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The Nonlinear Limit-Point/Limit-Circle Problem for Higher Order Equations

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Abstract. We describe the nonlinear limit-point/limit-circle problem for the n-th order differential equation

$$y^{(n)} + r(t)f(y, y', \dots, y^{(n-1)}) = 0.$$

The results are then applied to higher order linear and nonlinear equations. A discussion of fourth order equations is included, and some directions for further research are indicated.

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1 Background

In 1910, H. Weyl [21] studied eigenvalue problems for second order linear differential equations of the form

$$(p(t)y')' + r(t)y = \lambda y, \quad \text{Im } \lambda \neq 0,$$

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and he classified this linear equation to be of the *limit-circle type* if every solution y belongs to the class L^2, and to be of the *limit-point type* if at least one solution does not belong to L^2. This notion has been generalized to include even order self-adjoint linear differential equations and operators (see, for example, [5,6,7,8,9,14,15,16,17,18]), and more recently to nonlinear second order equations of the form

$$(a(t)y')' + q(t)f(y) = 0$$

(see the papers of Graef and Spikes [10,11,12,13,19,20]).

Here, we consider the n-th order nonlinear differential equation

$$y^{(n)} + r(t)f(y, y', \dots, y^{(n-1)}) = 0, \tag{E}$$

where $r \in L_{\text{loc}}[0, \infty)$,

$$r \text{ does not change sign on } [t_0, \infty),\ t_0 \geq 0, \tag{1}$$

$f : \mathbf{R}^n \to \mathbf{R}$ is continuous, and

$$x_1 f(x_1, \dots, x_n) \geq 0 \text{ on } \mathbf{R}^n. \tag{2}$$

We consider only those solutions of (E) that are continuable to all of $\mathbf{R}_+ = [0, \infty)$ and are not eventually identically zero. Such a solution is said to be *oscillatory* if it is has arbitrarily large zeros, and it is said to be *nonoscillatory* otherwise.

Definition 1. Equation (E) is of the *nonlinear limit-circle type* if every continuable solution y satisfies

$$\int_0^\infty y(t)f(y(t), y'(t), \dots, y^{(n-1)}(t))\, dt < \infty;$$

if there is at least one continuable solution y of (E) such that

$$\int_0^\infty y(t)f(y(t), y'(t), \dots, y^{(n-1)}(t))\, dt = \infty,$$

then equation (E) is said to be of the *nonlinear limit-point type*.

In this paper, we describe what is known for the higher order nonlinear limit-point/limit-circle problem and indicate a number of open questions for future research.

2 Motivation

Kauffman, Read, and Zettl [14, p. 95] noted that *there are no known examples of functions r such that*

$$y^{(4)} + r(t)y = 0. \tag{L$_4$}$$

is limit-circle, i.e., all solutions of (L$_4$) *are in* L^2. This leads to the following conjecture.

Conjecture 2. *The equation*

$$y^{(4k)} + r(t)y = 0 \tag{L_{4k}}$$

always has a solution $y \notin L^2[0,\infty)$*, i.e.,* (L_{4k}) *is never of the limit-circle type.*

As a consequence of our results, we will show that as long as r does not change sign, or r is an oscillatory function that is either bounded from above or bounded from below, then (L_{4k}) can never be a limit-circle equation. In addition, we will apply our results to the sublinear Emden-Fowler equation

$$y^{(4k)} + r(t)|y|^{\lambda}\operatorname{sgn} y = 0, \quad \lambda \in (0,1]$$

and show that this equation always has a solution $y \notin L^{1+\lambda}[0,\infty)$ provided r satisfies (1).

3 Main Results

We begin by presenting some sufficient conditions for equation (E) to be of the nonlinear limit-point type (see [4]).

3.1 The Case $r \leq 0$

Theorem 3. *Suppose* $r(t) \leq 0$ *on* $[t_0,\infty)$*,* (2) *holds, and there exist constants* $M > 0$ *and* $M_1 > 0$ *such that*

$$\frac{1}{x_1} \leq f(x_1,\dots,x_n) \leq M_1(1+x_1) \tag{C_1}$$

for $x_1 \geq M,\ x_i \in \mathbf{R},\ i = 2,\dots,n$*. Then* (E) *is of the nonlinear limit-point type.*

If we restrict our attention to equations of the form

$$y^{(n)} + r(t)f(y) = 0,$$

then (C_1) becomes

$$\frac{1}{u} \leq f(u) \leq M_1(1+u)$$

for $u \geq M > 0$, which is certainly true, for example, if f is an increasing function with

$$|f(u)| \leq A + B|u| \qquad \text{for large } u,$$

or if $f(u) = u^{\gamma}$ where $0 < \gamma \leq 1$ is the ratio of odd positive integers.

