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Periodic Problems for ODE's via Multivalued Poincaré Operators

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Abstract. We shall consider periodic problems for ordinary differential equations of the form

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(0) = x(a), \end{cases} \tag{I}$$

where $f : [0, a] \times R^n \to R^n$ satisfies suitable assumptions.
To study the above problem we shall follow an approach based on the topological degree theory. Roughly speaking, if on some ball of R^n, the topological degree of, associated to (I), multivalued Poincaré operator P turns out to be different from zero, then problem (I) has solutions.
Next by using the multivalued version of the classical Liapunov-Krasnoselskĭ guiding potential method we calculate the topological degree of the Poincaré operator P. To do it we associate with f a guiding potential V which is assumed to be locally Lipschitzean (instead of C^1) and hence, by using Clarke generalized gradient calculus we are able to prove existence results for (I), of the classical type, obtained earlier under the assumption that V is C^1.
Note that using a technique of the same type (adopting to the random case) we are able to obtain all of above mentioned results for the following random periodic problem:

$$\begin{cases} x'(\xi, t) = f(\xi, t, x(\xi, t)), \\ x(\xi, 0) = x(\xi, a), \end{cases} \tag{II}$$

where $f : \Omega \times [0, a] \times R^n \to R^n$ is a random operator satisfying suitable assumptions.
This paper stands a simplification of earlier works of F. S. De Blasi, G. Pianigiani and L. Górniewicz (see: [7], [8]), where the case of differential inclusions is considered.

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1 Background in geometric topology

Throughout this note $\mathbb{R}^n$, $n \geq 1$, is an n-dimensional real Euclidean space, with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. A closed (resp. open) ball in $\mathbb{R}^n$ with center x and radius $r \geq 0$ is denoted by $B^n(x,r)$ (resp. $B^n_0(x,r)$). Furthermore we put:

$$B^n(r) = B^n(0,r), \qquad B^n_0(r) = B^n_0(0,r),$$
$$S^{n-1}(r) = B^n(r) \setminus B^n_0(r), \qquad \mathbb{P}^n = \mathbb{R}^n \setminus \{0\}.$$

$\mathbb{Z}$ stands for the set of all integers.

For $A \subset \mathbb{R}^n$ we denote by $\overline{A}$, the closure of A. If $A \subset \mathbb{R}^n$ is nonempty, we put

$$|A|^- = \inf\{\|a\| \mid a \in A\}.$$

As usual, $\varphi : X \to Y$ (resp. $\varphi : X \multimap Y$) denotes a single valued (resp. multivalued) map φ from X to Y.

In the sequel, any topological space X we consider is always supposed to be metric. When the clarity is not affected, by "space X" we mean "topological metric space X".

A space X is called *contractible* if there is a continuous homotopy $h : X \times [0,1] \to X$ and a point $x_0 \in X$ such that:

$$h(x,0) = x \qquad \text{for every } x \in X,$$
$$h(x,1) = x_0 \qquad \text{for every } x \in X.$$

A nonempty compact space X is called an R_δ*-set* if there is a decreasing sequence $\{X_k\}$ of compact contractible spaces X_k such that

$$X = \bigcap_{k=1}^{+\infty} X_k.$$

A space X is called an *absolute neighbourhood retract* ($X \in ANR$) if, for every space Y and any closed set $C \subset Y$ and any continuous map $f : C \to X$, there is an open neighbourhood U of C in Y and a continuous map $g : U \to X$ such that:

$$g(x) = f(x) \qquad \text{for every } x \in C. \tag{3}$$

A space X is called an *absolute retract* ($X \in AR$) if, for any space Y and any closed $C \subset Y$ and any continuous map $f : C \to X$, there is a continuous map $g : Y \to X$ satisfying (3).

Clearly $X \in AR$ implies $X \in ANR$ (the converse is not true). Moreover, if $X, Y \in ANR$ then $X \times Y \in ANR$. It is also easy to verify that any $X \in AR$ is contractible.

A multivalued map $\varphi : X \multimap Y$ with nonempty values is called *upper semicontinuous* (u.s.c.), if $\{x \in X \mid \varphi(x) \subset U\}$ is open in X for each open $U \subset Y$.

