ARCHIVUM MATHEMATICUM (BRNO)
Tomus 34 (1998), 31–47

Behaviour of Solutions of Linear Differential Equations with Delay

Josef Diblík

Department of Mathematics, Faculty of Electrical Engineering and Computer Science, Technical University of Brno, Technická 8, 616 00 Brno, Czech Republic
Email: diblik@dmat.fee.vutbr.cz

Abstract. This contribution is devoted to the problem of asymptotic behaviour of solutions of scalar linear differential equation with variable bounded delay of the form

$$\dot{x}(t) = -c(t)x(t - \tau(t)) \tag{*}$$

with positive function $c(t)$. Results concerning the structure of its solutions are obtained with the aid of properties of solutions of auxiliary homogeneous equation

$$\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))]$$

where the function $\beta(t)$ is positive. A result concerning the behaviour of solutions of Eq. (*) in critical case is given and, moreover, an analogy with behaviour of solutions of the second order ordinary differential equation

$$x''(t) + a(t)x(t) = 0$$

for positive function $a(t)$ in critical case is considered.

AMS Subject Classification. 34K15, 34K25.

Keywords. Positive solution, oscillating solution, convergent solution, linear differential equation with delay, topological principle of Ważewski (Rybakowski's approach).

1 Introduction

This contribution is devoted to the problem of asymptotic behaviour of solutions of scalar linear differential equation with variable bounded delay of the form

$$\dot{x}(t) = -c(t)x(t - \tau(t)) \tag{1}$$

with positive function $c(t)$. Results concerning the structure of its solutions are obtained with the aid of properties of solutions of auxiliary homogeneous equation

$$\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))] \tag{2}$$

where the function $\beta(t)$ is positive. It is known that, supposing existence of a positive solution $x = \omega(t)$ of Eq. (1), the substitution $y(t) = x(t)/\omega(t)$ gives an equation of the type (2) where $\beta(t) \equiv c(t)\omega(t - \tau(t))/\omega(t)$. On the other hand equation of the type (1) can be obtained from Eq. (2) by means of transformation $y(t) = x(t)\exp\left(\int_{t_0}^{t} \beta(s)\, ds\right)$. This means that both equations (1) and (2) are equivalent in this sense. Eq. (2) has very suitable form for investigations since an obvious property (see Lemma 1 below), that any monotone initial function generates monotone solution, implies many further properties concerning behaviour of all solutions.

A result concerning the behaviour of solutions of Eq. (1) in critical case (when $\tau(t) \equiv \tau = \text{const}$ and $\lim_{t\to\infty} c(t) = 1/\tau e$) is given and, moreover, an analogy with behaviour of solutions of the second order ordinary differential equation

$$x''(t) + a(t)x(t) = 0 \tag{3}$$

when positive continuous function $a(t)$ satisfies the condition $\lim_{t\to\infty} t^2 a(t) = 1/4$ is showed. Comparisons with known results are given.

2 Convergence of solutions of Eq. (2)

Let us consider Eq. (2)

$$\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))]$$

where $\tau \in C(I_{-1}, \mathbb{R}^+)$, $I_{-1} = [t_{-1}, \infty)$, $t_{-1} \in \mathbb{R}$, $\mathbb{R}^+ = (0, \infty)$, $t - \tau(t)$ is an increasing function on I_{-1}, $\tau(t) \le r, t \in I_{-1}, 0 < r = \text{const}$ and $\beta \in C(I_{-1}, \mathbb{R}^+)$. Let us denote $I = [t_0, \infty)$, $I_1 = [t_1, \infty)$ where $t_0 = t_{-1} + \tau(t_0)$ and $t_1 = t_0 + \tau(t_1)$. The symbol "$\,\dot{}\,$" represents the right-hand derivative.

A function y is called a *solution of Eq.* (2) *corresponding to initial point* $t^* \in I$ if y is defined and is continuous on $[t^* - \tau(t^*), \infty)$, differentiable on $[t^*, \infty)$ and satisfies (2) for $t \ge t^*$. By a *solution* of (2) we mean a solution corresponding to some initial point $t^* \in I$. We denote $y(t^*, \varphi)(t)$ a solution of Eq. (2) corresponding to initial point $t^* \in I$ which is generated by continuous *initial* function $\varphi : [t^* -$

$\tau(t^*), t^*] \mapsto \mathbb{R}$. In the case of linear Eq. (2) solution $y(t^*, \varphi)(t)$ is unique on its maximal existence interval $D_{t^*,\varphi} = [t^*, \infty)$ ([20]).

By analogy we define these notions for Eq. (1) or for other classes of differential equations with delay. If in the text of the paper an initial point is not indicated, we suppose it equals t_0.

We say that a solution of Eq. (2) corresponding to initial point t^* is *convergent* or *asymptotically convergent* if it has a finite limit at $+\infty$.

Let us start with the following trivial lemma:

Lemma 1. (J. Diblík [8]) *Let the initial function $\varphi(t)$ be defined and continuous on $[t_{-1}, t_0]$ and*

$$\varphi(t) < \varphi(t_0) \tag{4}$$

or

$$\varphi(t) > \varphi(t_0)\,, \tag{5}$$

where $t \in [t_{-1}, t_0)$. Then the corresponding solution $y(t, \varphi)$ of Eq. (2) *is on I increasing in the case of inequality* (4) *or decreasing in the case of inequality* (5).