Remark 4. The left hand inequality in (C_1) is not unreasonable. For example, for third order equations, Bartušek and Došlá (see Theorem 3.3 and Remark 3.4 in [1]) proved that if $r(t) \leq -K < 0$ and there exists $\beta > \frac{3}{2}$ such that

$$|f(x_1,x_2,x_3)| \leq \frac{1}{|x_1|^{\beta}} \qquad \text{for } |x_1| \geq M > 0,$$

then (E) is of the nonlinear limit-circle type. Whether their result is true for $n > 3$ remains an open question.

The proof of Theorem 3, as well as the other theorems in this section, are somewhat long and technical in nature. They make use of an energy type function, some integral inequalities, and knowledge of the behavior of oscillatory solutions of (E). Hence, we will omit the proofs, and concentrate on the nature of the results.

3.2 The Case $r \geq 0$

In studying the asymptotic behavior of solutions of higher order equations, the order itself sometimes plays an important role. Observe that the set of positive integers can be divided into the three disjoint sets, $\{n : n = 4k,\ k = 1, 2, \dots\}$, $\{n : n = 2k+1,\ k = 1, 2, \dots\}$, and $\{n : n = 4k+2,\ k = 1, 2, \dots\}$.

Theorem 5. *If $n = 4k$, (2) holds, $r(t) \geq 0$ on $[t_0, \infty)$, and there exist constants $M_1 > 0$, $M_2 > 0$, and $\lambda \in (0, 1]$ such that*

$$M_1|x_1|^\lambda \leq |f(x_1, x_2, \dots, x_n)| \leq M_2(1 + |x_1|) \qquad \text{on } \mathbf{R}^n, \tag{C$_2$}$$

then (E) is of the nonlinear limit-point type.

Observe once again that if $f(x_1, x_2, \dots, x_n) = f(x_1) = x_1^\gamma$ with $0 < \gamma \leq 1$ the ratio of odd positive integers, then condition (C_2) is clearly satisfied.

Theorem 6. *If $n \geq 3$, (2) holds, and there exist constants $M > 0$, $M_1 > 0$, $M_2 > 0$, and $\lambda \in (0, 1]$ such that*

$$0 \leq r(t) \leq M,$$

and

$$M_1|x_1|^\lambda \leq |f(x_1, x_2, \dots, x_n)| \leq M_2|x_1|^\lambda \qquad \text{on } \mathbf{R}^n,$$

then (E) is of the nonlinear limit-point type.

Remark 7. The case $n = 3$ is contained in [1, Theorem 3.7] under a slightly weaker nonlinearity condition on f; the proof for $n \geq 4$ appears in [4, Theorem 3].

The following two theorems generalize the nonlinearity condition imposed on f in Theorem 6, but at the same time, restrict the values of n allowed.

Theorem 8. *Suppose $n = 2k + 1$, there exist constants $M_1 > 0$ and $M_2 > 0$ such that*

$$M_1 \leq r(t) \leq M_2,$$

and there is a positive constant M and a continuous function $g : \mathbf{R}_+ \to \mathbf{R}$ such that $g(0) = 0$, $g(x) > 0$ for $x > 0$, $\liminf_{x\to\infty} g(x) > 0$, and

$$g(|x_1|) \leq |f(x_1, \dots, x_n)| \leq M(1 + |x_1|) \qquad \text{on } \mathbf{R}^n.$$

Then (E) is of the nonlinear limit-point type.

Theorem 9. *Suppose that* $n = 4k$, (2) *holds, and that there exist constants* K_i, $i = 0, 1, 2, 3, 4$, *and* x^* *such that*

$$0 \le r(t) \le K_0\, t^{\delta} \ \textit{on}\ (t_0, \infty),$$
$$g_1(|x_1|) \le |f(x_1, \dots, x_n)| \le g_2(|x_1|) \ \textit{on}\ \mathbf{R}^n,$$

where $\delta = \frac{n+1}{n-2}$ *and*

$$g_1(x) = \begin{cases} K_1 x & \textit{for } x \in [0, x^*] \\ K_2 & \textit{for } x \in (x^*, \infty) \end{cases}$$

$$g_2(x) = \begin{cases} K_3 & \textit{for } x \in [0, x^*] \\ K_4 x & \textit{for } x \in (x^*, \infty). \end{cases}$$

Then (E) *is of the nonlinear limit-point type.*

Observe that in Theorems 6 and 8, $r(t)$ is bounded above, while in Theorem 9, $r(t)$ is allowed to grow with t.

