the typical convergence rates even if the coarse grid contains only one degree of freedom and the domain contains many very small geometric details (cf. [**3**]).

The purpose of this section was to elucidate some key principles how very coarse discretizations of domains having complicated micro-structures can be obtained. The consideration was quite elementary but it will turn out that the principles can be used to define a new class of finite elements which includes the advantages of the Shortley-Weller FDM but can be applied to a much bigger class of problems.

3 Composite Finite Elements

In this section we will introduce so-called *Composite Finite Elements*. First, we will explain how grids can be generated such that geometric coarsening is straightforward. Then, the finite element spaces are defined on these coarsened grids as subspaces of the fine grid space by specifying appropriate inter-grid prolongations. The following considerations do not depend on the space dimension and hence are formulated in an abstract way.

We start recalling some basic definitions of finite element spaces. Let τ denote a partitioning of a domain Ω into small elements $\{K_j\}_{1\leq j\leq n}$. The finite element space V corresponding to this grid is defined as

$$V = \left\{u \in \mathcal{C}^k\left(\Omega\right) : u\left.\right|_K \text{ is a polynomial of maximal degree } p \text{ for all } K\in\tau\right\}.$$

Let $\Theta = \{x_j\}_{1\leq j\leq N}$ denote the set of nodal points and $\{\Phi_i\}_{1\leq i\leq N}$ the corresponding Lagrangian nodal basis of V given by

$$\begin{aligned} &\Phi_i \in V, \\ &\Phi_i\left(x_j\right) = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.}\end{cases} \end{aligned} \tag{2}$$

Then, each function $u \in V$ has a unique basis representation by

$$u\left(x\right) = \sum_{i=1}^{N} \mathbf{u}_i \Phi_i\left(x\right) \tag{3}$$

with $\mathbf{u}_i = u(x_i)$. Equation (3) provides a canonical interpretation of a (discrete) vector of nodal values $\mathbf{u} \in \mathbf{R}^N$ as a finite element function.

In the following we will describe a method how a sequence of grids can be constructed such that geometric coarsening is straightforward.

3.1 Construction of the Grids and Definition of Composite Finite Elements

The following formal setting is illustrated in Figures 3-6.

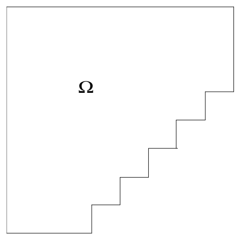

Fig. 3. Domain Ω containing a rough boundary piece.

First, we have to construct a sequence of auxiliary grids $\{\tilde{\tau}_\ell\}_{0\le\ell\le\ell_{\max}}$. Let Q_0 be a rectangle resp. a cuboid containing the domain Ω. Choose an arbitrary partitioning of Q_0 as the initial grid $\tilde{\tau}_0$. Refine $\tilde{\tau}_0$ for several times by any common refinement strategy as, e.g., combining the midpoint of triangles, the faces of hexahedrons, etc. to obtain a physically and logically nested sequence of grids $\{\tilde{\tau}_\ell\}_{0\le\ell\le\ell_{\max}}$.

This means that any element K of $\tilde{\tau}_\ell$ has a certain numbers of children given by

$$K' \in \tilde{\tau}_\ell \text{ is a } \textit{child} \text{ of } K \Leftrightarrow K' \subset K$$

and, vice versa, each element of $\tilde{\tau}_{\ell+1}$ has a uniquely determined parent in $\tilde{\tau}_\ell$. Note that the definition of $\tilde{\tau}_\ell$ does not include any adjustment process of the grids to the physical domain. However, in practical computations, one would generate grids $\tilde{\tau}_\ell$ which contain small elements in or near parts of Ω where a higher resolution is required. This can be done, e.g., by using error estimators or an a priori known grading function which controls the refinement strategy. We assume that $\tau_{\ell_{\max}}$ is fine enough such that nodal points lying close to the boundary of Ω can be moved onto the boundary without distorting the elements too much. Furthermore, we assume that there exists a subset of elements of the resulting grid which is a proper FE grid of the domain Ω. This mesh is denoted by $\tau_{\ell_{\max}}$. Note that the movement of grid points of $\tau_{\ell_{\max}}$ also is changing the shape of the elements on