This lemma establishes an obvious fact concerning monotony of solutions of Eq. (2). Immediately there arise the questions concerning the conditions for convergence and divergence of such solutions. In this section and in the next one we shall try some of these questions answered.

Theorem 2. (Convergence Criterion) (J. Diblík [6]) *For the convergence of all solutions of Eq.* (2), *corresponding to initial point t_0, a necessary and sufficient condition is that there exists function $k \in C(I_{-1}, \mathbb{R}^+)$ satisfying the integral inequality*

$$1 + k(t) \geq \exp\left[\int_{t-\tau(t)}^{t} \beta(s)k(s)\,ds\right] \tag{6}$$

on interval I.

The following corollary gives known sufficient condition for convergence of solutions of Eq. (2) which can be obtained as a consequence of Theorem 2 if $k(t) \equiv k = const$ where k is a sufficiently small positive number.

Corollary 3. *All solutions of Eq.* (2) *are convergent if*

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} \beta(s)\,ds < 1.$$

As further consequences we can obtain more accurate sufficient conditions for convergence if $\tau(t) \equiv \tau$ and $k(t) \equiv (t \ln^{\varepsilon} t)^{-1}$ where $\varepsilon > 1$ or $k(t) \equiv \varepsilon t^{-1}(1/\tau - L/t)^{-1}$ where ε is a small positive constant. These sufficient conditions were at first obtained by F. V. Atkinson and J. R. Haddock [2].

Corollary 4. *All solutions of Eq.* (2) *are convergent if*

$$\int_t^{t+\tau} \beta(s)\, ds \le 1 - \frac{\tau}{t} - \frac{L}{t \ln t}$$

for some $L > \tau$ *and all sufficiently large* t *or if*

$$\beta(t) \le \frac{1}{\tau} - \frac{L}{t}$$

where $L > 1/2$ *and* t *is sufficiently large.*

3 Divergence of solutions of Eq. (2)

It is easy to see that the nonexistence of the function $k \in C(I, \mathbb{R}^+)$ in Theorem 2 implies existence of divergent solutions of Eq. (2) and vice versa.

Theorem 5. (Divergence Criterion) (J. Diblík [6]) *Sufficient and necessary condition for existence of solution of Eq.* (2), *corresponding to initial point* t_0, *with property* $y(\infty) = \infty$ *is nonexistence of function* $k \in C(I_{-1}, \mathbb{R}^+)$ *satisfying the integral inequality* (6) *on interval* I.

A consequence of this criterion (if an additional property of $k(t)$ in (6) is taken into account (see [6])) is:

Corollary 6. (J. Diblík [6]) *For existence of solution of Eq.* (2), *corresponding to initial point* t_0, *with property* $y(\infty) = \infty$ *it is sufficient that*

$$\int_{t-\tau(t)}^{t} \beta(s)\, ds \ge 1, \quad t \in I. \tag{7}$$

Consider Eq. (2) where $\tau(t) \equiv 1$. Such type of equation was considered in the paper by S. N. Zhang [41] with connection of investigation of structure and asymptotic behaviour of solutions in *divergent* case (see an unpublished manuscript by F. V. Atkinson and S. N. Zhang of the identical title too). His main conditions (except condition $\beta(t) > 0$) are:

$$\int_t^{t+1} \beta(s)\, ds \ge 1, \quad \int_t^{t+1} \beta(s)\, ds \not\equiv 1, \quad t > t_0. \tag{8}$$

As we can see, these conditions are a special case of (7).

A more detailed sufficient condition for divergence which is sometimes suitable in the case when $\lim_{t\to\infty} \beta(t)\tau(t) = 1$ is given in the next theorem. (This case can be called *critical* in view of Corollary 3 and Corollary 6.)

Theorem 7. (J. Diblík [8]) *Eq.* (2) *has on* I_{-1} *a solution* $y = y(t)$ *with property* $y(\infty) = \infty$ *if*

$$\frac{1}{\tau(t)\beta(t)} - 1 \le \int_{t-\tau(t)}^{t} \left[\frac{1}{\tau(s)} - \beta(s)\right] ds,\ t \in I$$

and, moreover,

$$\int^{+\infty} \left[\frac{1}{\tau(s)} - \beta(s)\right] ds = +\infty.$$

4 Structure of solutions of Eq. (2) in convergent case

In the convergent case each solution has a finite limit. In this case we can give the estimate of the rate of convergence to this limit.

