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Additive groups connected with asymptotic stability of some differential equations*

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Abstract. The asymptotic behaviour of a Sturm-Liouville differential equation with coefficient $\lambda^2 q(s),\ s \in [s_0, \infty)$ is investigated, where $\lambda \in \mathbb{R}$ and $q(s)$ is a nondecreasing step function tending to ∞ as $s \to \infty$. Let S denote the set of those λ's for which the corresponding differential equation has a solution not tending to 0. It is proved that S is an additive group. Four examples are given with $S = \{0\}$, $S = \mathbb{Z}$, $S = \mathbb{D}$ (i.e. the set of dyadic numbers), and $\mathbb{Q} \subset S \subsetneqq \mathbb{R}$.

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1 Introduction and new results

In [1] F. V. Atkinson investigated the differential equations of the form

$$y''(s) + \left(\lambda^2 q(s) + \lambda \sqrt{q(s)}\, g(s)\right) y(s) = 0 \quad \lambda \in \mathbb{R},\ s \in (s_0, \infty)$$

with a coefficient $q(s) > 0$, which is continuous, nondecreasing and $\lim_{s\to\infty} q(s) = \infty$, and $\int^\infty |g(s)|\, ds < \infty$. He defined the set S of those λ's for which there exist a $g(s)$ and a solution $y(s)$ of this differential equation such that the relation $\lim_{s\to\infty} y(s) = 0$ does not hold. He found that S is an additive group and he gave examples when $S = \{0\}$, $S = \mathbb{Z}$.

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Here we consider the cases when $q(s)$ is a step function, i.e.

$$q(s) = k_i^2 \quad \text{for } s_i \le s < s_{i+1}, \quad i = 0, 1, \dots, \tag{1}$$

where $0 < k_0 < k_1 < \dots$, $\lim_{i\to\infty} k_i = \infty$ and we consider the differential equation

$$y''(s) + \lambda^2 q(s)\, y(s) = 0 \quad s \ge s_0,\ \lambda \in \mathbb{R}. \tag{2}$$

The function $y(s)$ is a solution of this differential equation if $y(s)$ is continuously differentiable, $y'(s)$ is piecewise continuously differentiable and it satisfies (2) on that pieces of interval.

In [3] we have shown that (2) has at least one solution for which $\lim_{s\to\infty} y(s) = 0$ holds provided $\lambda \ne 0$. It is a question whether all solutions of (2) tend to zero or there are some which do not do this. This property may depend heavily on the actual value of λ. Here we extend the Atkinson's result in the following way.

Theorem. *Let S denote the set of those λ's for which* (2) *has a solution $y_\lambda(s)$ such that the limit $\lim_{s\to\infty} y_\lambda(s) = 0$ does not hold. Then S is an additive group.*

The set S is never empty because $0 \in S$: for $\lambda = 0$ in (2) we have the solution $y_0(s) \equiv 1$ which does not tend to 0. On the other hand, if $\lambda \ne 0$ and $\lambda \in S$, then $-\lambda \in S$ because in (2) only the value λ^2 counts.

In [3] we have investigated similar problems and we have seen that the stability properties of differential equation (2) are equivalent to the stability of the difference equation

$$\begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = \mathcal{D}(d_i)\mathcal{E}(\lambda\omega_i) \begin{bmatrix} a_i \\ b_i \end{bmatrix}, \quad i = 0, 1, \dots, \tag{3}$$

where

$$d_i = \frac{k_i}{k_{i+1}}, \quad \omega_i = k_i(s_{i+1} - s_i), \quad \mathcal{D}(d) = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}, \quad \mathcal{E}(\omega) = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}. \tag{4}$$

Clearly, the sequences $\{d_i\}_{i=0}^\infty$, $\{\omega_i\}_{i=0}^\infty$ are subject to the restrictions

$$0 < d_i < 1, \quad \prod_{i=0}^{\infty} d_i = 0, \quad \sum_{i=0}^{\infty} \omega_i d_0 \dots d_{i-1} = \infty. \tag{5}$$

It is evident that if the sequences $\{d_i\}_{i=0}^\infty$, $\{\omega_i\}_{i=0}^\infty$ are given, satisfying (5), and knowing the initial data k_0 and s_0, we can reconstruct the function $q(s)$ of the form (1). Hence the correspondence between the differential equation (2) and the difference equation (3) is one to one.

We shall give examples for different additive groups S.