As usual $C([a,b], \mathbb{R}^n)$ stands for the Banach space of all continuous maps $x : [a,b] \to \mathbb{R}^n$, endowed with the norm of uniform convergence. Clearly it holds $C([a,b], \mathbb{R}^n) \in AR$.

Now, following [11], (see also [6,7,10]), we recall some definitions of the topological degree for multivalued maps. Applications to periodic problems (comp. (9)) will be given later, in Sections 3 and 4.

For any $X \in ANR$ we set

$$J(B^n(r), X) = \{F : B^n(r) \multimap X \mid F \text{ is u.s.c. with } R_\delta\text{-values}\}.$$

For any continuous $f : X \to \mathbb{R}^n$, when $X \in ANR$, we put

$$\begin{aligned} J_f(B^n(r), \mathbb{R}^n) = \{\varphi : B^n(r) \multimap \mathbb{R}^n \mid \varphi = f \circ F \text{ for some } F \in J(B^n(r), X),& \\ \text{and } \varphi(S^{n-1}(r)) \subset \mathbb{P}^n\}.& \end{aligned}$$

Finally, we define

$$\begin{aligned} CJ(B^n(r), \mathbb{R}^n) = \bigcup \{J_f(B^n(r), \mathbb{R}^n) \mid f : X \to \mathbb{R}^n \text{ is continuous,}& \\ \text{with } X \in ANR\}.& \end{aligned}$$

It is well known (see: [11,6,7,10]) that the notion of topological degree for multivalued maps in the class $CJ(B^n(r), \mathbb{R}^n)$ is available. To define it we need an appropriate notion of homotopy in $CJ(B^n(r), \mathbb{R}^n)$.

Definition 1. Let $\varphi_1, \varphi_2 \in CJ(B^n(r), \mathbb{R}^n)$ be two maps of the form

$$\begin{aligned} \varphi_1 = f_1 \circ F_1 \qquad & B^n(r) \overset{F_1}{\multimap} X \xrightarrow{f_1} \mathbb{R}^n \\ \varphi_2 = f_2 \circ F_2 \qquad & B^n(r) \overset{F_2}{\multimap} X \xrightarrow{f_2} \mathbb{R}^n. \end{aligned}$$

We say that φ_1 and φ_2 are *homotopic* in $CJ(B^n(r), \mathbb{R}^n)$ if there exists an u.s.c. R_δ-valued homotopy $\chi : B^n(r) \times [0,1] \multimap X$ and a continuous homotopy $h : X \times [0,1] \to \mathbb{R}^n$ satisfying:

(i) $\chi(u,0) = F_1(u), \quad \chi(u,1) = F_2(u)$ for every $u \in B^n(r)$,

(ii) $h(x,0) = f_1(x), \quad h(x,1) = f_2(x)$ for every $x \in X$,

(iii) for every $(u,\lambda) \in S^{n-1}(r) \times [0,1]$ and $x \in \chi(u,\lambda)$ we have $h(x,\lambda) \neq 0$.

The map $H : B^n(r) \times [0,1] \multimap \mathbb{R}^n$ given by

$$H(u,\lambda) = h(\chi(u,\lambda),\lambda)$$

is called a *homotopy* in $CJ(B^n(r), \mathbb{R}^n)$ between φ_1 and φ_2.

By using approximation results for multivalued maps (see: [11,6,7,10]) one can prove the following theorem concerning the construction of a degree for maps $\varphi \in CJ(B^n(r), \mathbb{R}^n)$.