Theorem 8. (J. Diblík [6]) *Let there be a function* $k \in C(I_{-1}, \mathbb{R}^+)$ *which satisfies the integral inequality* (6) *on* I. *Then for each solution* $y(t)$ *of Eq.* (2), *corresponding to initial point* t_0, *representation*

$$y(t) = K + \zeta(t) \tag{9}$$

holds on I_{-1}, *where* $K = y(\infty)$, *and* $\zeta(t)$ *is a vanishing function. Moreover,*

$$|\zeta(t)| < \psi(t),\ t \in I_1,$$

where $\psi(\infty) = 0$,

$$\psi(t) \equiv \delta e^{-\int\limits_{t_0-r}^{t} \beta(s)k(s)\,ds} - \delta e^{-\int\limits_{t_0-r}^{\infty} \beta(s)k(s)\,ds}$$

and δ *is a fixed positive number such that*

$$\delta > M \left\{ \min_{[t_0, t_0+r]} \left[\beta(t)k(t)e^{-\int_{t_0-r}^{t} \beta(s)k(s)\,ds}\right]\right\}^{-1},$$

where

$$M = \max_{[t_0, t_0+r]} |\dot{y}(t)|.$$

On the other hand, to each $K \in \mathbb{R}$ *there corresponds a solution* $y(t)$ *of Eq.* (2) *and a function of the type* $\zeta(t)$ *such that representation* (9) *holds and for* $K = 0$ *there is an indicated representation with positive function* $\zeta(t)$.

5 Structure of solutions of Eq. (2) in divergent case

Existence of a solution, tending to ∞, plays the main role in the characterization of the family of solutions of Eq. (2) in nonconvergent case. Let us state the following result concerning the structure formula for the solutions of Eq. (2). The unique assumption of it is the existence of a solution $y(t) = Y(t)$ of Eq. (2) with property $Y(\infty) = \infty$. This result generalizes the result by S. N. Zhang [41] (which is contained in the above mentioned manuscript of F. V. Atkinson and S. N. Zhang too) where the main assumptions are: $\beta(t) > 0$ and (8).

Theorem 9. (J. Diblík [8]) *Let $Y(t)$ be a solution of Eq.* (2) *on I_{-1} with property $Y(\infty) = \infty$. Then for each solution $y(t)$ of Eq.* (2), *corresponding to initial point t_0, representation*

$$y(t) = K \cdot Y(t) + \delta(t) \tag{10}$$

holds on I_{-1}, where $K \in \mathbb{R}$ is a constant, dependent on $y(t)$, and $\delta(t)$ is a bounded solution of (2) *on I_{-1} dependent on $y(t)$. This representation is unique* (*with respect to K and $\delta(t)$*). *On the other hand, to each $K \in \mathbb{R}$ there corresponds a solution $y(t)$ of Eq.* (2) *and a function of the type $\delta(t)$ such that representation* (10) *holds and for any real K, L, M the expression $K \cdot Y(t) + L + M\delta(t)$ gives a solution of Eq.* (2).

Remark 10. In the paper by J. Diblík [8] it is proved that (under certain conditions) bounded nonconstant and nonmonotone solutions of Eq. (2) exist.

6 Concluding remarks concerning the solutions of Eq. (2)

As an analysis of properties of solutions of Eq. (2) shows, the affirmations of the following theorems are equivalent. The indicated conjectures are included as some open problems.

Theorem 11. (Convergent case) *The following assertions are equivalent:*

1) *All solutions of Eq.* (2) *are convergent.*
2) *There is a function $k \in C(I_{-1}, \mathbb{R}^+)$ which satisfies the integral inequality* (6) *on I.*
3) *There is a convergent nonconstant and monotone solution of Eq.* (2).
4) *Solution of Eq.* (2) *with infinite limit does not exist.*
5) (***Conjecture***) *There is a convergent nonconstant and nonmonotone solution of Eq.* (2).
6) (***Conjecture***) *Divergent bounded solution of Eq.* (2) *does not exist.*

Theorem 12. (Divergent case) *The following assertions are equivalent:*

1) *There is a solution of Eq.* (2) *with an infinite limit.*
2) *A function $k \in C(I_{-1}, \mathbb{R}^+)$, which satisfies the integral inequality* (6) *on interval I, does not exist.*

3) *Each nonconstant monotone solution of Eq.* (2) *has infinite limit.*
4) (***Conjecture***) *A convergent nonconstant solution of Eq.* (2) *does not exist.*
5) (***Conjecture***) *There is divergent bounded solution of Eq.* (2).

7 Properties of solutions of Eq. (1)

Let us suppose that $c \in C(I_{-1}, \mathbb{R}^+)$. All assumptions with respect to the delay $\tau(t)$ remain the same as above.

As usual, a solution of Eq. (1) is called *oscillatory* if it has arbitrary large zeros. Otherwise it is called *non-oscillatory* (*positive* or *negative*).

At first we prove theorem concerning existence of positive solutions of Eq. (1)

$$\dot{x}(t) = -c(t)x(t - \tau(t))$$

with nonzero limit. In this theorem we shall suppose $\int^{\infty} c(s)\, ds < \infty$ and the point t_0 so large that $\int_{t_0-r}^{\infty} c(s)\, ds < 1$.