Example 1. Let $d_i < d_{i+1} < 1$ $(i = 0, 1, \dots)$ and $\lim_{i\to\infty} \omega_i = 0$ such that (5) is satisfied and

$$\sum_{i=0}^{\infty} (1 - d_{i+1})\omega_i^2 = \infty.$$

Then $S = \{0\}$.

Particularly, for $d_i = \frac{i+1}{i+2}$, $\omega_i = \frac{1}{\sqrt{\log(i+2)}}$ all the requirements of Example 1 are satisfied.

Example 2. Let $\omega_i = \pi$ and $d_i < d_{i+1} < 1$ with $\prod_{i=0}^{\infty} d_i = 0$. Then $S = \mathbb{Z}$.

Let $\mathbb{D}$ denote the set of dyadic numbers, i.e. the rational numbers of the form $n/2^m$ for all $n, m \in \mathbb{Z}$. Clearly, this set is an additive group.

Example 3. Let $\omega_i = 2^i\pi$ and $d_i = d \in [\frac{1}{2}, 1)$ be fixed. Then $S = \mathbb{D}$.

Example 4. Let $\omega_i = i!\,\pi$ and $d_i = d \in (0,1)$. Then $\frac{1}{2}e \notin S$, where $e = 2.718\ldots$ is the Euler number and $\mathbb{Q} \subset S \subsetneq \mathbb{R}$.

Open problem. For the case $S = \mathbb{R}$ we have no other example than the trivial one (see also in [1]) when $q(s)$ tends to a positive constant or $q(s) \equiv \mathit{const} > 0$. We guess that there is no example for $S = \mathbb{R}$ and $\lim_{s\to\infty} q(s) = \infty$.

In the next section we prepare the tools for the proof of the above theorem and examples and the proof itself will be carried out in Section 3.

2 Preliminaries

In [1] the proof goes on the Prüfer transformation technique. Also here we shall follow this way. First we consider the difference equation

$$\begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = \mathcal{D}(d_i)\mathcal{E}(\omega_i)\begin{bmatrix} a_i \\ b_i \end{bmatrix}, \quad i = 0, 1, \ldots, \tag{6}$$

with parameters d_i, ω_i as in (5). According to the results in [2], we know that the limit $\lim_{i\to\infty}(a_i^2 + b_i^2)$ exists for all solutions $\{[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}], [\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}], \ldots\}$. We say that the difference equation (6) is asymptotically stable if for all solutions $\lim_{i\to\infty}(a_i^2 + b_i^2) = 0$, otherwise we say that (6) is not asymptotically stable. Clearly, $\lambda \in S$ if and only if (3) is not asymptotically stable. Therefore we look for criteria to decide when a difference equation is asymptotically stable or not asymptotically stable.

Let r_i, φ_i be defined by

$$a_i = r_i \cos\varphi_i, \; b_i = -r_i \sin\varphi_i, \quad (r_i > 0). \tag{7}$$

Then $\{r_i\}_{i=0}^{\infty}$ is defined uniquely by $r_i = \sqrt{a_i^2 + b_i^2}$. Also φ_0 is unique if we make the restriction $0 \le \varphi_0 < 2\pi$. The desirable uniqueness of the values $\varphi_1, \varphi_2, \ldots$ will be guaranteed by a continuity consideration given later. By (6) we have

$$\begin{aligned} a_{i+1} &= r_{i+1}\cos\varphi_{i+1} = r_i \cos(\omega_i + \varphi_i), \\ b_{i+1} &= -r_{i+1}\sin\varphi_{i+1} = -d_i\, r_i \sin(\omega_i + \varphi_i), \end{aligned} \quad i = 0, 1, \ldots. \tag{8}$$

Hence

$$r_{i+1}^2 = r_i^2[1 - (1 - d_i^2)\sin^2(\omega_i + \varphi_i)], \quad i =, 0, 1, \ldots,$$

consequently

$$r_{i+1}^2 = r_0^2 \prod_{j=0}^{i} [1 - (1 - d_j^2)\sin^2(\omega_j + \varphi_j)].$$

Clearly, (6) is not asymptotically stable if and only if there exists an initial value φ_0 (and $r_0 = 1$), such that the sequences $\{\varphi_i\}_{i=0}^{\infty}$ and $\{d_i\}_{i=0}^{\infty}$ satisfy (8) and

$$\prod_{i=0}^{\infty} [1 - (1 - d_i^2)\sin^2(\omega_i + \varphi_i)] > 0,$$

or equivalently,

$$\sum_{i=0}^{\infty} (1 - d_i^2)\sin^2(\omega_i + \varphi_i) < \infty. \tag{9}$$

In this criterion only the knowledge of the sequence $\varphi_0, \varphi_1, \dots$ is important and we do not have to calculate the sequence $\{r_1, r_2, \dots\}$ to decide the asymptotic stability of the difference equation (6).

