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A Note on Asymptotic Expansion for a Periodic Boundary Condition

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Abstract. The aim of this contribution is to present a new result concerning asymptotic expansion of solutions of the heat equation with periodic Dirichlet–Neuman boundary conditions with the period going to zero in 3D.

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1 Introduction

In the recent paper [3, Filo–Luckhaus] we have determined the first two terms in the asymptotic expansion (with respect to a small parameter ε) of the solution $u_\varepsilon = u_\varepsilon(x,t)$ to the following problem:

$$\begin{aligned}
\frac{\partial u_\varepsilon}{\partial t} &= \Delta u_\varepsilon + f(x,t) && \text{in } \Omega \times (0,T),\\
\frac{\partial u_\varepsilon}{\partial \nu} &= \vartheta(x,t) - \sigma(x,t) u_\varepsilon && \text{on } n^\varepsilon \times (0,T),\\
u_\varepsilon &= 0 && \text{on } d^\varepsilon \times (0,T),\\
u_\varepsilon &= \varphi && \text{on } \Omega \times \{t=0\}.
\end{aligned} \tag{1}$$

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Here $\Omega \subset \mathbb{R}^2$ is a bounded domain whose boundary is given by a C^3 simple closed curve Γ,

$$\Gamma = \{(p(\tau), q(\tau));\ 0 \le \tau \le \pi\}, \qquad (p'(\tau))^2 + (q'(\tau))^2 = 1,$$

a is 2π periodic function such that

$$a(\sigma) = \begin{cases} 0 & : \quad \sigma \in [\pi - \delta, \pi + \delta] \\ 1 & : \quad \sigma \in [0, \pi - \delta) \cup (\pi + \delta, 2\pi] \end{cases}$$

for some $\delta \in (0, \pi)$,

$$n^\varepsilon = \left\{ x \in \Gamma;\ x = (p(\tau), q(\tau)),\ a\left(\frac{\tau}{\varepsilon}\right) = 1,\ 0 \le \tau \le \pi \right\},$$

$$d^\varepsilon = \left\{ x \in \Gamma;\ x = (p(\tau), q(\tau)),\ a\left(\frac{\tau}{\varepsilon}\right) = 0,\ 0 \le \tau \le \pi \right\}$$

and

$$\varepsilon^{-1} \qquad \text{is an even integer .}$$

We have shown, under certain smoothness assumptions on the data f, σ, ϑ and φ, that

$$u_\varepsilon = u + \varepsilon u^1 + \varepsilon \mathcal{O}(\varepsilon)\ , \tag{2}$$

where

$$\mathcal{O}(\varepsilon) \longrightarrow 0 \quad \text{strongly in } L_p(\Omega \times (0,T)) \quad \text{if} \quad \varepsilon \to 0$$

for any $p,\ 1 \le p < 4$ and

$$\frac{u_\varepsilon - u}{\varepsilon} \rightharpoonup \omega_0(\vartheta - \partial_\nu u) \quad \text{weakly in } L_2(\Gamma \times (0,T))\ . \tag{3}$$

The functions u and u^1 are solutions of the problems

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f(x,t) && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \Gamma \times (0,T), \\ u &= \varphi && \text{on } \Omega \times \{t = 0\}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} \frac{\partial u^1}{\partial t} &= \Delta u^1 && \text{in } \Omega \times (0,T), \\ u^1 &= \omega_0\left(\vartheta - \tfrac{\partial u}{\partial \nu}\right) && \text{on } \Gamma \times (0,T), \\ u^1 &= 0 && \text{on } \Omega \times \{t = 0\}, \end{aligned} \tag{5}$$

respectively. Here

$$\omega_0 \;=\; \frac{1}{\pi}\int_0^\pi \omega(x_1,0)\,dx_1 \;,$$

where $\omega = \omega(x_1, x_2)$ is the unique nonnegative 2π periodic (in the x_1 variable) solution of the following boundary value problem

$$\begin{aligned} \Delta\omega \;&=\; 0 && \text{in } \mathbb{R}^2_+, \\ a(x_1)\left(\frac{\partial\omega}{\partial x_2}(x_1,0)+1\right)&+(1-a(x_1))\omega(x_1,0)=0 && \text{for } x_1\in\mathbb{R}, \end{aligned}$$

satisfying

$$\|\omega\|_{L^\infty(\mathbb{R}^2_+)} + \int_0^\infty\int_0^\pi |\nabla\omega|^2(x_1,x_2)\,dx_1dx_2 < \infty \;.$$