Theorem 2. *There exists a map*

$$\mathrm{Deg} : CJ(B^n(r), \mathbb{R}) \to \mathbb{Z},$$

called topological degree function, satisfying the following properties:

(i) *If $\varphi \in CJ(B^n(r), \mathbb{R}^n)$ is of the from $\varphi = f \circ F$ with F single valued and continuous, then* $\mathrm{Deg}(\varphi) = \deg(\varphi)$, *when* $\deg(\varphi)$ *stands for the ordinary Brouwer degree of the single valued continuous map $\varphi : B^n(r) \to \mathbb{R}^n$.*

(ii) *If* $\mathrm{Deg}(\varphi) \neq 0$, *where $\varphi \in CJ(B^n(r), \mathbb{R}^n)$, then there exists $u \in B_0^n(r)$ such that $0 \in \varphi(u)$.*

(iii) *If $\varphi \in CJ(B^n(r), \mathbb{R}^n)$ and $\{u \in B^n(r) \mid 0 \in \varphi(u)\} \subset B_0^n(\widetilde{r})$ for some $0 < \widetilde{r} < r$, then the restriction $\widetilde{\varphi}$ of φ to $B^n(\widetilde{r})$ is in $CJ(B^n(\widetilde{r}), \mathbb{R}^n)$ and* $\mathrm{Deg}(\widetilde{\varphi}) = \mathrm{Deg}(\varphi)$.

(iv) *Let $\varphi_1, \varphi_2 \in CJ(B^n(r), \mathbb{R}^n)$ be two maps of the form*

$$\varphi_1 = f_1 \circ F_1, \qquad B^n(r) \overset{F_1}{\multimap} X \xrightarrow{f_1} \mathbb{R}^n$$

$$\varphi_2 = f_2 \circ F_2, \qquad B^n(r) \overset{F_2}{\multimap} Y \xrightarrow{f_2} \mathbb{R}^n,$$

where $X, Y \in ANR$. If there exists a continuous map $h : X \to Y$ such that the diagram

$$\begin{tikzcd} & X \arrow[dd, "h"] \arrow[dr, "f_1"] & \\ B^n(r) \arrow[ur, multimap, "F_1"] \arrow[dr, multimap, "F_2"'] & & R^n \\ & Y \arrow[ur, "f_2"'] & \end{tikzcd}$$

is commutative, that is $F_2 = h \circ F_1$ and $f_1 = f_2 \circ h$, then $\mathrm{Deg}(\varphi_1) = \mathrm{Deg}(\varphi_2)$.

(v) *If φ_1, φ_2 are homotopic in $CJ(B^n(r), \mathbb{R}^n)$, then* $\mathrm{Deg}(\varphi_1) = \mathrm{Deg}(\varphi_2)$.

2 Construction of a random topological degree

For details concerning this section we recommend [8] where the present state of a random topological degree and a random periodic problem for differential inclusions is presented.

Let (Ω, Σ) be a measurable space and $\varphi : \Omega \multimap X$ be a multivalued mapping with nonempty values; φ is called *measurable* if $\{\omega \in \Omega \mid \varphi(\omega) \subset A\} \in \Sigma$ for every closed A in a metric space X.

If X is a metric space we shall use the symbol $\mathcal{B}(X)$ to denote the *Borel σ-algebra* on X. Then $\Sigma \otimes \mathcal{B}(X)$ denotes the smallest σ-algebra on $\Omega \times X$ which contains all the sets $A \times B$, where $A \in \Sigma$ and $B \in \mathcal{B}(X)$. We say that a single valued map $v : X \to Y$ is a selection of $\varphi : X \multimap Y$ (written $v \subset \varphi$) provided $v(x) \in \varphi(x)$ for every $x \in X$.

The following lemma is crucial in what follows:

Lemma 3 ([8]). *Let X be a separable metric space, A be a closed subset of X and $\varphi : \Omega \times A \multimap X$ be a measurable mapping (with respect to the σ-algebra $\Sigma \otimes \mathcal{B}(A)$) with compact nonempty values. Assume further that for every $\omega \in \Omega$ the set $\operatorname{Fix} \varphi_\omega = \{x \in X \mid x \in \varphi(\omega, x)\}$ is compact and nonempty. Then the map $F : \Omega \multimap X$ defined by:*

$$F(\omega) = \operatorname{Fix} \varphi_\omega \qquad \text{for every } \omega \in \Omega,$$

has a measurable selection, where $\varphi_\omega(x) = \varphi(\omega, x)$ for every $x \in A$.

Sketch of proof. First, let us define the function $f : \Omega \times A \to [0, +\infty)$ as follows:

$$f(\omega, x) = \operatorname{dist}(x, \varphi(\omega, x)) = \inf\{d(x, y) \mid y \in \varphi(\omega, x)\}$$

for every $\omega \in \Omega$ and $x \in A$. Since φ is measurable one can get that f is measurable.