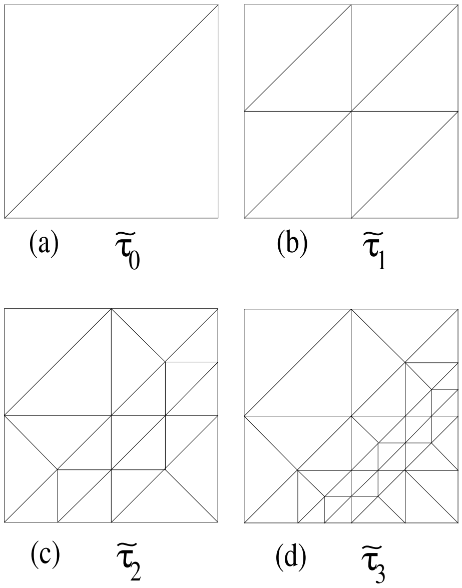

Fig. 4. Auxiliary grids $\{\tilde{\tau}_\ell\}_{0\leq\ell\leq3}$ which arise by refining a coarse grid with an appropriate refinement strategy. Note that no adjustment of the grid to the boundary of the domain takes place.

coarser levels. These distorted coarser grids are further reduced by cancelling all elements having zero cut with Ω. The resulting meshes are denoted by τ_ℓ.

The construction above implies that the elements are no longer physically nested. The situation, depicted in Figure 6, typically arise near the boundary where fine grid points have been moved.

Definition 1. An element $K \in \tau_\ell$ is said to be regular if the union of the (iterated) sons of K on the finest level is K.

Since $\tau_{\ell_{\max}}$ is a proper FE grid of Ω, the system matrix $\mathbf{L}_{\ell_{\max}}$ on this level is generated in the standard way. The coarser systems are defined recursively via the Galerkin product

$$\mathbf{L}_{\ell-1} = r_{\ell-1\leftarrow\ell}\mathbf{L}_\ell p_{\ell\leftarrow\ell-1}. \tag{4}$$

Since the restriction is again defined as the adjoint of $p_{\ell\leftarrow\ell-1}$ we have to specify only an appropriate choice of the inter-grid prolongation $p_{\ell\leftarrow\ell-1}$. This is done by using the interpretation (3) of a nodal vector as a grid function. A nodal vector $\mathbf{u}_\ell \in \mathbf{R}^{N_\ell}$ on level ℓ defines a continuous function u_ℓ by using the grid τ_ℓ and corresponding standard FE basis functions $\left\{\Phi_i^\ell\right\}_{1\leq i\leq N_\ell}$ (see (2)). The evaluation

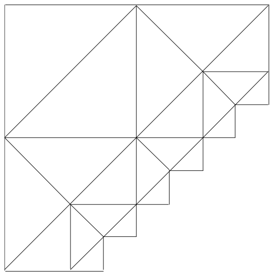

Fig. 5. Fine grid $\tau_{\ell_{\max}}$ with $\ell_{\max} = 3$. All triangles which lie outside of the domain are rejected. Note that in this example no movement of grid points was necessary.

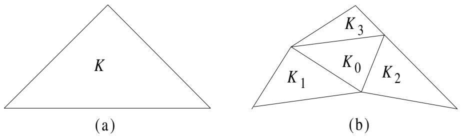

Fig. 6. Triangle K of τ_ℓ and logical children $\{K_i\}_{0 \leq i \leq 3}$ of the finer level $\ell + 1$.

of u_ℓ at the nodal points of the finer grid associates to any $\mathbf{u}_\ell \in \mathbf{R}^{N_\ell}$ a nodal vector $\mathbf{u}_{\ell+1} \in \mathbf{R}^{N_{\ell+1}}$. This defines the mapping $p_{\ell+1 \leftarrow \ell} : \mathbf{R}^{N_\ell} \to \mathbf{R}^{N_\ell + 1}$.