Let us introduce the continuous function $\Phi(d, \alpha)\colon (0, \infty) \times \mathbb{R} \to \mathbb{R}$ by the relations:

$$\begin{aligned} \Phi(1, \alpha) &= \alpha, \\ \Phi(d, k\frac{\pi}{2}) &= k\frac{\pi}{2}, \quad d > 0,\ k \in \mathbb{Z}, \\ \tan \Phi(d, \alpha) &= d \tan \alpha, \quad d > 0, \alpha \neq (2k+1)\frac{\pi}{2},\ k \in \mathbb{Z}. \end{aligned} \tag{10}$$

Clearly, $\Phi(d, \alpha)$ is strictly increasing function of α when d is fixed. Hence there exists its inverse $\Phi^{-1}(d, \alpha)$, too. Making use of the function $\Phi(d, \alpha)$, we have by (8)

$$\varphi_{i+1} = \Phi(d_i, \omega_i + \varphi_i), \quad i = 0, 1, \dots, \tag{11}$$

which defines uniquely the values of $\varphi_1, \varphi_2, \dots$.

Let the function $\sigma(d, \alpha, \beta)$ be defined on $(0, \infty) \times \mathbb{R}^2$ by one of the following (equivalent) relations:

$$\begin{aligned} \sigma(d, \alpha, \beta) &= \Phi^{-1}(d, \Phi(d, \alpha) + \Phi(d, \beta)) - \alpha - \beta, \\ \Phi(d, \alpha + \beta + \sigma(d, \alpha, \beta)) &= \Phi(d, \alpha) + \Phi(d, \beta). \end{aligned} \tag{12}$$

Clearly, we have $\sigma(1, \alpha, \beta) \equiv 0$. The most important property of this function is formulated as follows.

Lemma. *Let* $0 < d < 1$*, then*

$$|\sigma(d, \alpha, \beta)| \leq \frac{\pi}{2}\,(1 - d^2)\,|\sin \alpha|\,|\sin \beta|,$$

where the equality holds if and only if either $\sin \alpha = 0$ *or else* $\sin \beta = 0$.

The proof of this lemma will be given in the next section.

On asymptotic stability or non stability we can find sufficient conditions in [2] or in [3]. We recall them as follows.

Theorem A. *The difference equation* (6) *is asymptotically stable if*

$$\sum_{i=0}^{\infty} \min\{1-d_i, 1-d_{i+1}\} \sin^2 \omega_i = \infty.$$

Theorem B. *If the sum* $\sum_{i=0}^{\infty} |\sin \omega_i| < \infty$*, then the difference equation* (6) *is not asymptotically stable.*

Let $\mathcal{M}$ be a 2×2 (real) matrix and let $\mathbf{x} = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ with the norm $|\mathbf{x}| = \sqrt{a^2+b^2}$. Define the spectral norm $\|\mathcal{M}\|$ of the matrix $\mathcal{M}$ by

$$\|\mathcal{M}\| = \max_{|\mathbf{x}|=1} |\mathcal{M}\mathbf{x}|.$$

Consider the difference equation

$$\begin{bmatrix} \hat{a}_{i+1} \\ \hat{b}_{i+1} \end{bmatrix} = \mathcal{M}_i \begin{bmatrix} \hat{a}_i \\ \hat{b}_i \end{bmatrix} \quad i = 0, 1, \ldots, \tag{13}$$

where $\mathcal{M}_i$ is nonsingular 2×2 matrix for $i = 0, 1, \ldots$. We say that (13) is an ℓ_1-perturbation of (6) if

$$\sum_{i=0}^{\infty} \|\mathcal{M}_i - \mathcal{D}(d_i)\mathcal{E}(\omega_i)\| < \infty \tag{14}$$

holds. Here we recall another result from [2, Theorem 6 and Remark 1, Proposition 3]:

Theorem C. *Suppose* (13) *is an* ℓ_1*-perturbation of* (6)*. Then these difference equations are either both asymptotically stable or both not asymptotically stable.*

3 Proofs

We start with the proof of Lemma because we have to apply it to the proof of Theorem.