Theorem 13. *Eq.* (1) *has a positive solution with nonzero limit if and only if*

$$\int^{\infty} c(t)\, dt < \infty. \tag{11}$$

Proof. Without loss of generality we shall suppose that $\int_{t_0-r}^{\infty} c(t)\, dt = m < 1$. Let us define $\omega(t)$, where $t \in I$, as the set of functions $\lambda \in C([t-r, t], \mathbb{R})$ such that

$$\varphi_1(t+\theta) < \lambda(t+\theta) < \varphi_2(t+\theta)$$

for all $\theta \in [-r, 0)$ where

$$\varphi_1(t) \equiv 1 + \delta_1 \int_t^{\infty} c(s)\, ds, \quad \varphi_2(t) \equiv 1 + \delta_2 \int_t^{\infty} c(s)\, ds, \quad t \in I_{-1},$$

$\delta_1, \delta_2 = \text{const}, 0 < \delta_1 < 1; 1/(1-m) < \delta_2$ and either $\lambda(t) = \varphi_1(t)$ or $\lambda(t) = \varphi_2(t)$. Let us define function

$$W(t, x) \equiv (x - \varphi_1(t)) \cdot (x - \varphi_2(t)), \quad t \in I_{-1}$$

and find the sign of derivative of this function along the solutions of Eq. (1) on the set $\omega(t)$ for each $t \in I$. We obtain

$$\frac{dW(t,x)}{dt} =$$

$$-(c(t)x(t-\tau(t)) + \varphi_1'(t)) \cdot (x - \varphi_2(t)) - (x - \varphi_1(t)) \cdot (c(t)x(t-\tau(t)) + \varphi_2'(t)).$$

For each $\lambda \in \omega$, such that $\lambda(t) = \varphi_1(t)$, $t \in I$, we have

$$\left.\frac{dW(t,x)}{dt}\right|_{x=\lambda} = -(c(t)\lambda(t-\tau(t)) + \varphi_1'(t))(\varphi_1(t) - \varphi_2(t)) >$$

$$> (\delta_2 - \delta_1)c(t)\left(1 - \delta_1 + \delta_1 \int_{t-\tau(t)}^{\infty} c(s)\,ds\right)\int_t^{\infty} c(s)\,ds > 0$$

and for each $\lambda \in \omega$, such that $\lambda(t) = \varphi_2(t)$, $t \in I$, we get

$$\left.\frac{dW(t,x)}{dt}\right|_{x=\lambda} = -(\varphi_2(t) - \varphi_1(t))(c(t)\lambda(t-\tau(t)) + \varphi_2'(t)) >$$

$$> (\delta_2 - \delta_1)c(t)\left(\delta_2 - 1 - \delta_2 \int_{t-\tau(t)}^{\infty} c(s)\,ds\right)\int_t^{\infty} c(s)\,ds > 0.$$

Therefore in both cases, for $t \in I$, the following is true:

$$\left.\frac{dW(t,x)}{dt}\right|_{x=\lambda} > 0.$$

Now, by the topological method of T. Ważewski (see, for instance, [38]) in the adaptation which is suitable for the retarded functional differential equations (given by K. P. Rybakowski [36]), there is a solution of Eq. (1) $x = \tilde{x}(t), t \in I$ such that $\tilde{x}(t) \in \omega(t)$ for each $t \in I$. From the form of the set $\omega(t)$ it follows that $\varphi_1(t) < \tilde{x}(t) < \varphi_2(t)$ on I_{-1} and, moreover, $\lim_{t\to\infty} \tilde{x}(t) = 1$ since $\lim_{t\to\infty} \varphi_1(t) = \lim_{t\to\infty} \varphi_2(t) = 1$. The details of the application of the topological principle are omitted because they can be found e.g. in [8,9,10], [36].

Now, let us suppose that $\int^{\infty} c(t)\,dt = \infty$. If there is a positive solution $x = \tilde{x}(t), t \in I_{-1}$, of Eq. (1) with nonzero limit $\tilde{x}(\infty) = L > 0$, then integration of this equation with limits t_0 and ∞ gives

$$L - \tilde{x}(t_0) = -\int_{t_0}^{\infty} c(s)\tilde{x}(s - \tau(t))\,ds. \tag{12}$$

We obtain a contradiction since the left hand side of (12) is equal to a negative number although the right hand side is equal to $-\infty$. The theorem is proved.

Corollary 14. *As it follows from the proof of Theorem 13 if* (11) *holds then there is a solution $x = x(t)$ of Eq.* (1) *on I_{-1} such that*

$$1 + \delta_1 \int_t^{\infty} c(s)\,ds < x(t) < 1 + \delta_2 \int_t^{\infty} c(s)\,ds,$$

where $t \in I_{-1}, \delta_1, \delta_2 =$ const $, \delta_1 \in (0,1), \delta_2 \in (1/(1-m), \infty)$ and $m = \int_{t_0-r}^{\infty} c(s)\,ds$.

Remark 15. As it follows from Theorem 13, each positive solution of Eq. (1) tends to zero if $\int^{\infty} c(t)\,dt = \infty$.