This prolongation can be interpreted as a convex interpolation in the following way. Let x be a nodal point of the grid $\tau_{\ell+1}$ which lies in a coarser element $K \in \tau_\ell$. Then, the prolonged nodal value at x is given by standard FE interpolation on K using the coarse-grid nodal values on K

$$\mathbf{u}_{\ell+1}(x) = \sum_{y \in \Theta_\ell \cap \bar{K}} \alpha_y(x)\, \mathbf{u}_\ell(y)$$

where $\alpha_y(x)$ are the coefficients of the FE interpolation.

In the case of homogeneous Dirichlet boundary conditions, we have to modify $p_{\ell \leftarrow \ell-1}$ such that $x \in \partial\Omega$ implies that $\mathbf{u}_{\ell+1}(x) = 0$ (see [12]).

The FE system matrices were generated recursively by using (4). Alternatively, it is possible to define a finite element space along with an appropriate basis such that the corresponding stiffness matrix equals $\mathbf{L}_\ell$. For this, let us consider the grid τ_ℓ and let x_j denote a nodal point of τ_ℓ. Define the unit nodal vector corresponding to this point by

$$\mathbf{e}_i = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Using the prolongation operators iteratively we can associate a fine grid nodal vector $\tilde{\mathbf{e}}_i$ with $\mathbf{e}_i$ by

$$\tilde{\mathbf{e}}_i = p_{\ell_{\max}\leftarrow\ell_{\max}-1} p_{\ell_{\max}-1\leftarrow\ell_{\max}-2} \cdots p_{\ell+1\leftarrow\ell}\mathbf{e}_i.$$

The finite element interpolation of the fine grid vector $\tilde{\mathbf{e}}_i$ links any $\mathbf{e}_i$ with a continuous function on Ω by

$$e_i\left(x\right) := \sum_{i=1}^{N_{\ell_{\max}}} \tilde{\mathbf{e}}_i \Phi_i^{\ell_{\max}}\left(x\right). \tag{5}$$

Note that e_i is a polynomial on each **fine**-grid element (provided $\Phi_i^{\ell_{\max}}$ are piecewise polynomials) while this is not true in general for the coarse grid elements. The *Composite Finite Element Space* is defined by

$$V_\ell = \operatorname{span}\left\{e_i\left(x\right) : 1 \leq i \leq N_\ell\right\}.$$

Remark 2. From the definition it follows that the Composite Finite Element Spaces are nested: $V_\ell \subset V_{\ell+1}$.

3.2 Approximation Property of V_ℓ

In many cases, the error analysis of Galerkin discretizations of PDEs leads to an estimate of the form

$$\left\|u - u_\ell\right\|_{H^1(\Omega)} \leq \left(1 + \frac{C_S}{\gamma}\right) \operatorname{dist}\left(u, V_\ell\right),$$

where u_ℓ denotes the solution of the Galerkin discretization and

$$\operatorname{dist}\left(u, V_\ell\right) := \inf_{w_\ell \in V_\ell} \left\|u - w_\ell\right\|_{H^1(\Omega)}.$$

The stability constant γ and continuity constant C_S mainly depend on the PDE on the continuous level. Obviously, the approximation property of the FE space, which is employed for the Galerkin discretization, plays a key role in the error estimate. In the following, we state that under relatively weak assumptions the asymptotic approximation property of finite elements carries over to composite finite element spaces independent of the (low) dimension of V_ℓ. The proof of the theorem was worked out in detail in [8] while more general situations as, e.g., the 3-d case and more general elements are treated in [13].