Proof of the Lemma. Suppose that $\tan\alpha$ and $\tan\beta$ are defined (i.e. $\alpha \not\equiv \frac{\pi}{2} \pmod{\pi}$ and $\beta \not\equiv \frac{\pi}{2} \pmod{\pi}$). Let $\alpha_1 = \Phi(d, \alpha)$, $\beta_1 = \Phi(d, \beta)$. Again we suppose that $\alpha_1 + \beta_1 \not\equiv \frac{\pi}{2} \pmod{\pi}$. Then by (10), (12) we have

$$\tan(\alpha_1 + \beta_1) = d\tan(\alpha + \beta + \sigma) = d\frac{\tan(\alpha+\beta) + \tan\sigma}{1 - \tan(\alpha+\beta)\tan\sigma} =$$
$$= \frac{\tan\alpha_1 + \tan\beta_1}{1 - \tan\alpha_1\, \tan\beta_1} = d\frac{\tan\alpha + \tan\beta}{1 - d^2\tan\alpha\tan\beta},$$

therefore

$$\tan\sigma = \tan\sigma(d, \alpha, \beta) = -(1-d^2)\frac{\sin\alpha\, \sin\beta\, \sin(\alpha+\beta)}{1 + (1-d^2)\sin\alpha\, \sin\beta\, \cos(\alpha+\beta)}. \tag{15}$$

Also by this formula it is clear that $\sigma(1,\alpha,\beta)\equiv 0$ and $\sigma(d,\alpha,\beta)$ is defined for all $(\alpha,\beta)\in\mathbb{R}^2$ if $d\in(0,1]$, i.e. $|\sigma(d,\alpha,\beta)|<\frac{\pi}{2}$.

By (15) it follows that

$$\sigma(d,\alpha,\beta)=\sigma(d,\beta,\alpha),\quad \sigma(d,\alpha+\pi,\beta)=\sigma(d,\alpha,\beta),\quad \sigma(d,-\alpha,-\beta)=-\sigma(d,\alpha,\beta).$$

Thus it is sufficient to prove our Lemma for $0\le|\beta|\le\alpha\le\frac{\pi}{2}$. If $\beta=0$, the statement is trivial. Let $0<\beta\le\alpha\le\frac{\pi}{2}$. First we show that $|\sigma(s,\alpha,-\beta)|\le|\sigma(d,\alpha,\beta)|$ or

$$(1-d^2)\frac{\sin\alpha\,\sin\beta\,\sin(\alpha-\beta)}{1-(1-d^2)\sin\alpha\,\sin\beta\,\cos(\alpha-\beta)}\le(1-d^2)\frac{\sin\alpha\,\sin\beta\,\sin(\alpha+\beta)}{1+(1-d^2)\sin\alpha\,\sin\beta\,\cos(\alpha+\beta)}$$

or simplifying by $(1-d^2)\sin\alpha\,\sin\beta$:

$$(1-d^2)\sin\alpha\,\sin\beta\,\sin 2\alpha\le 2\cos\alpha\,\sin\beta$$

whence the equality holds if $\alpha=\frac{\pi}{2}$, and the sharp inequality $(1-d^2)\sin^2\alpha<1$ in other cases.

Introducing the quantity $x=\frac{\pi}{2}(1-d^2)\sin\alpha\,\sin\beta$, we have to show by (15) that

$$|\tan\sigma|=\frac{\frac{2}{\pi}x\sin(\alpha+\beta)}{1+\frac{2}{\pi}x\cos(\alpha+\beta)}<\tan x=\frac{\sin x}{\cos x},\quad 0<x<\frac{\pi}{2}$$

or equivalently

$$\sin(\alpha+\beta-x)<\frac{\pi}{2}\frac{\sin x}{x}.$$

The function on the right hand side is strictly decreasing and only at $x=\frac{\pi}{2}$ would attain the value 1, and this fact proves our Lemma. $\square$