Now, it is obvious that the graph

$$\Gamma_F = \{(\omega, x) \in \Omega \times X \mid x \in F(\omega)\}$$

of F is equal to

$$f^{-1}(0) = \{(\omega, x) \in \Omega \times A \mid f(\omega, x) = 0\}.$$

It implies that Γ_F is a measurable subset of $\Omega \times X$ and consequently by virtue of Aumann's selection theorem there exists a measurable selection of F. □

Definition 4. A multivalued map $\varphi : \Omega \times Y \multimap X$ with compact nonempty values is called a *random operator* provided φ is measurable and satisfies the following condition:

$$\varphi(\omega, \cdot) : Y \multimap X \text{ is u.s.c.for every } \omega \in \Omega. \tag{4}$$

Now, assume that $Y \subset X$ and $\varphi : \Omega \times Y \multimap X$ is a random operator. We say that φ has a *random fixed point* provided there exists a single valued measurable map $\xi : \Omega \to Y$, called the random fixed point of φ, such that:

$$\xi(\omega) \in \varphi(\omega, \xi(\omega)) \qquad \text{for every } \omega \in \Omega.$$

We let

$$\operatorname{Fix}^{ra}(\varphi) = \{\xi : \Omega \to Y \mid \xi \text{ is a random fixed point of } \varphi\}.$$

In view of Lemma 3 it is easy to see that in many cases existence of deterministic fixed points implies existence of random fixed points. Namely, we have:

Proposition 5. *Let X be a separable AR-space and $\underline{\varphi : \Omega \times X} \multimap X$ be a random operator. Assume further that φ has R_δ-values and $\overline{\varphi_\omega(X)}$ is compact for every $\omega \in \Omega$. Then φ has a random fixed point.*

Sketch of proof. In fact, in view of Schauder Fixed Point Theorem (see [10]) we get that $\operatorname{Fix} \varphi_\omega$ is compact and nonempty. Then the map $F : \Omega \multimap X$, $F(\omega) = \operatorname{Fix} \varphi_\omega$ has a measurable selection $\xi \subset \mathcal{F}$ (comp. Lemma 3). Of course, $\xi \in \operatorname{Fix}^{ra}(\varphi)$. $\square$

Note (comp. [4,10,11]) that Proposition 5 can be formulated in many other cases.

Below, we would like to show that the topological degree theory considered in Section 1 can be taken up onto the random case (see: [7,8]).

According to Section 1 we shall use the following notations.

For any $X \in ANR$ we let:

$$J^{ra}(\Omega \times B^n(r), X) = \{F : \Omega \times \mathcal{B}^n(r) \multimap X \mid F \text{ is random operator with } R^\delta\text{-values}\};$$

for any continuous $f : X \to R^n$ we let

$$J_f^{ra}(\Omega \times B^n(r), \mathbb{R}^n) = \{\varphi : \Omega \times B^n(r) \multimap \mathbb{R}^n \mid \varphi = f \circ F \text{ for some } F \in J^{ra}(\Omega \times B^n(r), X) \text{ and } \varphi(\Omega \times S^{n-1}(r) \subset \mathbb{P}^n\};$$

finally, we define

$$CJ^{ra}(\Omega \times B^n(r), \mathbb{R}^n) = \bigcup\{J_f^{ra}(\Omega \times B^n(r), \mathbb{R}^n) \mid f : X \to \mathbb{R}^n \text{ is continuous and } X \in ANR\}.$$

In the set $CJ^{ra}(\Omega \times B^n(r), \mathbb{R}^n)$ we can introduce the appropriate notion of homotopy (comp. Section 1 for deterministic case or [8]).