4 Applications of Main Results

Our first corollary concerns equation (E) and is an immediate consequence of Theorems 3 and 5.

Corollary 10. *If* $n = 4k$, *and* (1)–(2) *and* (C_2) *hold, then* (E) *is of the nonlinear limit-point type.*

Next, we apply our results to the equation

$$y^{(4k)} + r(t)y = 0 \qquad (L_{4k})$$

and obtain a positive answer to the conjecture raised in Section 2.

Corollary 11. *If* $r(t)$ *satisfies* (1) *or is an oscillatory function that is either bounded from above or bounded from below, then* (L_{4k}) *is not limit-circle.*

Proof. If r satisfies (1), then the conclusion follows immediately from Corollary 10. Suppose that r is an oscillatory function that is bounded from below. Then there exists a constant $K > 0$ such that $r(t) \ge -K$. By Corollary 10,

$$y^{(4k)} + (r(t) + K)y = 0$$

is not limit-circle. By a result of Naimark [16, §23, Theorem 1, p.192], it follows that the equation

$$y^{(4k)} + (r(t) + K + q(t))y = 0$$

is not limit-circle whenever q is a measurable and essentially bounded function. Thus, letting $q = -K$ we obtain that (L_{4k}) is also not of the limit-circle type. A similar argument holds if $r(t)$ is bounded from above.

Remark 12. Corollary 11 does not follow from Fedorjuk [9, Theorem 5.1] because additional assumptions on the integrability of the derivatives of r would be needed.

As another application of our results, we consider the Emden-Fowler equation

$$y^{(n)} + r(t)|y|^{\lambda} \operatorname{sgn} y = 0, \quad \lambda \in (0, 1]. \tag{E-F}$$

From Theorems 3–6, we have the following corollary (see [4]).

Corollary 13. (a) *If* $n = 4k$ *and* (1) *holds, then* (E-F) *always has a solution* $y \notin L^{1+\lambda}[0, \infty)$.

(b) *Suppose* $n = 2k + 1$ *or* $n = 4k + 2$. *If either* $r(t) \le 0$ *or* $0 \le r(t) \le M$, *then* (E-F) *always has a solution* $y \notin L^{1+\lambda}[0, \infty)$.

5 More on Fourth Order Equations

Now that we have seen that equation (L_4) is not a limit-circle equation (the only possibile exception being if r is an oscillatory function that is unbounded from above and below), it seems appropriate to ask if there are other fourth order equations that are of the limit-circle type. This leads us to the study of fourth order equations in self-adjoint form, namely,

$$y^{(4)} - (p(t)y')' + r(t)f(y) = 0, \tag{SA}$$

where p, $r : [0, \infty) \to \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ are continuous, and $uf(u) \ge 0$ on $\mathbf{R}$ (see [3]). For equation (SA), the definitions of nonlinear limit-point and limit-circle take the following form.

Definition 14. Equation (SA) is of the *nonlinear limit-circle type* if every continuable solution y satisfies

$$\int_0^{\infty} y(t)f(y(t))\, dt < \infty,$$

and if there is at least one continuable solution y such that

$$\int_0^{\infty} y(t)f(y(t))\, dt = \infty,$$

then equation (SA) is said to be of the *nonlinear limit-point type.*

We have the following result in the case where f is sublinear, that is, there exists $K > 0$ such that

$$\frac{1}{|y|} \le |f(y)| \le 1 + |y| \quad \text{for } |y| \ge K. \tag{C$_3$}$$

Theorem 15. *Let* (C_3) *hold.*

(a) *If* $r(t) \le 0$ *and either*
  (i) $p(t) \ge 0$*, or*
  (ii) $p(t) \le 0$ *and* $I(p) = \int_0^\infty s|p(s)|ds < \infty$,
  then (SA) *is of the nonlinear limit-point type.*

(b) *If* $r(t) \ge 0$ *is bounded,* $p(t) \not\equiv 0$*, and* $I(p) < \infty$*, then* (SA) *is of the nonlinear limit-point type.*

A special case of equation (SA), namely, the self-adjoint linear equation

$$My \equiv y^{(4)} - (p(t)y')' + r(t)y = 0 \qquad \text{(SAL)}$$

plays an important role in the spectral theory of singular differential operators (see, for example, [5,6,7,8,9,16]) in which the so called deficiency index is defined as follows.

