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Bifurcation of Periodic and Chaotic Solutions in Discontinuous Systems

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Abstract. Chaos generated by the existence of Smale horseshoe is the well-known phenomenon in the theory of dynamical systems. The Poincaré-Andronov-Melnikov periodic and subharmonic bifurcations are also classical results in this theory. The purpose of this note is to extend those results to ordinary differential equations with multivalued perturbations. We present several examples based on our recent achievements in this direction. Singularly perturbed problems are studied as well. Applications are given to ordinary differential equations with both dry friction and relay hysteresis terms.

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1 Introduction

There are many concrete problems in mechanics with non-smooth nonlinearities. Such discontinuities arise in the context of modelling Coulomb friction [1,3,9,19]. One of the simplest examples for such problems is provided by the pendulum with dry friction given by

$$\ddot{x} + x + \mu \operatorname{sgn} \dot{x} = \psi(t)\,. \tag{1}$$

Here $\operatorname{sgn} r = r/|r|$ for $r \in \mathbb{R} \setminus \{0\}$. Equation (1) is studied for the periodic case in [7] and for the almost periodic case in [8]. Also a rather complete picture of the asymptotic behaviour of (1) is derived. The range of μ is found such that nontrivial almost periodic and periodic motions exist. The question of uniqueness of such motions is studied as well.

The numerical analysis is presented in the papers [26,27,28] for a mechanical model of a friction-oscillator with simultaneous self- and external excitation given by the equation of motion

$$\ddot{x} + x = F_R(v_r) + u_0 \cos \Omega t\,, \tag{2}$$

where u_0, Ω are positive constants, $v_r = v_0 - \dot{x}$ is a relative velocity and F_R is the friction force defined by

$$\begin{aligned} &F_R(v_r) = \mu(v_r) F_N \operatorname{sgn}(v_r) \quad \text{for the slip mode} \quad v_r \neq 0 \\ &F_R(v_r) = x(t) - u_0 \cos \Omega t \quad \text{for the stick mode} \quad v_r = 0\,. \end{aligned}$$

Here $\mu(v_r)$ is a friction coefficient and F_N is the normal force. Three different types of friction coefficients $\mu(v_r)$ are studied in [26,27,28] including the Coulomb one $\mu(v_r) = 1$, $v_r \neq 0$. The bifurcation behaviour and the routes to chaos of (2) are investigated for a wide range of parameters. The influence of these three types of friction coefficients is described and the admissibility of smoothing procedures is examined by comparing results gained for non-smooth and smoothed friction coefficients. These papers [26,27,28] present a nice introduction to the phenomenon of dry friction problem as well.

The boundedness of solutions of the equation

$$\ddot{x} + x + \mu \operatorname{sgn} x = \psi(t) \tag{3}$$

is studied in [23] as well as the existence of infinitely many periodic and quasiperiodic solutions of (3) is established for all $\mu > 0$ sufficiently large.

By using Lyapunov exponents, the qualitative analysis for (1) and for a similar friction-oscillator is given in [20] and [21], respectively. A numerical analysis of the same friction-oscillator is presented in [22].

Finally, let us note that equations of the type (1) also appear in electrical engineering (see [1, Chap. III]), related problems are studied in control systems (see [31]) as well, and dry friction problems were investigated already in [29], [30].

This note is based on recent results derived in the papers [10,11,12,13,14,15]. We focus on concrete examples rather than presenting theoretical results. In Section 2, we give examples with chaotic solutions. Section 3 deals with bifurcation of periodic solutions for a friction-oscillator. A problem with small relay hysteresis is studied in Section 4.

In this paper, the dry friction is modelled by the Coulomb law [9], [19] which includes a static coefficient of friction μ_s and a dynamic coefficient of friction μ_d. If $\mu_s = \mu_d = \mu$, then the friction law may be written as $\dot{x} \to \mu \operatorname{sgn} \dot{x}$. On the other hand, since usually $\mu_s > \mu_d$, the smooth approximation of $\operatorname{sgn} r$ given, for instance, by

$$\Phi(r) = \frac{1}{\pi}\big(7 \arctan 8sr - 5 \arctan 4sr\big), \quad s \gg 1$$

seems to be physically more relevant than the mathematically convenient approximation of the form

$$r \to \frac{2}{\pi} \arctan sr, \quad s \gg 1 .$$

The function Φ has two symmetric spikes at $r = \pm\frac{\sqrt{6}}{8s}$ of the values

$$\pm\frac{1}{\pi}\left(7 \arctan \sqrt{6} - 5 \arctan \frac{\sqrt{6}}{2}\right) \doteq \pm 1,2261344 .$$