Moreover, we have demonstrated, that

$$\left\|\frac{u_\varepsilon - u}{\varepsilon} - w_\varepsilon(\vartheta-\partial_\nu u)\right\|_{L_2(\Gamma\times(0,T))} \le C\,\sqrt{\varepsilon}$$

for

$$w_\varepsilon(x)\equiv\omega\left(\frac{\tau(x)}{\varepsilon},\frac{\delta(x)}{\varepsilon}\right)$$

where the functions τ, δ are defined for $x \in \overline{\Omega}$ sufficiently close to Γ such that $\delta(x) = dist(x, \Gamma)$ and

$$p'(\tau(x))(x_1 - p(\tau(x))) + q'(\tau(x))(x_2 - q(\tau(x))) = 0\;.$$

In addition,

$$\frac{u_\varepsilon - u}{\varepsilon} - w_\varepsilon\mathcal{G} \rightharpoonup u^1 - \omega_0\mathcal{G}$$

weakly in $V_2^{1,0}(\Omega \times (0, T))$, where

$$\mathcal{G}(x,t)\equiv\vartheta(x,t)-\xi(x)\partial_\nu u(p(\tau(x)),q(\tau(x)),t)$$

and ξ is a cutoff function that equals 1 in a neighbourhood of Γ and $\xi(x) = 0$ for any $x \in \Omega$, $dist(x, \Gamma) \ge \delta_0$ for some positive δ_0.
For definitions of function spaces we refer to [5, Ladyzenskaja at al.].

It is the aim of this contribution to present a generalization of the previous result to the case of more space dimensions developed in [4, Luckhaus–Filo].

2 Motivation

Our original goal was to study flow problems in porous media with a part of the boundary covered by a fluid. For one incompressible fluid in porous medium one has to solve the equation

$$\frac{\partial\theta(p)}{\partial t} = \nabla \cdot (k(\theta(p))(\nabla p + e)), \tag{6}$$

where p is the unknown pressure, θ the water content, k the conductivity of the porous medium, and $-e$ the direction of gravity (see [1, Bear], for mathematical treatment of (6) [2, Alt - Luckhaus], for example).

The part of the boundary, which is covered by the fluid and where the infiltration takes place is supposed to behave like a impervious layer with many small holes. It is assumed that the holes are distributed uniformly and create a periodic structure with period ε. The pressure is supposed to be 0 on the holes, where the fluid infiltrates into the porous medium, and the condition $(k(\theta(p))(\nabla p+e))\cdot\nu = 0$ is assumed to be satisfied on the impervious part of the boundary. As the period and the diameter of the hole is of order ε and the domain occupied by the porous medium is large, it is natural to let $\varepsilon \to 0$ and to ask on the behaviour of solutions to (6).

However, since this nonlinear problem was not yet treatable, we have studied the heat equation, i.e. equation (6) with

$$\theta(p) \equiv p, \quad k(\theta(p)) \equiv 1 \quad \text{and} \quad e = 0 \ .$$

3 Model Problem in $\mathbb{R}^3$

Let Λ be the square in $\mathbb{R}^2$, i.e. $\Lambda \equiv (0, 2\ell) \times (0, 2\ell)$ for some positive ℓ and $\theta\colon \mathbb{R}^2 \to \mathbb{R}_+$, $\mathbb{R}_+ \equiv (0,\infty)$ be a smooth function, say, $C^3(\mathbb{R}^2)$, even and 2ℓ-periodic in each of its variable. Points in $\mathbb{R}^3$ are denoted by $x = (\bar{x}, x_3)$ $\bar{x} = (x_1, x_2)$ and we define

$$\Omega \equiv \{x \in \mathbb{R}^3 \mid \ \bar{x} \in \Lambda,\ \theta(\bar{x}) < x_3 < d\}$$

for some positive d greater than the maximum of the function θ and define

$$\Gamma \equiv \{x \in \partial\Omega \mid \ x_3 = \theta(\bar{x}),\ \bar{x} \in \Lambda\}.$$

Now let $\mathcal{F} = \{\bar{x} \in \Lambda \mid \ |\bar{x} - \bar{\ell}| \le \delta\}$, $\bar{\ell} = (\ell, \ell)$ for some $0 < \delta < \ell$ and set

$$\overline{a}(\bar{x}) = \begin{cases} 0 & : \quad \bar{x} \in \mathcal{F} \\ 1 & : \quad \bar{x} \in \Lambda \setminus \mathcal{F}. \end{cases}$$