Definition 16. The equation

$$y^{(4)} - (p(t)y')' + r(t)y = \lambda y, \quad \mathrm{Im}\ \lambda \ne 0, \qquad (\mathrm{SAL}_\lambda)$$

is said to be *limit-*ν if it has ν linearly independent solutions in $L^2(0,\infty)$. The differential expression M has the *deficiency index* (ν,ν) if (SAL_λ) is limit-ν.

It is known from the spectral theory of linear operators that $\nu \in \{2,3,4\}$ for equation (SAL_λ). When $\nu = 2$, (SAL_λ) is said to be *limit-point*; when $\nu = 4$, (SAL_λ) is said to be *limit-circle*. Note that this agrees with our Definition 14 above.

We will make use of the following two results from spectral theory. The first describes the relationship between equations (SAL) and (SAL_λ), and enables us to give criteria under which (SAL) is not limit-circle.

Lemma 17. (Naimark [16, Theorem 4, p.93]) *Equation* (SAL_λ) *is limit-*4 *if and only if equation* (SAL) *has all its solutions belonging to* $L^2(0,\infty)$.

Lemma 18. (Naimark [16, §23, Theorem 1, p.192]) *Let* q *be a real, measurable, essentially bounded function on* $\mathbf{R}_+$*. Then the deficiency index of the expression* M *is not changed by adding the function* q *to* r.

The following conjecture is still open (see, e.g., Paris and Wood [17] or Schultz [18]).

Conjecture 19. *Real formally self-adjoint expressions with nonnegative coefficients are not limit-circle.*

Kauffman [15] proved this conjecture in the case where the coefficients are finite sums of real multiples of real powers satisfying certain other conditions. We can provide additional information about this conjecture with our next result.

Theorem 20. *The equation* (SAL) *is not limit-circle, equivalently,* (SAL_λ) *is either limit-2 or limit-3, or equivalently, the deficiency index of M is either (2,2) or (3,3), if any one of the following conditions is satisfied:*

(i) $r(t) \le 0$ *and* $p(t) \ge 0$,
(ii) $r(t) \le 0$, $p(t) \le 0$, *and* $I(p) = \int_0^\infty s|p(s)|ds < \infty$, *or*
(iii) r *is bounded.*

Proof. Parts (i) and (ii) follow immediately from Theorem 15 and Lemma 17. To prove (iii), first observe that the equation

$$y^{(4)} - (p(t)y')' = 0$$

is never of the limit-circle type since $y(t) \equiv 1 \notin L^2$ is a solution. Hence,

$$y^{(4)} - (p(t)y')' + r(t)y = 0$$

is not limit-circle by Lemma 18.

Note 21. Results analogous to Theorems 15 and 20 for self-adjoint equations of order $n > 4$ are not yet known.

We conclude this section with the following open problem.

Problem 22. Under what conditions, such as $|r(t)| \le |R(t)|$ for all $t > t_0$, is the following statement true.
If

$$y^{(4)} - (p(t)y')' + R(t)y = 0$$

is not limit-circle, then

$$y^{(4)} - (p(t)y')' + r(t)y = 0$$

is not limit-circle.

To be of interest, it should be assumed that $r(t)$ is an unbounded function (see Theorem 20). Moreover, if $p(t) \equiv 0$, then $r(t)$ should be assumed to be oscillatory as well (see Corollary 11).

6 Concluding Remarks

We conclude this paper by noting the implication of the above results on the study of the nonlinear limit-point/limit-circle problem. Nonlinear equations of the form

$$y^{(n)} + r(t)f(y) = 0 \tag{NL}$$

have always been popular objects of study; this has been especially true for second order equations. As a consequence of Corollary 11, unless r is an unbounded oscillatory function, it would not be possible to find sufficient conditions for equation

(NL) to be of the nonlinear limit-circle type if the conditions on the nonlinear function f include linear functions as a special case. This is not the case for second order equations as can be seen from the work of Graef et al. [10,11,12,13,19,20]. Finally, it would be interesting to examine the relationships, if any, between the nonlinear limit-point/limit-circle property and the boundedness, oscillation, or convergence to zero of solutions. These interconnections for second order equations were studied in [10,11,12,13], but for higher order equations, it remains an open question.

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