Proof of the Theorem. We have to show that if $\lambda,\mu\in S$ (and $\lambda+\mu\neq 0$), then $\lambda+\mu\in S$. According to (3) and (9) there exist φ_0 and ψ_0 such that for the sequences $\{\varphi_i\}_{i=0}^{\infty}$, $\{\psi_i\}_{i=0}^{\infty}$ defined by (11):

$$\varphi_{i+1}=\Phi(d_i,\lambda\omega_i+\varphi_i),\quad \psi_{i+1}=\Phi(d_i,\mu\omega_i+\psi_i)$$

satisfy the relations

$$\begin{aligned}&\sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\lambda\omega_i+\varphi_i)<\infty,\\ &\sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\lambda\mu_i+\psi_i)<\infty.\end{aligned}\tag{16}$$

Let $\sigma_i=\sigma(d_i,\lambda\omega_i+\varphi_i,\mu\omega_i+\psi_i)$ be defined by (12) and consider the difference equation

$$\begin{bmatrix}a_{i+1}\\ b_{i+1}\end{bmatrix}=\begin{bmatrix}1&0\\0&d_i\end{bmatrix}\begin{bmatrix}\cos\bar\omega_i&\sin\bar\omega_i\\-\sin\bar\omega_i&\cos\bar\omega_i\end{bmatrix}\begin{bmatrix}a_i\\ b_i\end{bmatrix}\quad i=0,1,\ldots,\tag{17}$$

where $\bar{\omega}_i = (\lambda+\mu)\omega_i + \sigma_i$. Let $\bar{\varphi}_i = \varphi_i + \psi_i$. Then by definition of $\bar{\omega}_i$ and by (12) we obtain

$$\begin{aligned}\bar{\varphi}_{i+1} &= \varphi_{i+1} + \psi_{i+1} = \Phi(d_i, \lambda\omega_i + \varphi_i) + \Phi(d_i, \mu\omega_i + \psi_i) = \\ &= \Phi(d_i, \lambda\omega_i + \varphi_i + \mu\omega_i + \psi_i + \sigma_i) = \Phi(d_i, (\lambda+\mu)\omega_i + \sigma_i + \bar{\varphi}_i) = \\ &= \Phi(d_i, \bar{\omega}_i + \bar{\varphi}_i).\end{aligned}$$

Now the difference equation (17) is not asymptotically stable because it has a solution not tending to 0. To see this we apply relation (9). We find

$$\begin{aligned}\sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\bar{\omega}_i + \bar{\varphi}_i) &= \sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\lambda\omega_i + \varphi_i + \mu\omega_i + \psi_i + \sigma_i) \le \\ &\le 3\sum_{i=0}^{\infty}(1-d_i^2)\left[\sin^2(\lambda\omega_i+\varphi_i) + \sin^2(\mu\omega_i+\psi_i) + \sin^2\sigma_i\right] = \\ &= 3\sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\lambda\omega_i+\varphi_i) + 3\sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\mu\omega_i+\psi_i) + \\ &\quad + 3\sum_{i=0}^{\infty}(1-d_i^2)\sin^2\sigma_i.\end{aligned}$$

The first two terms are convergent because of (16). By Lemma we have

$$\sin^2\sigma_i \le \sigma_i^2 \le \frac{\pi^2}{4}(1-d_i^2)^2\sin^2(\lambda\omega_i+\varphi_i)\sin^2(\mu\omega_i+\psi_i),$$

hence also the third term is convergent. Thus we have got

$$\sum_{i=0}^{\infty}(1-d_i^2)\sin^2(\bar{\omega}_i+\bar{\varphi}_i) < \infty,$$

which implies the existence of a solution of (17) not tending to 0.

To complete the proof, we show that (17) is an ℓ_1-perturbation of the difference equation

$$\begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = \mathcal{D}(d_i)\mathcal{E}((\lambda+\mu)\omega_i)\begin{bmatrix} a_i \\ b_i \end{bmatrix} \quad i = 0, 1, \ldots. \tag{18}$$

By Theorem C we have to estimate the spectral norm of the difference of the coefficient matrices:

$$\begin{aligned}\|\mathcal{D}(d_i)\left[\mathcal{E}(\bar{\omega}_i) - \mathcal{E}((\lambda+\mu)\omega_i)\right]\| &\le \|\mathcal{D}(d_i)\|\,\|\mathcal{E}((\lambda+\mu)\omega_i)\|\,\|\mathcal{E}(\sigma_i)-\mathcal{E}(0)\| \le \\ &\le 1\cdot 1\cdot\sqrt{\sin^2\sigma_i + (1-\cos\sigma_i)^2} \le |\sigma_i|\end{aligned}$$