8 Structure formulas for solutions of Eq. (1)

Theorem 16. *Let us suppose the existence of a positive solution $x = \tilde{x}(t)$ of Eq.* (1) *on I_{-1}. Then every solution $x = x(t)$ of Eq.* (1) *is by a unique way represented either by the formula*

$$x(t) = \tilde{x}(t)(K + \zeta(t)), \tag{13}$$

where $K \in \mathbb{R}$ is a constant, dependent on $x(t)$, and $\zeta(t)$, $\zeta(\infty) = 0$ is a continuous function defined on I_{-1} dependent on $x(t)$, or by the formula

$$x(t) = \tilde{x}(t)(KY(t) + \delta(t)) \tag{14}$$

where $Y(t)$ is a continuous increasing function which is the same for each $x(t)$, $Y(\infty) = \infty$, $K \in \mathbb{R}$ is a constant, dependent on $x(t)$, and $\delta(t)$ is a bounded continuous function defined on I_{-1} dependent on $x(t)$. On the other hand, to each $K \in \mathbb{R}$ there corresponds a solution of $x(t)$ Eq. (1) *and a function of the type $\zeta(t)$ (if in* (13) *$K = 0$, then there is a representation of a solution $x(t)$ with positive function $\zeta(t)$) or of the type $\delta(t)$ such that either formula* (13) *holds or formula* (14) *is valid. Moreover, in this case the representation* (14) *gives a solution of Eq.* (1) *if $\delta(t)$ is shifted by any constant or is equal to any constant.*

Proof. Let us introduce a new variable $y(t)$ by means of formula

$$y(t) = x(t)/\tilde{x}(t)$$

where $x(t)$ is any solution of Eq. (1). Then $y(t)$ satisfies the equation of the type of Eq. (2), i.e. the equation

$$\dot{y}(t) = \frac{c(t)\tilde{x}(t-\tau(t))}{\tilde{x}(t)}\,[y(t) - y(t-\tau(t))]. \tag{15}$$

We can conclude that either there is a positive function $k(t)$ on I_{-1} which satisfies the integral inequality (6) on I if

$$\beta(t) \equiv \frac{c(t)\tilde{x}(t-\tau(t))}{\tilde{x}(t)}$$

or such function does not exist. This means: either the convergence criterion (Theorem 2) holds or the divergence criterion (Theorem 5) is valid. If the first case occurs, then formula (13) immediately follows from Theorem 8 (formula (9)). If we deal with the second possibility, then Theorem 9 is true and the representation (14) follows immediately from formula (10). The theorem is proved.

Remark 17. Let us suppose that Theorem 16 holds. Then there are two linearly independent positive solutions of Eq. (1) $x_1(t), x_2(t)$ on I_{-1}, defined in the case (13) as

$$x_1(t) = \tilde{x}(t), \;\; x_2(t) = \tilde{x}(t)\zeta(t)$$

(the existence of a positive function $\zeta(t)$ follows from Lemma 1) and in the case (14) as

$$x_1(t) = \tilde{x}(t)Y(t), \;\; x_2(t) = \tilde{x}(t).$$

Obviously $\lim_{t\to\infty} x_2(t)/x_1(t) = 0$. Then formula (14) turns into $x(t) = Kx_1(t) + O(x_2(t))$. In the next theorem it is shown that this formula covers both representations (13), (14).

Theorem 18. *Let there be a positive solution $x = \tilde{x}(t)$, $t \in I_{-1}$, of Eq.* (1)*. Then there are two positive solutions $x_1(t)$, $x_2(t)$, $t \in I_{-1}$, of Eq.* (1) *such that $\lim_{t\to\infty} x_2(t)/x_1(t) = 0$. Moreover, every solution $x = x(t)$, $t \in I_{-1}$, of Eq.* (1) *is represented by the formula*

$$x(t) = Kx_1(t) + O(x_2(t)), \;\; t \in I_{-1}, \tag{16}$$

where $K \in \mathbb{R}$ depends on $x(t)$.

Proof. In view of Theorems 9, 16, Lemma 1 and Remark 17 it is sufficient to prove formula (16) if representation (13) holds. Let us introduce a new variable $y(t)$ by means of formula

$$y(t) = x(t)/(\tilde{x}(t)\zeta(t))$$

where $x(t)$ is any solution of Eq. (1) and $\zeta(t) > 0$. Proceeding as above, we conclude that for corresponding equation of the type (15) the structure formula (10) holds. This means

$$y = \tilde{K}\;\tilde{Y}(t) + \tilde{\delta}(t)$$

where the sense of $\tilde{K}, \tilde{Y}(t)$ and $\tilde{\delta}(t)$ is the same as the sense of $K, Y(t)$ and $\delta(t)$ in formula (10). The representation (13) can be written in the form

$$x(t) = \tilde{x}(t)\zeta(t)(\tilde{K}\;\tilde{Y}(t) + \tilde{\delta}(t)).$$

This representation is simultaneously the representation of the type (14) for which the affirmation was proved in Remark 17. The theorem is proved.

Remark 19. For previous results in this direction we refer to the papers by E. Kozakiewicz [28,29,30] and the book of A. D. Myshkis [32]. Note, except this, that (if Theorem 18 holds) any oscillating solution $x = x(t)$ of (1) satisfies relation $x(t) = O(x_2(t))$ and, consequently, tends to zero if $t \to \infty$.