Now, observe that if $\varphi \in CJ^{ra}(\Omega \times B^n(r), \mathbb{R}^n)$, then $\varphi_\omega \in CJ(\Omega \times B^n(r), \mathbb{R}^n)$ for every $\omega \in \Omega$ and, consequently, topological degree $\operatorname{Deg}(\varphi_\omega)$ of φ_ω is well defined (see Section 1 or [7]). Therefore we are allowed to define:

Definition 6. We define a multivalued map $\mathcal{D} : CJ^{ra}(\Omega \times B^n(r), \mathbb{R}^n) \multimap \mathbb{Z}$ by putting

$$\mathcal{D}(\varphi) = \{\operatorname{Deg}(\varphi_\omega) \mid \omega \in \Omega\}.$$

Then the map $\mathcal{D}$ is called the *random topological degree* on $CJ^{ra}(\Omega \times B^n(r), \mathbb{R}^n)$; we say that the random topological degree $\mathcal{D}(\varphi)$ of φ is different from zero (written $\mathcal{D}(\varphi) \neq 0$) if and only if $\operatorname{Deg}(\varphi_\omega) \neq 0$ for every $\omega \in \Omega$.

Finally, let us remark that Theorem 2 holds true for random operators. We recommend also [6] and [10] for further possible consequences of the random topological degree.

3 The Poincaré operator

In this section we define the Poincaré translation map along trajectories of ordinary differential equations ([1,2,3,4,5,7,8,9,10,11,12]): A map $f : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is called *Carathéodory* if it satisfies:

$$t \to f(t,x) \text{ is measurable for every } x \in \mathbb{R}^n, \tag{5}$$

$$x \to f(t,x) \text{ is continuous for almost all } t \in [0,a], \tag{6}$$

$$\|f(t,x)\| \leq \mu(t)(1+\|x\|) \text{ for every } (t,x) \in [0,a] \times \mathbb{R}^n,$$
$$\text{where } \mu : [0,a] \to [0,+\infty) \text{ is an integrable function.} \tag{7}$$

For a Carathéodory map $f : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ we shall consider the following two problems:

Cauchy problem

$$\begin{cases} x'(t) = f(t,x(t)), \\ x(0) = x_0, \end{cases} \tag{8}$$

and

Periodic problem

$$\begin{cases} x'(t) = f(t,x(t)), \\ x(0) = x(a), \end{cases} \tag{9}$$

where a solution $x : [0,1] \to \mathbb{R}^n$ is an absolutely continuous map such that:

$$x'(t) = f(t,x(t)) \text{ for almost all } t \in [0,a].$$

For each $x_0 \in \mathbb{R}^n$ we denote by

$$S^f(x_0) = \{x : [0,1] \to R^n \mid x \text{ is a solution of } (8)\}$$

and by

$$P^f = \{x : [0,a] \to R^n \mid x \text{ is a solution of } (9)\}.$$

We recall the well known result so called Aronszajn Theorem (comp. [7,10,12]).

Theorem 7 (Aronszajn). *If $f : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map, then the map*

$$S_f : \mathbb{R}^n \multimap C([0,a], \mathbb{R}^n)$$

defined by $S_f(x) = S^f(x)$, for every $x \in R^n$, is an u.s.c. map with R_δ-values, where $C([0,a], \mathbb{R}^n)$ is a Banach space of continuous mappings with the usual max-norm.

Now, we wish to study problem (9). To do it we shall consider the diagram:

$$\mathbb{R}^n \overset{S_f}{\multimap} C([0,a], \mathbb{R}^n) \xrightarrow{e_t} \mathbb{R}^n,$$

where for every $t \in [0,a]$ the map e_t is defined by $e_t(x) = x(0) - x(t)$.

For any $t \in [0,a]$ the map $P_f^t = e_t \circ S_f$ is called the *Poincaré translation map* associated to the problem (8).

The following proposition is self-evident:

Proposition 8. *Problem* (9) *has a solution if and only if there is $x \in \mathbb{R}^n$ such that $0 \in P_f^a(x)$.*

In what follows we can assume, without loss of generality, that $0 \notin P_f^a(x)$ for every $x \in \mathbb{R}^n$ such that $\|x\| = r$ for some $r > 0$. Then we have:

Theorem 9. *If* $\operatorname{Deg}(P_f^a) \neq 0$*, then* $P^f \neq \emptyset$*, where we consider* P_f^a *as a mapping in* $CJ(B^n(r), \mathbb{R}^n)$ *for* r *given above.*

Proof. Since S_f is u.s.c. with R_δ-values and e_a is continuous, we have that

$$P_f^a \in CJ(B^n(r), \mathbb{R}^n), \text{ for } X = C([0,a], \mathbb{R}^n)$$

being an AR-space. □

Therefore by our assumption $\operatorname{Deg}(P_f^a)$ on $B^n(r)$ is well defined. Consequently, our result follows from Theorem 2, (ii) and Proposition 8.