because $\bar{\omega}_i = (\lambda+\mu)\omega_i + \sigma_i$ and $\mathcal{E}(\alpha+\beta) = \mathcal{E}(\alpha)\mathcal{E}(\beta)$. By Lemma and by (16) we conclude that

$$\sum_{i}^{\infty}|\sigma_i| \le \frac{\pi}{2}\sum_{i=0}^{\infty}(1-d_i^2)\left(\sin^2(\lambda\omega_i+\varphi) + \sin^2(\mu\omega_i+\psi_i)\right) < \infty,$$

i.e. the difference equation (18) is not asymptotically stable. Finally we observe that this difference equation corresponds to the differential equation

$$y''(s) + (\lambda+\mu)^2 q(s) y(s) = 0,$$

hence $\lambda+\mu \in S$. □

Proof of Example 1. Let $\lambda \neq 0$, then we have $\lim_{i\to\infty} \lambda\omega_i = 0$. Let i_0 be sufficiently large integer such that $|\lambda\omega_i| < \frac{\pi}{2}$ for $i \geq i_0$. Applying the inequality $\sin x/x > 1/\sqrt{2}$ for $|x| < \frac{\pi}{2}$, we obtain

$$\sum_{i=0}^{\infty} (1-d_{i+1}) \sin^2 \lambda\omega_i \geq \frac{\lambda^2}{2} \sum_{i=i_0}^{\infty} (1-d_{i+1})\omega_i^2 = \infty,$$

hence by Theorem A we conclude that $\lambda \notin S$, which proves that $S = \{0\}$. □

Proof of Example 2. Let $\lambda = k \in \mathbb{Z}$, then

$$\sum_{i=0}^{\infty} |\sin k\omega_i| = \sum_{i=0}^{\infty} |\sin k\pi| = 0,$$

and by Theorem B $k \in S$, i.e. $\mathbb{Z} \subset S$.

If $\lambda \notin \mathbb{Z}$, then $\sin \lambda\pi \neq 0$ and

$$\sum_{i=0}^{\infty} (1-d_{i+1}) \sin^2 \lambda\pi = \sin^2 \lambda\pi \sum_{i=1}^{\infty} (1-d_i) = \infty$$

because by (5) the restriction $\prod_{i=0}^{\infty} d_i = 0$ is equivalent to $\sum_{i=0}^{\infty}(1-d_i) = \infty$. By Theorem A all solutions of (3) tend to 0 if $\lambda \notin \mathbb{Z}$, consequently for these λ's we have $\lambda \notin S$, which proves this example. □

Proof of Example 3. The restriction $d \in [\frac{1}{2}, 1)$ is justified by the requirement in (5): $\sum_{i=0}^{\infty} 2^i \pi d^i = \pi \sum_{i=0}^{\infty} (2d)^i = \infty$. Let $\lambda = \frac{1}{2^n}$, $n \in \mathbb{N}$. Then

$$\sum_{i=0}^{\infty} |\sin \lambda\omega_i| = \sum_{i=0}^{\infty} |\sin \frac{2^i}{2^n}\pi| = \sum_{i=0}^{n-1} |\sin 2^{i-n}\pi| < \infty$$

and by Theorem B $\frac{1}{2^n} \in S$, consequently $\mathbb{D} \subset S$.

Since $1 \in S$ and S is an additive group, it is sufficient to show that if $\lambda \notin \mathbb{D}$, $\lambda \in (0,1)$, then $\lambda \notin S$. A real number λ in $(0,1)$ can be represented in the form

$$\lambda = \sum_{n=1}^{\infty} \frac{e_n}{2^n}, \qquad \text{where } e_n \in \{0,1\}.$$

Then the condition $\lambda \notin \mathbb{D}$ is equivalent to the restriction that in the sequence e_1, e_2, e_3, … there are infinitely many 0's and 1's. We claim that

$$\sum_{i=0}^{\infty} \sin^2 2^i \lambda\pi = \infty. \tag{19}$$

We prove this in indirect way. If this sum is convergent, then $\lim_{i\to\infty}\sin^2 2^i\lambda\pi = 0$ and there exists index $k \geq 1$ such that $\sin^2 2^i\lambda\pi < \frac{1}{4}$ or $|\sin 2^i\lambda\pi| < \frac{1}{2}$ for $i = k, k+1, \dots$. Since

$$\sin 2^i\lambda\pi = \sin\left(\sum_{n=1}^{\infty}\frac{e_n}{2^n}2^i\pi\right) = \pm\sin\left(\sum_{n=i+1}^{\infty}\frac{e_n}{2^{n-i}}\right)\pi. \tag{20}$$

Taking into account the bound $|\sin 2^i\lambda\pi| < \frac{1}{2} = \sin\frac{\pi}{6}$ for $i \geq k$, we have two possibilites: (1): $e_{k+1} = 0$, (2): $e_{k+1} = 1$.