Theorem 3. *Let Ω be a 2-d domain with Lipschitz boundary, τ_ℓ denote a triangulation, and $h_\ell := \max_{\Delta\in\tau_\ell} \operatorname{diam}\Delta$ the step size of τ_ℓ. We assume that Φ_i of* (2) *are the piecewise linear "hat"-functions and*

(a) *τ_ℓ is quasi-uniform, i.e., $h_\ell \leq C \operatorname{diam}\Delta$, for all $\Delta \in \tau_\ell$,*

(b) τ_ℓ *is shape-regular, i.e.,* $\sup\{\operatorname{diam} S : S$ *is a ball contained in* $\Delta\} \geq Ch_\ell$ *for all* $\Delta \in \tau_\ell$,
(c) $h_{\ell+1} \leq \frac{2}{3} h_\ell$
(d) *the prolongation process is local, i.e.,* $\operatorname{diam}(\operatorname{supp} e_i) \leq Ch_\ell$ *with* e_i *given by* (5),

with constants independent of ℓ *and* $\ell_{\max}$.
Then, for all $u \in H^2(\Omega)$ *there exists* $u_\ell \in V_\ell$ *such that*

$$\|u - u_\ell\|_{H^m(\Omega)} \leq Ch_\ell^{2-m} \|u\|_{H^2(\Omega)}, \qquad m \in \{0, 1\}. \tag{6}$$

Proof. The proof is essentially given in [8]. The only thing to check is that Assumption (d) above implies Assumption 2 in [8]. Since this is purely technical but straightforward we skip this detail here.

Hence, we have shown that V_ℓ has the asymptotic approximation property starting with extremely few degrees of freedom. In view of Figure 4(a), this means that the Galerkin discretization with composite finite elements on the grid τ_0 for the Poisson problem on Ω (cf. fig. 3) with Neumann boundary conditions satisfies

$$\|u - u_0\|_1 \leq Ch_0 \|u\|_2$$

with $h_0 = \operatorname{diam} \Omega$. The function u_0 is a function which lives only on the physical domain Ω, while the four degrees of freedom associated with u_0 are located at the corners of the square formed by the two coarse-grid triangles. Estimate (6) means that one is already in the asymptotic range, i.e., the error on the grid τ_1 is expected to be only half of the error of u_0.

Since the spaces V_ℓ are nested they are also well-suited to be used for defining coarse-grid approximations for multi-grid methods. The approximation property for multi-grid methods (cf. [4, Chapter 6]) directly follows from this fact.

3.3 Complexity of Composite Finite Elements

In this subsection we will investigate the complexity of generating the system matrix corresponding to the space V_ℓ. We recall that we assumed that the step sizes of the sequence of grids τ_ℓ satisfy

$$O(\operatorname{diam} \Omega) = h_0 > h_1 = \frac{h_0}{2} > h_2 = \frac{h_0}{4} > \ldots > h_\ell =: H > \ldots > h_{\ell_{\max}} =: h.$$

We assumed here for simplicity that the step size is reduced by a factor 2 in each step, while other contraction rates can be treated in the same way. If one is interested in the generation of the whole sequence of system matrices $\{\mathbf{L}_\ell\}_{0 \leq \ell \leq \ell_{\max}}$ one could use the Galerkin products. The complexity of generating the system matrix on the finest level is $O\left(h_{\ell_{\max}}^{-d}\right)$ where $d = 2, 3$ denotes the space dimension. Since the prolongation and restriction operators are local in the sense that the evaluation per nodal point requires $O(1)$ operations, we obtain that the generation of $\mathbf{L}_{\ell-1}$

from $\mathbf{L}_\ell$ needs $O\left(h_\ell^{-d}\right)$ operations. Together one obtains that the complexity of generating all system matrices is given by

$$\sum_{\ell=0}^{\ell_{\max}} h_\ell^{-d} = O\left(h_{\ell_{\max}}^{-d}\right),$$

i.e., does not increase the asymptotic complexity.