Denote by $a(\bar{x})$ for $\bar{x} \in \mathbb{R}^2$ the 2ℓ-periodic extension of the function $\overline{a}$ on the whole $\mathbb{R}^2$. Let $\varepsilon^{-1} = 2^k$ for $k \in \{0, 1, 2, \cdots\}$, define

$$\begin{aligned} \mathcal{D}^\varepsilon &\equiv \{x \in \Gamma \mid \ a(\varepsilon^{-1}\bar{x}) = 0\}, & \mathcal{D}^\varepsilon_T &\equiv \mathcal{D}^\varepsilon \times (0, T), \\ \mathcal{N}^\varepsilon &\equiv \{x \in \Gamma \mid \ a(\varepsilon^{-1}\bar{x}) = 1\}, & \mathcal{N}^\varepsilon_T &\equiv \mathcal{N}^\varepsilon \times (0, T), \\ D &\equiv \{\bar{x} \in \mathbb{R}^2 \mid \ a(\bar{x}) = 0\}, & N &\equiv \{\bar{x} \in \mathbb{R}^2 \mid \ a(\bar{x}) = 1\}. \end{aligned}$$

and for simplicity of notation we put $\partial_t u \equiv \partial u/\partial t$, $\partial_\nu u \equiv \partial u/\partial \nu$ etc.

Consider now the problem

$$\begin{aligned} \partial_t u_\varepsilon &= \Delta u_\varepsilon + f_\varepsilon(x,t) && \text{in } \Omega_T, \\ \partial_\nu u_\varepsilon &= \vartheta_\varepsilon(x,t) - \sigma_\varepsilon(x,t) u_\varepsilon && \text{on } \mathcal{N}^\varepsilon_T, \\ u_\varepsilon &= 0 && \text{on } \mathcal{D}^\varepsilon_T, \\ \partial_\nu u_\varepsilon &= 0 && \text{on } (\partial\Omega \setminus \Gamma)_T, \\ u_\varepsilon &= u_0^\varepsilon && \text{on } \Omega \times \{t=0\} \end{aligned} \tag{7}$$

under the following assumptions:

(A) $f_\varepsilon, f, f^1 \in L_2(\Omega_T)$ and such that

$$\frac{f_\varepsilon - f}{\varepsilon} \rightharpoonup f^1 \qquad \text{in} \quad L_2(\Omega_T);$$

(B) σ_ε, $\partial_t \sigma_\varepsilon \in L_\infty(\Gamma_T)$ for any ε and there exists a positive constant C independent of ε such that $\|\sigma_\varepsilon\|_{L_\infty(\Gamma_T)} \le C$;

(C) $\vartheta_\varepsilon, \vartheta$, $\partial_t \vartheta_\varepsilon \in L_2(\Gamma_T)$ and such that

$$\vartheta_\varepsilon \rightharpoonup \vartheta \qquad \text{in} \quad L_2(\Gamma_T);$$

(D) u_0^ε, $u_0 \in W_2^1(\Omega)$, $u_0 = 0$ on Γ, $u_0^\varepsilon = 0$ on $\mathcal{D}^\varepsilon$, $u^1 \in L_2(\Omega)$ and such that

$$\frac{u_0^\varepsilon - u_0}{\varepsilon} \rightharpoonup u_0^1 \qquad \text{in} \quad L_2(\Omega).$$

We prove that asymptotic expansion (2) holds in the sense that

$$\mathcal{O}(\varepsilon) \longrightarrow 0$$

weakly in $L_2(\Omega_T)$ and strongly in $L_2(\Omega^*_T)$ for any subdomain $\Omega^* \subset \Omega$ with a positive distance from Γ, and, comparing to (3),

$$\frac{u_\varepsilon - u}{\varepsilon}(x,t) \rightharpoonup \omega_0(x)\left(\vartheta(x,t) - \partial_\nu u(x,t)\right) \tag{8}$$