In order to show that $\operatorname{Deg}(P_f^a) \neq 0$ we shall use the *guiding potential* introduced by Liapunov (comp. [16,17,18]) and subsequently developed by Krasnoselskǐ (comp. [13]), Mawhin [14] and others (see: [7,8,10,15]).

4 Guiding potentials

The guiding potential method has been recently employed in [10,15] to study periodic problems (9). Unlike these papers, where the guiding potential V is supposed to be C^1, here we assume that V is only locally Lipschitzean (see: [7]). We recall some notions of Clarke generalized gradient calculus [9] we shall need later.

Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitzean function. For $x_0 \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, let $V^0(x_0, v)$ be the generalized directional derivative of V at x_0 in the direction v, that is

$$V^0(x_0, v) = \limsup_{\substack{x \to x_0 \\ t \to 0+}} \frac{V(x+tv) - V(x_0)}{t}.$$

Then the *generalized gradient* $\partial V(x_0)$ of V at x_0 is defined by

$$\partial V(x_0) = \{y \in \mathbb{R}^n \mid \langle y, v\rangle \leq V^0(x_0, v) \ \text{ for every } \ v \in \mathbb{R}^n\}.$$

The map $\partial V : \mathbb{R}^n \multimap \mathbb{R}^n$ defined by the above equality is u.s.c. with nonempty compact convex values ([9, pp. 27, 29]). If, in addition, V is also convex, then $\partial V(x_0)$ coincides with the subdifferential of V at x_0 in the sense of convex analysis ([10, p. 36]), that is

$$\partial V(x_0) = \{y \in \mathbb{R}^n \mid \langle y, x - x_0\rangle \leq V(x) - V(x_0) \ \text{ for every } \ x \in \mathbb{R}^n\}.$$

If V is C^1, then $\partial V(x_0)$ reduces to the singleton set $\{\operatorname{grad} V(x_0)\}$ ([10, p. 33]).

Definition 10. Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitzean. If, for some $r_0 > 0$, V satisfies

$$\langle \partial V(x), \partial V(x) \rangle^- > 0 \quad \text{for every } \|x\| \geq r_0, \tag{10}$$

then V is called a *direct potential*, where $\langle \partial V(x), \partial V(x) \rangle^- = \inf\{\langle u, v \rangle \mid u, v \in \partial V(x)\}$. If, in addition, V is convex and instead of (4.1) satisfies

$$0 \notin \partial V(x) \qquad \text{for every } \|x\| \geq r_0, \tag{11}$$

then V is called a *direct convex potential*.

Observe that Definition 10 implies (11), the converse is not true in general. Moreover, if V is C^1, then either (10) or (11) is equivalent to saying that $\operatorname{grad} V(x) \neq 0$ for every $\|x\| \geq r_0$. In view of that, the above definition can be interpreted as a generalization of the definition of a direct potential V in the sense of Krasnoselskiĭ [13] (see also [14,15,18]), where V is assumed to be a C^1 function.

Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a direct potential. Observe that $\partial V \in CJ(B^n(r), \mathbb{R}^n)$ if $r \geq r_0$, and thus, by Theorem 2, the topological degree $\operatorname{Deg}(\partial V)$ is well defined and independent of r. Hence, it makes sense to define the index $\operatorname{Ind}(V)$ of the direct potential V, by putting

$$\operatorname{Ind}(V) = \operatorname{Deg}(\partial V),$$

when ∂V stands for the restriction of ∂V to $B^n(r), r \geq r_0$. Of course the definition of $\operatorname{Ind}(V)$ is analogous if V is direct convex potential.