Example 20. Let us consider the equation of the type Eq. (1)

$$\dot{x}(t) = -(1/t)x(t-1). \tag{17}$$

In the papers by J. Diblík [9], [10] it was shown that asymptotic behavior of two linearly independent positive solutions $x_1(t), x_2(t)$ of Eq. (17) is given by relations

$$|x_1(t) - (t-1)^{-1}| < (t-1)^{-2}$$

and

$$\exp[-3(t+1/4)\ln(t+1/4)] < x_2(t) < \exp[-(t/2+1/8)\ln(t-1/4)].$$

Then, by Theorem 18, the representation (16) holds.

Example 21. For the equation of the type Eq. (1)

$$\dot{x}(t) = -(1/e\tau)x(t-\tau) \tag{18}$$

where $\tau = \text{const}$ it is known that there are two asymptotically different positive solutions, namely $x_1(t) = t\exp(-t/\tau)$, $x_2(t) = \exp(-t/\tau)$. In accordance with Theorem 18 the representation (16) holds and each solution is representable in the form

$$x(t) = te^{-t/\tau}\left(K + O\left(1/t\right)\right).$$

9 Existence of positive solution of Eq. (1)

In D. Zhou [42], L. H. Erbe, Q. Kong, B. G. Zhang [16] or J. Diblík [7], [11] some criterions for existence of positive solution of Eq. (1) are given. Let us give one of them which will be used in the sequel.

Theorem 22. (L. H. Erbe, Q. Kong, B. G. Zhang [16], p. 29) *Eq.* (1) *has a positive solution with respect to* t_0 *if and only if there exists a continuous function* $\lambda(t)$ *on* I_{-1} *such that* $\lambda(t) > 0$ *on* I *and*

$$\lambda(t) \ge c(t)e^{\int_{t-\tau(t)}^{t} \lambda(s)\, ds}, \quad t \in I. \tag{19}$$

A very well known sufficient condition, given (under various slightly different assumption for (1) or for modified classes of this equation) by many authors (see, e.g., L. H. Erbe, Q. Kong, B. G. Zhang [16], K. Gopalsamy [17], I. Györi, G. Ladas [18], I. Györi, M. Pituk [19], R. G. Koplatadze, T. A. Chanturija [25], M. Pituk [35]) is a consequence of this criterion:

Corollary 23. (L. H. Erbe, Q. Kong, B. G. Zhang [16], p. 29) *If*

$$\int_{t-\tau(t)}^{t} c(s)\, ds \le 1/e, \quad t \in I \tag{20}$$

then Eq. (1) *has a positive solution with respect to* t_0.

This consequence gives that, in the case $\tau(t) \equiv \text{const}$ for existence of a positive solution with respect to t_0 of Eq. (1), the inequality

$$c(t) \le 1/e\tau, \quad t \in I_{-1} \tag{21}$$

is sufficient. In the next section the case

$$\lim_{t\to\infty} c(t) = \frac{1}{\tau e}$$

is considered.

10 Behaviour of solutions of Eq. (1) in critical case

Y. Domshlak [13], [14] was the first who noticed that among the equations of the form (1) with $\lim_{t\to\infty} c(t) = 1/\tau e$ there exist equations such that all their solutions are oscillatory in spite of the fact that the corresponding limiting equation (18) admits a non-oscillatory solution (see Example 21). This situation is called critical.

Let us give an improvement of the last sufficient condition (21) together with the sufficient condition for oscillation of all solutions of Eq. (1).
Let us denote

$$\ln_p t = \underbrace{\ln\ln\ldots\ln}_{p} t,\ \ p \geq 1,$$

if $t > \exp_{p-2} 1$, where

$$\exp_p t \equiv (\underbrace{\exp(\exp(\ldots\exp}_{p} t))),\ p \geq 1,$$

$\exp_0 t \equiv t$ and $\exp_{-1} t \equiv 0$. Moreover, let us define $\ln_0 t \equiv t$. Instead of expressions $\ln_0 t, \ln_1 t$, we will write only t and $\ln t$ in the sequel. The following holds:

Theorem 24. (J. Diblík [11])
A) Let us assume that $\tau(t) \equiv \tau = \text{const}$,

$$c(t) \leq c_p(t) \tag{22}$$

for $t \to \infty$ *and an integer* $p \geq 0$, *where*

$$c_p(t) \equiv \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \frac{\tau}{8e(t\ln t\ln_2 t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\ldots\ln_p t)^2}\ .$$

Then there is a positive solution $x = x(t)$ *of Eq.* (1). *Moreover,*

$$x(t) < e^{-t/\tau}\sqrt{t\ln t\ln_2 t\ldots\ln_p t}$$

as $t \to \infty$.
B) Let us assume that $\tau(t) \equiv \tau = \text{const}$,

$$c(t) \geq c_{p-1}(t) + \frac{\theta\tau}{8e(t\ln t\ln_2 t\ldots\ln_p t)^2} \tag{23}$$

for $t \to \infty$, *an integer* $p \geq 1$ *and a constant* $\theta > 1$. *Then all solutions of Eq.* (1) *oscillate.*

The proof of the part A) of this theorem can be made with the aid of Theorem 22. Indeed, it is easy to see that the inequality (19), where $c(t) \equiv c_p(t)$, has (for sufficiently large t) a solution

$$\lambda(t) = \frac{1}{\tau} - \frac{1}{2t} - \frac{1}{2t\ln t} - \frac{1}{2t\ln t\ln_2 t} - \cdots - \frac{1}{2t\ln t\ln_2 t\ldots\ln_p t}\ .$$