(weakly in) in $L_2(\Gamma_T)$. Here, similarly as above (see (4) and (5) above) u is the unique solution of the problem

$$\begin{aligned} \partial_t u &= \Delta u + f(x,t) && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ \partial_\nu u &= 0 && \text{on } (\partial\Omega \setminus \Gamma)_T, \\ u &= u_0 && \text{on } \Omega \times \{t=0\}, \end{aligned} \tag{9}$$

u^1 is the unique very weak solution of the problem

$$\begin{aligned}
\partial_t u^1 &= \Delta u^1 + f^1(x,t) && \text{in } \Omega_T, \\
u^1 &= \omega_0(x)\left(\vartheta(x,t) - \partial_\nu u(x,t)\right) && \text{on } \Gamma_T, \\
\partial_\nu u^1 &= 0 && \text{on } (\partial\Omega \setminus \Gamma)_T, \\
u^1 &= 0 && \text{on } \Omega \times \{t=0\},
\end{aligned} \tag{10}$$

and the function $\omega_0(x)$ is defined for $x \in \Gamma$ as follows:

$$\omega_0(x) \equiv \frac{1}{\ell^2}\int_0^\ell \int_0^\ell \varpi(x;\bar{y},0)\, d\bar{y}\ ,$$

$\varpi = \varpi(x;y)$ is the unique bounded nonnegative solution of the problem

$$\begin{aligned}
&\sum_{k=1}^{3} \frac{\partial}{\partial y_k}\left(\sum_{j=1}^{3} \gamma_{jk}(x)\frac{\partial \varpi}{\partial y_j}(x;y)\right) \ = \ 0 && y \in \mathbb{R}^3_+\ , \\
&\varpi(x;\bar{y},0) = 0 && \bar{y} \in D\ , \\
&-\frac{\partial \varpi}{\partial y_3}(x;\bar{y},0) \ = \ 1 && \bar{y} \in N\ ,
\end{aligned} \tag{11}$$

where

$$\mathbf{C}(x) = (\gamma_{jk})_{j,k=1,2,3}\ ,$$

$$\mathbf{C}(x) \equiv \frac{1}{\sqrt{1+a_1^2+a_2^2}} \begin{pmatrix} 1+a_2^2 & -a_1a_2 & 0 \\ -a_2a_1 & 1+a_1^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ ,$$

and

$$a_j \equiv \frac{\partial \theta}{\partial x_j}(\bar{x})\ .$$

The function ϖ is 2ℓ-periodic in each of its variables y_1, y_2 and it is demonstrated that

$$\varpi(x;y) \ = \ \omega(x;\mathbf{E}^{-1}(x)y)\,,$$

where $\omega(x;z)$ is for each $x \in \Gamma$ the harmonic function in $z \in \mathbb{R}^2_+$ such that

$$a(\widehat{\mathbf{E}}(x)\bar{z})\left(\frac{\partial \omega}{\partial z_3}(x;\bar{z},0) + \lambda\right) + \left(1 - a(\widehat{\mathbf{E}}(x)\bar{z})\right)\omega(x;\bar{z},0) \ = \ 0 \tag{12}$$

and

$$\mathbf{E}^{-1}(x) \equiv \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \frac{a_2}{\sqrt{a_1^2+a_2^2}} & -\frac{a_1}{\sqrt{a_1^2+a_2^2}} & 0 \\ \frac{a_1}{\sqrt{a_1^2+a_2^2}} & \frac{a_2}{\sqrt{a_1^2+a_2^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{\mathbf{E}}(x) \equiv \begin{pmatrix} \frac{a_2}{\sqrt{a_1^2+a_2^2}} & \frac{a_1}{\sqrt{a_1^2+a_2^2}} \\ -\frac{a_1}{\sqrt{a_1^2+a_2^2}} & \frac{a_2}{\sqrt{a_1^2+a_2^2}} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} ,$$

$$\lambda(x) = \left(1 + a_1^2 + a_2^2\right)^{1/4} .$$

4 A priori estimates

The first and basic step to prove the validity of the expansion (2) consists of a priori estimates, that can be summarized in the following