In some situations, however, one is interested only in the generation of a coarse-grid matrix $\mathbf{L}_\ell$ corresponding to a step size $H = h_\ell$ but would like to resolve the geometric details with a smaller step size $h = h_{\ell_{\max}}$. The following observation plays the key role. In the regular situation, where no grid points have been moved in the adjustment of the auxiliary fine grid $\tilde{\tau}_{\ell_{\max}}$ to the domain, the matrix $\mathbf{L}_\ell$ defined by the Galerkin product coincides with the matrix assembled directly on the grid τ_ℓ using the standard "coarse" finite element basis functions Φ_i^ℓ. Hence, the complexity of generating $\mathbf{L}_\ell$ is of order H^{-d}. This means that for elements $K \in \tau_m$ which are not distorted during the refinement process, i.e., are regular in the sense of Definition 1, the corresponding portions of $\mathbf{L}_\ell$, can be generated directly by using the standard FE basis function Φ_i^ℓ on K. Since the adjustment of elements to the boundary only takes place near the boundary nearly all elements are not distorted during the refinement process and there, the system matrix can be generated without prolonging up to the finest level $\ell_{\max}$.

In typical situation, only $O\left(h_m^{1-d}\right)$ elements of τ_m intersects the boundary of Ω and have to be refinement further. The computation of the so-called element matrix on an element $K \in \tau_m$ requires $O(1)$ operations. Symbolically, the algorithm reads as follows.

1. On $\tau_\ell : O\left(h_\ell^{-d}\right)$ elements are regular, i.e., not distorted on finer levels and the computation of the corresponding portions (element matrices) of $\mathbf{L}_\ell$ requires $O\left(h_\ell^{-d}\right)$ operations. $O\left(h_\ell^{1-d}\right)$ elements have to be refined further.
2. On $\tau_{\ell+1} : O\left(h_\ell^{1-d}\right)$ elements are involved. The computation of $O\left(h_\ell^{1-d}\right)$ corresponding portions of $\mathbf{L}_{\ell+1}$ needs $O\left(h_\ell^{1-d}\right)$ operations, while $O\left(h_{\ell+1}^{1-d}\right)$ elements have to be refinement further.

   $\vdots$
3. On $\tau_{\ell_{\max}} : O\left(h_{\ell_{\max}-1}^{1-d}\right)$ elements are involved. The computation of $O\left(h_{\ell_{\max}}^{1-d}\right)$ remaining element matrices needs $O\left(h_{\ell_{\max}-1}^{1-d}\right)$ operations.

The total operation count for generating $\mathbf{L}_\ell$ sums up to $O\left(H^{-d}\right) + O\left(h^{1-d}\right)$. A typical mesh which arise by this procedure is depicted in Figure 7. For a detailed study of the complexity of composite finite elements and implementation details, we refer to [7].

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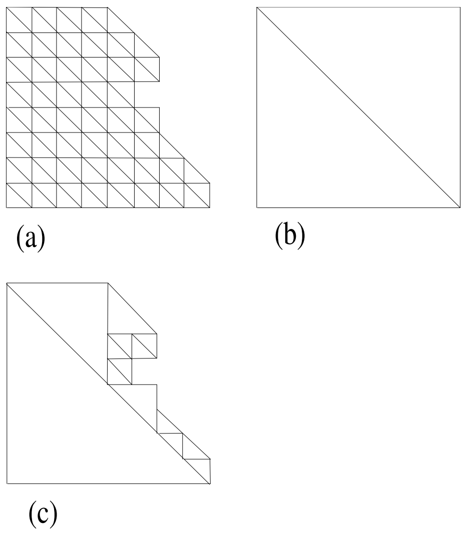

Fig. 7. Picture (a) shows the domain Ω together with the fine triangulation $\tau_{\ell_{\max}}$. In (b), the coarse triangulation τ_0 is depicted. Figure (c) shows the triangles corresponding to different grids which have to be generated in order to compute the entries of $\mathbf{L}_0$. Note that on any of the depicted triangles in (c) the usual FE basis functions are employed.

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