In process of verification it is necessary to find an asymptotic representation of the right hand side of inequality (19). After this, as usual, we compare the coefficients of identical functional terms on left hand side and on right hand side. The following equalities for determination of coefficients of the functional terms indicated below are valid:

$$
\begin{aligned}
1 \quad &: \qquad 1/\tau = 1/\tau,\\
1/(t\ln t\ldots\ln_j t),\ 0\le j\le p \quad &: \qquad -1/2 = -1/2,\\
1/(t\ln t\ldots\ln_j t)^2,\ 0\le j\le p \quad &: \qquad 0 = \tau/8-\tau/8,\\
1/[(t\ln t\ldots\ln_s t)^2(\ln_{s+1} t\ldots\ln_j t)],\ 0<s<j<p \quad &: \qquad 0 = \tau/8-\tau/8\,.
\end{aligned}
$$

For the next asymptotic smaller terms we have

$$0 \ge -\frac{\tau^2}{16t^3} - \frac{\tau^2}{16t^3} + o\left(\frac{1}{t^3}\right) = -\frac{\tau^2}{8t^3} + o\left(\frac{1}{t^3}\right).$$

This inequality holds for $t \to \infty$. The verification is ended.

In the paper by J. Diblík [11] this part is proved by another equivalent way.

The proof of the part B) is made in cited paper by using the method of Y. Domshlak. In this part, Theorem 24 generalizes Theorem 3 of the recent paper by Y. Domshlak and I. P. Stavroulakis [14]).

Remark 25. The behaviour of solutions in the critical case was investigated by many authors. For example, the papers (except the above mentioned ones) by Li Bingtuan [3], [4], by Á. Elbert and I. P. Stavroulakis [15], by J. Jaroš and I. P. Stavroulakis [23], by E. Kozakiewicz [26], [27] and by J. Werbowski [39] are devoted to this case. We refer to these papers for further bibliography (and history) concerning this question.

Problem 26. An analogy of Theorem 24 is not yet given if inequalities (22), (23) are substituted by inequalities (or by a slightly modified inequalities) obtained from (22), (23) by integrating with limits $t-\tau$ and t, i.e. an analogy is not given if corresponding inequalities are given in terms of the integral average $\int_{t-\tau}^{t} c(s)\,ds$ of the function $c(t)$ instead in terms of values of the function $c(t)$ itself. The first step in this direction is inequality (20). This can serve as a motivation for further investigations in this direction. (As far as this question in the oscillation case is concerned, we refer to the paper [14].) See this situation with an analogous one in Corollary 4.

Remark 27. Let us observe that if inequality (23) holds, then integral inequality (19) has not a positive solution, satisfying conditions indicated in Theorem 22. Note, moreover, that in the papers by F. Neuman (e.g. [33], [34]) a theoretical possibility is given for transformation of an equation with *variable* delays to an equation of the same class with *constant* delays. This perhaps can serve as a possibility of generalization of Theorem 24 if the delay is not constant.

11 Comparison with behaviour of solutions of Eq. (3) in critical case

Let us define functions

$$\mu_p(t) \equiv t \ln t \ln_2 t \dots \ln_p t,$$

$$a_p(t) \equiv \frac{1}{4}\left(\frac{1}{t^2} + \frac{1}{(t\ln t)^2} + \dots + \frac{1}{(t\ln t \dots \ln_{p-1} t)^2} + \frac{1+A}{(t \ln t \dots \ln_p t)^2}\right),$$

where $p \geq 0$, $A \in \mathbb{R}$ and t is sufficiently large.

Lemma 28. *The equation of the type of* (3)

$$x''(t) + a_p(t)x(t) = 0, \ p \geq 0, \tag{24}$$

has following linearly independent solutions:
A)

$$x_1(t) = \sqrt{\mu_p(t)} \sin\left(\frac{a}{2}\ln_{p+1} t\right), \ \ x_2(t) = \sqrt{\mu_p(t)} \cos\left(\frac{a}{2}\ln_{p+1} t\right),$$

if $A = a^2, a > 0, p \geq 0$;
B)

$$x_1(t) = \sqrt{\mu_p(t)}\,, \ \ x_2(t) = \sqrt{\mu_p(t)}\,\ln_{p+1} t,$$

if $A = 0, p \geq 0$;
C)

$$x_1(t) = \sqrt{\mu_{p-1}(t)}\,(\ln_p t)^{\lambda_1}\,, \ \ x_2(t) = \sqrt{\mu_{p-1}(t)}\,(\ln_p t)^{\lambda_2} \ \text{ for } \ p \geq 1,$$

and

$$x_1(t) = t^{\lambda_1}, \ \ x_2(t) = t^{\lambda_2} \ \text{ for } \ p = 0$$

if $A < 0$ *and* λ_1, λ_2 *are roots of the quadratic equation*

$$\lambda^2 - \lambda + (1 + A)/4 = 0, \ \text{ i.e. } \ \lambda_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{-A}\right).$$

Proof. It is easy to verify this affirmation by means of substitution of the expressions $x_1(t), x_2(t)$ into Eq. (24).