Theorem 1. *Assume that* (A)–(D) *are satisfied. Then there exists a positive constant* C*, independent of* ε*, such that*

$$\max_{0\le t\le T} \int_\Omega |u_\varepsilon - u|^2(x,t)\,dx + \int_0^T \int_\Omega |\nabla(u_\varepsilon - u)|^2(x,t)\,dx\,dt \le C\varepsilon \ ,$$

$$\int_0^T \int_\Gamma |u_\varepsilon - u|^2(x,t)\,dH^2(x)\,dt + \int_0^T \int_\Omega |u_\varepsilon - u|^2(x,t)\,dx\,dt \le C\varepsilon^2 \ ,$$

$$\max_{0\le t\le T} \int_\Omega |u_\varepsilon - u|^2(x,t)\phi(x)\,dx + \int_0^T \int_\Omega |\nabla(u_\varepsilon - u)|^2(x,t)\phi(x)\,dx\,dt \le C\varepsilon^2$$

and

$$\operatorname*{ess\,sup}_{0\le t\le T} \int_\Omega |\nabla(u_\varepsilon - u)|^2(x,t)\phi^3(x)\,dx + \int_0^T \int_\Omega |\partial_t(u_\varepsilon - u)|^2(x,t)\phi^3(x)\,dx\,dt \le C\varepsilon^2 \ ,$$

where ϕ is the principal eigenfunction of the problem

$$\begin{aligned} &\Delta\phi + \mu\phi = 0 && \text{in} \quad \Omega, \\ &\phi = 0 && \text{on} \quad \Gamma, \\ &\partial_\nu \phi = 0 && \text{on} \quad \partial\Omega \setminus \Gamma, \end{aligned}$$

with the corresponding principal eigenvalue $\mu = \mu_1 > 0$.

In the proof of Theorem 1 the following proposition plays an important role.

Proposition 2. *Let $v \in W_2^{1,0}(\Omega_T)$ be such that $v = 0$ on $\mathcal{D}_T^\varepsilon$. Then*

$$\int_0^T \int_\Gamma |v(x,t)|^2 dH^2(x) dt \le C\varepsilon \int_0^T \|v\|^2_{W_2^{1/2}(\Gamma)}(t)\ dt$$

and

$$\|v\|_{L_2(\Gamma_T)} \le c\|v\|_{W_2^{1,0}(\Omega_T)} \sqrt{\varepsilon}\ ,$$

where the positive constants C, c do not depend on ε and v.

Proof (of Proposition 2). We set

$$V(y,t) \equiv v(x(y),t),\ x(y) = (y_1, y_2, \theta(\bar y) + (d - \theta(\bar y))y_3/(d - \theta_0))$$

for $\bar y = (y_1, y_2) \in \Lambda$, $y_3 \in (0, d - \theta_0)$ and $\theta_0 = \max_{\bar x \in \overline{\Lambda}} \theta(\bar x)$. Note that

$$v(x,t) = V(y(x),t), \qquad y(x) = (x_1, x_2, (d - \theta_0)(x_3 - \theta(\bar x))/(d - \theta(\bar x)))$$

and $V(\bar y, 0, t) = 0$ for any $\bar y \in \Lambda$ such that $a(\varepsilon^{-1}\bar y) = 0$. Then it is not difficult to see that

$$\int_0^T \int_\Lambda |V(\bar y, 0, t)|^2 d\bar y\, dt \le \frac{\varepsilon \ell^3}{\delta^2 \pi} \int_0^T \int_\Lambda \int_\Lambda \frac{|V(\bar y, 0, t) - V(\bar z, 0, t)|^2}{|\bar y - \bar z|^3}\, d\bar y\, d\bar z\, dt\ .$$

As

$$\int_0^T \int_\Gamma |v(x,t)|^2 dH^2(x)\, dt = \int_0^T \int_\Lambda |V(\bar y, 0, t)|^2 \sqrt{1 + |\overline{\nabla}\theta(\bar y)|^2}\, d\bar y\, dt$$

and $\|V\|^2_{W_2^{1/2}(\Lambda)} \le c\|v\|^2_{W_2^{1/2}(\Gamma)} \le C\|v\|^2_{W_2^1(\Omega)}$, the assertion of Proposition 2 follows.

Proof (of Theorem 1). Note first that $u_\varepsilon - u$ is a solution of the problem

$$\begin{aligned} \partial_t(u_\varepsilon - u) &= \Delta(u_\varepsilon - u) + (f_\varepsilon - f)(x,t) && \text{in } \Omega_T, \\ \partial_\nu(u_\varepsilon - u) &= g_\varepsilon(x,t) && \text{on } \mathcal{N}_T^\varepsilon, \\ u_\varepsilon - u &= 0 && \text{on } \mathcal{D}_T^\varepsilon, \\ \partial_\nu(u_\varepsilon - u) &= 0 && \text{on } (\partial\Omega \setminus \Gamma)_T, \\ u_\varepsilon - u &= u_0^\varepsilon - u_0 && \text{on } \Omega \times \{t = 0\}, \end{aligned} \tag{13}$$