Proposition 11 ([6]). *If* $V : \mathbb{R}^n \to \mathbb{R}^n$ *is a direct potential* (*or a direct convex potential*) *satisfying* $\lim_{\|x\| \to +\infty} V(x) = +\infty$, *then* $\operatorname{Ind}(V) = 1$.

A connection between the notion of direct potential and ordinary differential equations is given by the following:

Definition 12. Let $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map. A direct potential $V : \mathbb{R}^n \to \mathbb{R}$ is called a guiding potential for f if there exists $r_0 > 0$ such that:

$$\langle f(t, x), \partial V(x) \rangle^- \geq 0 \ \text{ for every } t \ \text{ and } \ \|x\| \geq r_0. \tag{12}$$

Our main result of this section is the following:

Theorem 13. *Assume that* $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ *is a Carathéodory map and* $V : \mathbb{R}^n \to \mathbb{R}$ *is a guiding potential for* f *such that* $\operatorname{Ind}(V) \neq 0$. *Then* $P^f \neq \emptyset$.

In the proof of Theorem 13 we proceed analogously as in the proof of (4.4) in [12] (comp. also [6]). By the homotopy property of the topological degree finally we obtain $\operatorname{Deg}(P_f^a) = |\operatorname{Ind}(V)|$ and therefore our result follows from Theorem 9.

All technical details are omitted here and we left them to the reader.

5 The random case

In this section we would like point out that all results of section 3 and 4 can be formulated in the random case. We shall restrict our considerations to showing formulations and some ideas of the proofs (we recommend for more details [8]).

Definition 14. Let $f : \Omega \times [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ be a single valued map such that

$$f(\,\cdot\,,\,\cdot\,,x) : \Omega \times [0,a] \to \mathbb{R}^n \text{ is measurable,} \tag{13}$$

$$f(\omega,t,\,\cdot\,) : \mathbb{R}^n \to \mathbb{R}^n \text{ is continuous,} \tag{14}$$

$\|f(\omega,t,x)\| \le \mu(\omega,t)(1+\|x\|)$ for every t, ω and x, where $\mu\colon \Omega \times [0,a] \to [0,+\infty)$ is a map such that $\mu(\,\cdot\,,t)$ is measurable and $\mu(\omega,\,\cdot\,)$ is Lebesgue integrable. (15)

Then f is called a random Carathéodory operator.

Now, for given random Carathéodory operator and a measurable map $\xi_0 : \Omega \to \mathbb{R}^n$ we consider the following Cauchy problem:

$$\begin{cases} x'(\omega,t) = f(\omega,t,x(\omega,t)) \\ x(\omega,0) = \xi_0(\omega), \end{cases} \tag{16}$$

where the solution $x : \Omega \times [0,a] \to \mathbb{R}^n$ is a map such that $x(\,\cdot\,,t)$ is measurable, $x(\omega,\,\cdot\,)$ is absolutely continuous and the derivative $x'(\omega,t)$ is considered with respect to t.

Moreover we consider the following *random periodic problem*:

$$\begin{cases} x'(\omega,t) = f(\omega,t,x(\omega,t)), \\ x(\omega,t) = x(\omega,a). \end{cases} \tag{17}$$

To solve it we need the random Poincaré translation operator. Observe that for every $\omega \in \Omega$ and $y \in \mathbb{R}^n$ we can consider the following Cauchy problem:

$$\begin{cases} x'(t) = f_\omega(t,x(t)) = f(\omega,t,x(t)), \\ x(0) = y. \end{cases} \tag{18}$$

Now, we define the following map:

$$P : \Omega \times \mathbb{R}^n \multimap C([0,a],\mathbb{R}^n),$$
$$P(\omega,y) = S^{f_\omega}(y).$$

We have:

Proposition 15 ([8]). *Under the above assumptions the map P is a random operator.*

Note that for the proof of Proposition 15 we need the Fubini and Kuratowski Ryll-Nardzewski Selection Theorem (see [8]).

Consequently, the map

$$P_a = e_a \circ P \tag{19}$$

is called the random Poincaré operator associated to (17).

Now we are able to formulate Proposition 8, Theorem 9 and results of Section 4 for the random periodic problem. We refer to do it to the reader (comp. [8]).

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