Moreover, $\Phi(r)$ is close to 1 or -1 when $r > 0$ or $r < 0$, respectively, tends off 0. Summarizing, we can take for any $\eta \geq 0$, $\zeta \geq 1$, $0 < \kappa \leq 1$ the multivalued function $\mathrm{Sgn}_{\eta,\zeta,\kappa}\, r$ defined by

$$\mathrm{Sgn}_{\eta,\zeta,\kappa}\, r = \begin{cases} -1 & \text{for } r < -\eta \,, \\ [-\zeta, -\kappa] & \text{for } -\eta \leq r < 0 \,, \\ [-\zeta, \zeta] & \text{for } r = 0 \,, \\ [\kappa, \zeta] & \text{for } 0 < r \leq \eta \,, \\ 1 & \text{for } r > \eta \,. \end{cases}$$

The term $\mathrm{Sgn}_{\eta,\zeta,\kappa}\, \dot{x}$ can be viewed as an extension for modelling dry friction including static and dynamic frictions as well.

2 Chaos in Dry Friction Problems

Dry friction forces acting on a moving particle due to its contact to walls have in certain situations the form $\mu(x)(g_0(\dot{x}) + \mathrm{sgn}\, \dot{x})$, where x is displacement from the rest state, $\dot{x}$ is velocity, μ and g_0 are non-negative bounded continuous, and $\mathrm{sgn}\, r = r/|r|$ for $r \in \mathbb{R} \setminus \{0\}$, see [1,5,19]. If there is also damping , restoring and external forces, the following equation is studied

$$\ddot{x} + g(x) + \mu_1 \,\mathrm{sgn}\, \dot{x} + \mu_2 \dot{x} = \mu_3 \psi(t) \,, \tag{4}$$

where $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ are small parameters, $g \in C^2(\mathbb{R}, \mathbb{R})$, $g(0) = 0$, $g'(0) < 0$, $\psi \in C^1(\mathbb{R}, \mathbb{R})$ and ψ is periodic.

If a smooth small perturbation is included in (4), then by using a method developed in dynamical systems (see [18]), it would be possible to show the existence of chaos for such ordinary differential equations.

By introducing the multivalued mapping

$$\mathrm{Sgn}\, r = \begin{cases} \mathrm{sgn}\ r & \text{for } r \neq 0 \,, \\ [-1, 1] & \text{for } r = 0 \,, \end{cases}$$

(4) is rewritten as follows

$$\ddot{x} + g(x) + \mu_2 \dot{x} - \mu_3 \psi(t) \in -\mu_1 \,\mathrm{Sgn}\, \dot{x} \,. \tag{5}$$

By a solution of any first order differential inclusion we mean a function which is absolute continuous and satisfying differential inclusion almost everywhere.

We assume the existence of a homoclinic solution ω of $\ddot{x}+g(x)=0$ such that $\lim_{t\to\pm\infty}\omega(t)=0$ and $\omega(0)>0$.

Lemma 1. ([14]) *There is a unique $t_0\in\mathbb{R}$ satisfying $\dot{\omega}(t_0)=0$. Consequently, $\dot{\omega}(t)>0$, $\forall\, t<t_0$ and $\dot{\omega}(t)<0$, $\forall\, t>t_0$.*

We consider a mapping M_μ, $\mu=(\mu_1,\mu_2,\mu_3)$, of the form

$$M_\mu(\alpha)=-2\omega(t_0)\mu_1-\mu_2\int\limits_{-\infty}^{\infty}\dot{\omega}^2(s)\,ds+\mu_3\int_{-\infty}^{\infty}\dot{\omega}(s)\psi(s+\alpha)\,ds\,.$$

Since $\omega(0)>0$, Lemma 1 implies $\omega(t_0)>0$. By putting

$$A(\alpha)=\int_{-\infty}^{\infty}\dot{\omega}(s)\psi(s+\alpha)\,ds\,,$$

we arrive at

$$M_\mu(\alpha)=A(\alpha)\mu_3-2\omega(t_0)\mu_1-\mu_2\int\limits_{-\infty}^{\infty}\dot{\omega}^2(s)\,ds\,.$$

We note that A is periodic and C^1-smooth. We put $\bar{m}=\min A$, $\bar{M}=\max A$. By applying results of [12], we obtain the following theorem.