where $g_\varepsilon(x,t) = \vartheta_\varepsilon(x,t) - \sigma_\varepsilon(x,t)u_\varepsilon - \partial_\nu u$. Testing the problem (13) by $u_\varepsilon - u$ and applying Proposition 2 we arrive at

$$|u_\varepsilon - u| \equiv \max_{0\le t\le T} \|(u_\varepsilon - u)(t)\|_{L_2(\Omega)} + \|\nabla(u_\varepsilon - u)\|_{L_2(\Omega_T)} \le \\ \|u_0^\varepsilon - u_0\|_{L_2(\Omega)} + 2\|f_\varepsilon - f\|_{L_2(\Omega_T)} + C\|g_\varepsilon\|_{L_2(\Gamma_T)}\sqrt{\varepsilon}\,.$$

As, however, $\|u_\varepsilon - u\|_{L_2(\Gamma_T)} \le C|u_\varepsilon - u|\sqrt{\varepsilon}$, due to our assumptions (A) and (D) we get $\|u_\varepsilon - u\|_{L_2(\Gamma_T)} \le C\varepsilon$.

Multiplying now the equation in the problem (13) by $(u_\varepsilon - u)\phi$ and integrating over Ω one easily gets the third estimate of Theorem 1. Denote next

$$U(y,t) \equiv (u_\varepsilon - u)(x(y),t) \quad \text{for} \quad y \in \Omega^* \equiv \Lambda \times (0, d-\theta_0)\,.$$

Then we obtain

$$\int_{\Omega^*} |U(y,t)|^2 dy \le C_\eta \int_\Lambda \int_\eta^{d-\theta_0} |U(\bar{y}, y_3, t)|^2 y_3\, dy_3\, d\bar{y} + C\int_{\Omega^*} |\partial_{y_3} U(y,t)|^2 y_3\, dy$$

for any $t \in (0,T)$ and fixed $\eta \in (0, d-\theta_0)$. It is very well known that there exist positive constants c, C such that $c \le -\partial_\nu \phi \le C$ on Γ. This together with the above estimate yield the estimate $\|u_\varepsilon - u\|_{L_2(\Omega_T)} \le C\varepsilon$. The last estimate we obtain by multiplying the equation in the problem (13) by $\phi^3\partial_t(u_\varepsilon - u)$ and by integrating.

The essentiall part of the proof of the convergence (8) is the uniqueness of the problem

$$\Delta_z \omega(x;z) = 0 \qquad \text{in } \mathbb{R}^3_+ \tag{14}$$

with the boundary condition (12) in the following class of solutions.

Definition 3. By a solution of Problem (14), (12) we mean a function $\omega \in W^{1,2}_{loc}(\mathbb{R}^3_+)$ satisfying

$$\begin{gathered} \int_0^R \int_{B_2(\bar{y},L)} |\nabla\omega|^2(\bar{x},x_3)\, d\bar{x}dx_3 \le CL^2, \\ \int_0^R \int_{B_2(\bar{y},L)} |\omega|^2(\bar{x},x_3)\, d\bar{x}dx_3 \le CL^2(R^2+R), \\ \int_{B_2(\bar{y},L)} |\omega|^2(\bar{x},0)\, dx' \le CL^2 \end{gathered} \tag{15}$$

for any $\bar{y} \in \mathbb{R}^2$ (the positive constant C does not depend on $\bar{y}$,L, R), and the integral identity

$$\int_{\mathbb{R}^3_+} \nabla\omega(x)\nabla\psi(x)\, dx \;= \mu \int_{\mathbb{R}^2} \psi(\bar{x},0)\, d\bar{x}$$

for any $\psi \in W^1_{2,loc}(\mathbb{R}^3_+)$, $\psi = 0$ on $\Gamma_D \equiv \{x = (\bar{x}, 0) \mid a(\widehat{\mathbf{E}}(\bar{x})) = 0\}$ with compact support in $\overline{\mathbb{R}}^3_+$. Note that $B_2(\bar{y}, L) = \{\bar{x} \in \mathbb{R}^2 \mid |\bar{x} - \bar{y}| < L\}$.

This problem was obtained as a limit as $\varepsilon \to 0$ after applying rescaling arguments for $(u_\varepsilon - u)/\varepsilon$ in any point $x \in \Gamma$.

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