(1) We claim that $e_{k+1} = 0$ implies $e_{k+2} = 0$. Suppose the contrary, i.e. $e_{k+2} = 1$, then $\frac{1}{4} \leq \sum_{n=k+1}^{\infty}\frac{e_n}{2^{n-k}} < \frac{1}{2}$ and by (20) $\sin\frac{\pi}{4} \leq |\sin 2^k\lambda\pi| < \sin\frac{\pi}{2}$ which contradicts the restriction $|\sin 2^i\lambda\pi| < \frac{1}{2}$ for $i = k, k+1, \dots$. Repeating this argumentation, we find that $e_i = 0$ for $i = k+1, k+2, \dots$, hence $\lambda \in \mathbb{D}$, which was excluded.

(2) Similarly, we claim that $e_{k+1} = 1$ implies $e_{k+2} = 1$. Again, we suppose the contrary, i.e. let $e_{k+2} = 0$. Then $\frac{1}{2} \leq \sum_{n=k+1}^{\infty}\frac{e_n}{2^{n-k}} < \frac{1}{2} + \sum_{n=k+3}^{\infty}\frac{1}{2^{n-k}} = \frac{5}{4}$ and by (20) we find $|\sin 2^k\lambda\pi| > \sin\frac{5\pi}{4} > \frac{1}{2}$ contradicting our assumption on k. Consequently, we must have $e_i = 1$ for all $i \geq k+1$, which again contradicts the assumption $\lambda \notin \mathbb{D}$.

Thus we have proved that the sum in (19) is indeed, divergent. Then Theorem A implies the asymptotic stability of (3), hence $\lambda \notin \mathbb{D}$ implies $\lambda \notin S$, which completes the proof of the relation $S = \mathbb{D}$. □

Proof of Example 4. Let $n \in \mathbb{N}$, $n \neq 0$. Let $\lambda = \frac{1}{n}$. Since

$$\sum_{i=0}^{\infty}\left|\sin\frac{i!\,\pi}{n}\right| = \sum_{i=0}^{n-1}\left|\sin\frac{i!\,\pi}{n}\right| < \infty$$

by Theorem B we conclude $\frac{1}{n} \in S$, hence $\mathbb{Q} \subset S$ because $\mathbb{Q}$ is the smallest additive group which contains all the reciprocals $\frac{1}{n}$, $\quad n = 1, 2, \dots$.

We are going to show that $\frac{1}{2}e \notin S$. Consider the sum $\sum_{i=0}^{\infty}\sin^2(i!\,e\frac{\pi}{2})$! We have for $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{i!} + \frac{1}{(i+1)!} + \frac{1}{(i+2)!} + \dots$

$$\begin{aligned} i!\,e &= i!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{(i-2)!}\right) + i + 1 + \frac{1}{i+1} + \frac{1}{(i+1)(i+2)} + \dots = \\ &= 2k_i + i + 1 + \frac{1}{i+\theta_i}\,, \quad 0 < \theta_i < 1,\ k_i \in \mathbb{N},\ i \geq 2, \end{aligned}$$

therefore

$$\begin{aligned} \sum_{i=0}^{\infty}\sin^2\left(i!\,e\frac{\pi}{2}\right) &= \sum_{i=0}^{\infty}\sin^2\left(\frac{i+1}{2}\pi + \frac{\pi}{2(i+\theta_i)}\right) \geq \\ &\geq \sum_{i=0}^{\infty}\sin^2\left(\frac{2i+1}{2}\pi + \frac{\pi}{2(2i+\theta_{2i})}\right) = \sum_{i=0}^{\infty}\cos^2\frac{\pi}{2(2i+\theta_{2i})} = \infty, \end{aligned}$$

hence by Theorem A $\frac{e}{2} \notin S$, and $S \neq \mathbb{R}$, i.e. S is a proper subset of $\mathbb{R}$. However, it is still an open problem whether the relation $\mathbb{Q} = S$ holds. □

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