Theorem 2. *Assume that A has critical points only at maximums and minimums. Then there is an open subset $\mathcal{R}$ of $\mathbb{R}^3$ of all sufficiently small (μ_1,μ_2,μ_3) satisfying $\mu_3\neq 0$ together with*

$$\bar{m}<\frac{2\omega(t_0)\mu_1+\mu_2\int\limits_{-\infty}^{\infty}\dot{\omega}^2(s)\,ds}{\mu_3}<\bar{M}\,,$$

on which equation (4) has chaotic solutions in the following sense:

If $J\colon\mathcal{E}=\big\{E:\ E\in\{0,1\}^{\mathbb{Z}}\big\}\to\mathcal{E}$ is the Bernoulli shift defined by $J\big(\{e_j\}_{j\in\mathbb{Z}}\big)=\{\tilde{e}_j\}_{j\in\mathbb{Z}}$, $\tilde{e}_j=e_{j+1}$, then for any $\mu\in\mathcal{R}$ and $m\in\mathbb{N}$ sufficiently large, (4) possesses a family of solutions $\big\{x_{m,E}\big\}_{E\in\mathcal{E}}$ such that

(i) *$E\to x_{m,E}$ is injective;*

(ii) *$x_{m,J(E)}(t)$ is orbitally close to $x_{m,E}(t+\varOmega m)$, where $\varOmega>0$ is the period of ψ.*

To be more precise, we consider Duffing-type equation (4) with $g(x)=-x+2x^3$, $\psi(t)=\cos t$. Hence (4) has the form

$$\ddot{x}-x+2x^3+\mu_1\operatorname{sgn}\dot{x}+\mu_2\dot{x}=\mu_3\cos t\,. \tag{6}$$

Then (see [18]), $\omega(t) = \operatorname{sech} t$. So we have

$$A(\alpha) = \int_{-\infty}^{\infty} \dot{\operatorname{sech}}\, s \, \cos(s+\alpha)\, ds = \pi \operatorname{sech} \frac{\pi}{2} \, \sin \alpha \,,$$

and $\bar{M} = -\bar{m} = \pi \operatorname{sech} \frac{\pi}{2}$, $t_0 = 0$, $\omega(t_0) = 1$, $\int\limits_{-\infty}^{\infty} \dot{\omega}^2(s)\, ds = 2/3$.

Corollary 3. *Equation* (6) *has chaotic solutions in the sense of Theorem* 2 *provided that the parameters* μ_1, μ_2, μ_3 *are sufficiently small satisfying*

$$0 < 3\pi|\mu_3| \, sech \frac{\pi}{2} < |6\mu_1 + 2\mu_2| \,.$$

The next example is a modification of (4)

$$\ddot{x} + \delta g(x) + \frac{\eta}{\sqrt{\delta}} \dot{x} + \psi(t) \operatorname{sgn} \dot{x} = 0 \,, \tag{7}$$

where $g \in C^2(\mathbb{R}, \mathbb{R})$, $g(0) = 0$, $g'(0) < 0$, $\delta > 0$ is a large parameter, $\psi \in C^1(\mathbb{R}, (0, \infty))$ is periodic and η is a constant. We assume the existence of a homoclinic solution ω of $\ddot{x} + g(x) = 0$ such that $\lim\limits_{t \to \pm\infty} \omega(t) = 0$ and $\omega(0) < 0$. Then again there is a unique $t_0 \in \mathbb{R}$ satisfying $\dot{\omega}(t_0) = 0$. Consequently, $\dot{\omega}(t) < 0$, $\forall\, t < t_0$, $\dot{\omega}(t) > 0$, $\forall\, t > t_0$ and $\omega(t_0) < 0$.

The equation (7) is rewritten in the form

$$\begin{aligned} &\varepsilon \dot{x} = y, \quad \varepsilon = \sqrt{1/\delta} \\ &\varepsilon \dot{y} = -g(x) - \varepsilon^2 \big(\eta y + \psi(t) \operatorname{sgn} y\big) \,. \end{aligned} \tag{8}$$

Hence (8) is a singularly perturbed discontinuous problem. Results of [12] imply the next theorem.

Theorem 4. *If the function*

$$M(\alpha) = 2\psi(\alpha)\omega(t_0) - \eta \int_{-\infty}^{\infty} \dot{\omega}^2(s)\, ds$$

has a simple root, then for any $\delta > 0$ *sufficiently large, equation* (7) *has chaotic solutions in the sense of Theorem* 2.

We consider again the Duffing-type equation (7) of the form

$$\ddot{x} + \delta(-x + 2x^3) + \frac{\eta}{\sqrt{\delta}} \dot{x} + (2 + \cos\, t) \operatorname{sgn} \dot{x} = 0 \,. \tag{9}$$

Hence $g(x) = -x + 2x^3$, $\psi(t) = 2 + \cos\, t$, $\omega(t) = -\operatorname{sech} t$, $t_0 = 0$, $\omega(t_0) = -1$. Consequently, Theorem 4 gives the next corollary.

Corollary 5. *Equation* (9) *has chaotic solutions in the sense of Theorem* 2 *provided that* $\delta > 0$ *is sufficiently large and* $\eta \in (-9, -3)$.

3 Bifurcation of Periodic Solutions

Consider a mass attached to a mechanical device on a moving ribbon with a speed $v_0 > 0$. If there is also an external force and damping then the resulting differential equation [1,3,9] has the form

$$\ddot{x} + q(x) + \mu_1 \operatorname{sgn}(\dot{x} - v_0) + \mu_2 \dot{x} = \mu_3 \sin \omega t\,, \tag{10}$$

where $\operatorname{sgn} r$ corresponds to the dry friction between the mass and ribbon, $q \in C^2(\mathbb{R}, \mathbb{R})$ represents the force of the mechanical device and μ_1, μ_2, μ_3, $\omega > 0$ are constants. Since $\operatorname{sgn} r$ is discontinuous in $r = 0$, (10) is considered as a perturbed differential inclusion of the form

$$\dot{x} = y, \quad \dot{y} \in -q(x) - \mu_1 \operatorname{Sgn}(y - v_0) - \mu_2 y + \mu_3 \sin \omega t\,. \tag{11}$$

Moreover, we assume

(i) There are numbers $0 < c < e$ and a C^2-mapping $\gamma\colon (c, e) \times \mathbb{R} \to \mathbb{R}$ such that $\gamma(\theta, t)$ has the minimum period θ in t, $\dot{\gamma}(\theta, 0) = 0$ and $\gamma(\theta, \cdot)$ is a solution of $\ddot{x} + q(x) = 0$.

If $c < 2\pi/\omega < e$ then we take

$$B(\alpha) = \int_0^{2\pi/\omega} \sin \omega(t + \alpha) \dot{\gamma}(2\pi/\omega, t)\, dt\,.$$

Since B is periodic, we put $\tilde{m} = \min B$, $\tilde{M} = \max B$.

Theorem 6. *Let $v_0 > 0$ be sufficiently small and let B have only critical points at minimums and maximums. If (i) holds and $c < 2\pi/\omega < e$, then for any sufficiently small $\mu = (\mu_1, \mu_2, \mu_3)$, $\mu_3 \neq 0$ satisfying*

$$\tilde{m} < \frac{1}{\mu_3}\Big(\mu_2 \int_0^{2\pi/\omega} \dot{\gamma}^2(2\pi/\omega, t)\, dt + 2\mu_1 \big|\gamma(2\pi/\omega, \pi/\omega) - \gamma(2\pi/\omega, 0)\big|\Big) < \tilde{M}\,,$$

equation (10) *has a $2\pi/\omega$-periodic solution in a neighbourhood of the family $\gamma(\theta, t)$, $\theta \in (c, e)$ from (i).*

Proof. We apply Corollary 3.2 of [10]. The formula (3.6) of [10] has the form

$$\begin{aligned} M_{\mu, v_0}(\alpha) = \Big\{ & \int_0^{2\pi/\omega} h(s) \dot{\gamma}(2\pi/\omega, s)\, ds \ \mid\ h \in L^2(0, 2\pi/\omega)\,, \\ & h(t) \in \mu_1 \operatorname{Sgn}(\dot{\gamma}(2\pi/\omega, t) - v_0) + \mu_2 \dot{\gamma}(2\pi/\omega, t) - \mu_3 \sin \omega(t + \alpha) \\ & \quad \text{a.e. on} \quad [0, 2\pi/\omega] \Big\}\,. \end{aligned}$$

Lemma 5.5. of [10] gives that $\dot\gamma(2\pi/\omega, 0) = 0$, $\dot\gamma(2\pi/\omega, \pi/\omega) = 0$ and $\dot\gamma(2\pi/\omega, t) \neq 0\ \forall t \in (0, 2\pi/\omega) \setminus \{\pi/\omega\}$. Since $\ddot\gamma + g(\gamma) = 0$, we see that $\ddot\gamma(2\pi/\omega, 0) \neq 0$, $\ddot\gamma(2\pi/\omega, \pi/\omega) \neq 0$. Consequently for $v_0 > 0$ sufficiently small, $\dot\gamma(2\pi/\omega, t) = v_0$ has the only solutions $t_1(v_0) + k2\pi/\omega$, $t_2(v_0) + k2\pi/\omega$, $t_1(v_0) < t_2(v_0)$, where $k \in \mathbb{Z}$. Moreover, either $t_1(0) = 0$, $t_2(0) = \pi/\omega$ or $t_1(0) = \pi/\omega$, $t_2(0) = 2\pi/\omega$, and $t_{1,2}$ are smooth and $\dot\gamma(2\pi/\omega, t) > v_0$ on $(t_1(v_0), t_2(v_0))$; $\dot\gamma(2\pi/\omega, t) < v_0$ on $(t_2(v_0), t_1(v_0) + 2\pi/\omega)$. Hence we obtain

$$\begin{aligned} M_{\mu,v_0}(\alpha) =& \mu_2 \int_0^{2\pi/\omega} \dot\gamma^2(2\pi/\omega, t)\, dt + 2\mu_1(\gamma(2\pi/\omega, t_2(v_0)) - \gamma(2\pi/\omega, t_1(v_0)) \\ &- \mu_3 \int_0^{2\pi/\omega} \sin\omega(t+\alpha)\dot\gamma(2\pi/\omega, t)\, dt\,. \end{aligned}$$

We note

$$M_{\mu,0}(\alpha) = \mu_2 \int_0^{2\pi/\omega} \dot\gamma^2(2\pi/\omega, t)\, dt + 2\mu_1\big|\gamma(2\pi/\omega, \pi/\omega) - \gamma(2\pi/\omega, 0)\big| - \mu_3 B(\alpha)\,.$$

The assumptions of our theorem imply that M_{μ,v_0} changes the sign on $\mathbb{R}$ for v_0 sufficiently small and for μ given in the theorem. Consequently, Corollary 3.2 of [10] can be applied to (10). The proof is finished. □

We refer the reader for more examples to [10].

4 Systems with Small Relay Hysteresis

In this section, we deal with relay hysteresis [2,24,25]. So there is given a pair of real numbers $\alpha < \beta$ (thresholds) and a pair of real-valued continuous functions $h_o \in C([\alpha, \infty), \mathbb{R})$, $h_c \in C((-\infty, \beta], \mathbb{R})$ such that $h_o(u) \geq h_c(u)\,\forall\, u \in [\alpha, \beta]$. Moreover, we suppose that h_o, h_c are bounded on $[\alpha, \infty)$, $(-\infty, \beta]$, respectively.

For a given continuous input $u(t)$, $t \geq t_0$, one defines the output $v(t) = f(u)(t)$ of the relay hysteresis operator as follows

$$f(u)(t) = \begin{cases} h_o(u(t)) & \text{if} \quad u(t) \geq \beta\,, \\ h_c(u(t)) & \text{if} \quad u(t) \leq \alpha\,, \\ h_o(u(t)) & \text{if} \quad u(t) \in (\alpha, \beta) \quad \text{and} \quad u(\tau(t)) = \beta\,, \\ h_c(u(t)) & \text{if} \quad u(t) \in (\alpha, \beta) \quad \text{and} \quad u(\tau(t)) = \alpha\,, \end{cases}$$

where $\tau(t) = \sup\{s :\ s \in [t_0, t],\ u(s) = \alpha \text{ or } u(s) = \beta\}$. If $\tau(t)$ does not exist (i.e. $u(\sigma) \in (\alpha, \beta)$ for $\sigma \in [t_0, t]$), then $f(u)(\sigma)$ is undefined and we have to initially set the relay open or closed when $u(t_0) \in (\alpha, \beta)$. Of course, when either $h_o(\beta) > h_c(\beta)$ or $h_o(\alpha) > h_c(\alpha)$ then $f(u)$ is generally discontinuous.

Let us consider the problem

$$\dddot{y} - \ddot{y} + \dot{y} - y = \mu f(y)\,, \tag{12}$$

where $\mu \in \mathbb{R}$ and f is of the form

$$\alpha = -\delta, \quad \beta = \delta, \quad \delta > 0, \quad h_o = g + p, \quad h_c = g - p$$

with $p > 0$ constant and $g \in C^1(\mathbb{R}, \mathbb{R})$.

Theorem 7. *If $\theta_0 > \delta$ is a simple root of the function*

$$4p\Big(\sqrt{1 - \frac{\delta^2}{\theta^2}} - \frac{\delta}{\theta}\Big) + \int_0^{2\pi} g(\theta \sin t) \sin t\, dt\,,$$

then there is a constant $K > 0$ such that for any μ sufficiently small there are θ_μ, ω_μ, $|\theta_0 - \theta_\mu| \le K|\mu|$, $|\omega_0 - \omega_\mu| \le K|\mu|$, $\omega_0 = -\frac{2\delta p}{\pi\theta_0^2}$ and a $2\pi(1 + \mu\omega_\mu)$-periodic solution y_μ of (12) *satisfying*

$$\sup_{t\in\mathbb{R}} \Big| y_\mu(t) - \theta_\mu \sin \frac{t}{1 + \mu\omega_\mu} \Big| \le K|\mu|\,.$$

Proof. We apply Theorem 2.2 of [15], by taking

$$\mathcal{O} = (\delta, \infty), \quad \phi_1(t) = \psi_1(t) = \sin t, \quad \theta > \delta\,,$$

$$\phi_2(t) = \psi_2(t) = \cos t, \quad \eta(\theta, t) = \theta \sin t, \quad t_0 = \arcsin \frac{\delta}{\theta}\,.$$

The formula (2.8) of [15] has the form

$$M(\omega, \theta) = \big(M_1(\omega, \theta), M_2(\omega, \theta)\big)\,,$$

where

$$M_1(\omega, \theta) = \int_0^{2\pi} \omega(\theta \cos t + 2\theta \sin t - 3\theta \cos t) \sin t\, dt + \int_{t_0}^{t_0+\pi} (g(\theta \sin t) + p) \sin t\, dt$$

$$+ \int_{t_0+\pi}^{t_0+2\pi} (g(\theta \sin t) - p) \sin t\, dt = 2\pi\theta\omega + 4p\sqrt{1 - \frac{\delta^2}{\theta^2}} + \int_0^{2\pi} g(\theta \sin t) \sin t\, dt\,,$$

$$M_2(\omega, \theta) = \int_0^{2\pi} \omega(\theta \cos t + 2\theta \sin t - 3\theta \cos t) \cos t\, dt + \int_{t_0}^{t_0+\pi} (g(\theta \sin t) + p) \cos t\, dt$$

$$+ \int_{t_0+\pi}^{t_0+2\pi} (g(\theta \sin t) - p) \cos t\, dt = -2\pi\theta\omega - 4\frac{\delta p}{\theta}\,.$$

Clearly (ω_0, θ_0) is a simple zero of M. Consequently, [15, Theorem 2.2] implies the result. The proof is finished. □

More examples are given in [15].

References

1. Andronow, A. A., Witt, A. A. and Chaikin, S. E., *Theorie der Schwingungen I*, Akademie Verlag, Berlin, 1965
2. Bliman, P. A. and Krasnosel'skii, A. M., Periodic solutions of linear systems coupled with relay, Proc. 2nd. World Congr. Nonl. Anal., Athens – 96, Nonl. Anal., Th., Meth., Appl., **30** (1997), 687–696
3. Butenin, N. V., Nejmark, Y. I. and Fufaev, N. A., *An Introduction to the Theory of Nonlinear Oscillations*, Nauka, Moscow, 1987, (in Russian)
4. Chicone, C., Lyapunov-Schmidt reduction and Melnikov integrals for bifurcation of periodic solutions in coupled oscillators, J. Differential Equations, **112** (1994), 407–447
5. Deimling, K., *Multivalued Differential Equations'*, W. De Gruyter, Berlin, 1992
6. Deimling, K., Multivalued differential equations and dry friction problems, in *Proc. Conf. Differential and Delay Equations, Ames, Iowa 1991* (A. M. Fink, R. K. Miller, W. Kliemann, eds.), World Scientific, Singapore 1992, 99–106
7. Deimling, K. and Szilágyi, P., Periodic solutions of dry friction problems, Z. angew. Math. Phys. (ZAMP), **45** (1994), 53–60
8. Deimling, K., Hetzer, G. and Shen, W., Almost periodicity enforced by Coulomb friction, Adv. Differential Equations, **1** (1996), 265–281
9. den Hartog, J. P., *Mechanische Schwingungen*, 2nd ed., Springer-Verlag, Berlin, 1952
10. Fečkan, M., Bifurcation of periodic solutions in differential inclusions, Appl. Math., **42** (1997), 369–393
11. Fečkan, M., Bifurcation from homoclinic to periodic solutions in singularly perturbed differential inclusions, Proc. Royal Soc. Edinburgh, **127A** (1997), 727–753
12. Fečkan, M., Chaotic solutions in differential inclusions: Chaos in dry friction problems, Trans. Amer. Math. Soc. (to appear)
13. Fečkan, M., Bifurcation from homoclinic to periodic solutions in ordinary differential equations with multivalued perturbations, J. Differential Equations, **130** (1996), 415-450
14. Fečkan, M., Chaos in ordinary differential equations with multivalued perturbations: applications to dry friction problems, Proc. 2nd. World Congr. Nonl. Anal., Athens – 96, Nonl. Anal., Th., Meth., Appl., **30** (1997), 1355–1364
15. Fečkan, M., Periodic solutions in systems at resonances with small relay hysteresis, Math. Slovaca (to appear)
16. Fečkan, M. and Gruendler, J., Bifurcation from homoclinic to periodic solutions in ordinary differential equations with singular perturbations, preprint
17. Gruendler, J., Homoclinic solutions for autonomous ordinary differential equations with nonautonomous perturbations, J. Differential Equations, **122** (1996), 1–26
18. Guckenheimer, J. and Holmes, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983
19. Kauderer, H., *Nichtlineare Mechanik*, Springer-Verlag, Berlin, 1958
20. Kunze, M., On Lyapunov exponents for non-smooth dynamical systems with an application to a pendulum with dry friction, preprint, 1997
21. Kunze, M. and Michaeli, B., On the rigorous applicability of Oseledet's ergodic theorem to obtain Lyapunov exponents for non-smooth dynamical systems, submitted to the Proc. 2nd. Marrakesh Inter. Conf. Differential Eq., Ed. A. Vanderbauwhede, 1995
22. Kunze, M. and Küpper, T., Qualitative bifurcation analysis of a non-smooth friction-oscillator model, Z. angew. Math. Phys. (ZAMP), **48** (1997), 87–101

23. Kunze, M., Küpper, T. and You, J., On the application of KAM theory to non-smooth dynamical systems, Differential Equations, **139** (1997), 1–21
24. Macki, J. W., Nistri, P. and Zecca, P., Mathematical models for hysteresis, SIAM Review, **35** (1993), 94–123
25. Macki, J. W., Nistri, P. and Zecca, P., Periodic oscillations in systems with hysteresis, Rocky Mountain J. Math. **22** (1992), 669–681
26. Popp, K., Some model problems showing stick-slip motion and chaos, ASME WAM, Proc. Symp. Friction-Induced Vibration, Chatter, Squeal and Chaos (R. A. Ibrahim and A. Soom, eds.) **DE–49** (1992), 1–12
27. Popp, K., Hinrichs, N. and Oestreich, M., Dynamical behaviour of a friction oscillator with simultaneous self and external excitation, Sādhanā **20, 2–4** (1995), 627–654
28. Popp, K. and Stelter, P., Stick-slip vibrations and chaos, Philos. Trans. R. Soc. London A **332** (1990), 89–105
29. Reissig, R., Erzwungene Schwingungen mit zäher Dämpfung und starker Gleitreibung. II., Math. Nachr. **12** (1954), 119–128
30. Reissig, R., Über die Stabilität gedämpfter erzwungener Bewegungen mit linearer Rückstellkraft, Math. Nachr. **13** (1955), 231–245
31. Rumpel, R. J., Singularly perturbed relay control systems, preprint, 1996
32. Rumpel, R. J., On the qualitative behaviour of nonlinear oscillators with dry friction, ZAMM **76** (1996), 665–666