Let us formulate the known result concerning oscillatory and nonoscillatory properties of all solutions of Eq. (3) which can be proved by standard arguments with the aid of Lemma 28 and Sturmian Comparison Method (see e.g. [21]).

Theorem 29. *Let* $a \in C(I, \mathbb{R}^+)$. *All solutions of Eq.* (3) *oscillate on* I *if* $a(t) \geq a_p(t)$, $t \in I$ *for some integer* $p \geq 0$ *and* $A > 0$. *If* $a(t) \leq a_p(t)$, $t \in I$ *for some* $p \geq 0$ *and for* $A = 0$ *then Eq.* (3) *is nonoscillatory on* I.

Remark 30. Theorem 24 is an analogy of Theorem 29 since there is a parallel between oscillatory and nonoscillatory properties of solutions of Eq. (1) and Eq. (3). Previous analogues in the case of equations with delay (for $p = 0$ and for $p = 1$) with classical Kneser's theorem [24], [37] and with result due to Hille [22], [37] were given in the cited paper by Y. Domshlak and I. P. Stavroulakis [14]. Note, except this, that conditions concerning functions $a(t)$ and $c(t)$ are very similar. Comparison functions $a_p(t)$ and $c_p(t)$ consist of the same functional terms and differ only in their multipliers and in additive constant.

Remark 31. Some close problems for similar classes of equations and systems of equations (with respect to Eq. (1) and Eq. (2)) are considered e.g. by O. Arino, M. Pituk [1], by J. Čermák [5], by T. Krisztin [31] and for equations with impulses by A. Domoshnitsky, M. Drakhlin [12] and by Yu Jiang, Yan Jurang [40].

Acknowledgment

This research has been supported by the Grant Agency of the Czech Republic under Grant No 201/96/0410.

References

1. O. Arino, M. Pituk, Convergence in asymptotically autonomous functional differential equations, University of Veszprém, Department of Mathematics and Computing, preprint No. 065 (1997).
2. F. V. Atkinson, J. R. Haddock, Criteria for asymptotic constancy of solutions of functional differential equations, *J. Math. Anal. Appl.* **91** (1983), 410-423.
3. Li Bingtuan, Oscillation of first order delay differential equations, *Proc. Amer. Math. Soc.* **124** (1996), 3729-3737.
4. Li Bingtuan, Oscillations of delay differential equations with variable coefficients, *J. Math. Anal. Appl.* **192** (1995), 312-321.
5. J. Čermák, On the asymptotic behaviour of solutions of certain functional differential equations, to appear in *Math. Slovaca.*
6. J. Diblík, A criterion for convergence of solutions of homogeneous delay linear differential equations, submitted.
7. J. Diblík, A criterion for existence of positive solutions of systems of retarded functional differential equations, to appear in *Nonlin. Anal., T. M. A.*
8. J. Diblík, Asymptotic representation of solutions of equation $\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))]$, *J. Math. Anal. Appl.*, **217** (1998), 200–215.
9. J. Diblík, Existence of solutions with prescribed asymptotic for certain systems retarded functional differential equations, *Siberian Mathematical Journal*, **32** (1991), (No 2), 222–226.
10. J. Diblík, Existence of decreasing positive solutions of a system of linear differential equations with delay, *Proceedings of the Fifth International Colloquium on Differential Equations,* VSP, Eds.: D. Bainov and V. Covachov, 83–94, 1995.
11. J. Diblík, Positive and oscillating solutions of differential equations with delay in critical case, *J. Comput. Appl. Math.*, **88** (1998), 185–202.

12. A. Domoshnitsky, M. Drakhlin, Nonoscillation of first order differential equations with delay, *J. Math. Anal. Appl.,* **206** (1997), 254–269.
13. Y. Domshlak, *Sturmian Comparison Method in Investigation of the Behavior of Solutions for Differential-Operator Equations,* ELM, Baku, USSR, 1986. (In Russian)
14. Y. Domshlak, I. P. Stavroulakis, Oscillation of first-order delay differential equations in a critical case, *Applicable Analysis,* **61** (1996), 359–371.
15. Á. Elbert, I. P. Stavroulakis, Oscillation and non-oscillation criteria for delay differential equations, *Proc. Amer. Math. Soc.,* **123** (1995), 1503–1510.
16. L. H. Erbe, Q. Kong, B. G. Zhang, *Oscillation Theory for Functional Differential Equations,* Marcel Dekker, Inc., 1995.
17. K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics,* Kluwer Academic Publishers, 1992.
18. I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations,* Clarendon Press, 1991.
19. I. Györi, M. Pituk, Comparison theorems and asymptotic equilibrium for delay differential and difference equations, *Dynamic Systems and Application,* **5** (1996), 277–302.
20. J. K. Hale, S. M. V. Lunel, *Introduction to Functional Differential Equations,* Springer-Verlag, 1993.
21. P. Hartman, *Ordinary Differential Equations,* John Wiley & Sons, 1964.
22. E. Hille, Non-oscillation theorems, *Trans. Amer. Math. Soc.,* **64** (1948), 232–252.
23. J. Jaroš, I. P. Stavroulakis, Oscillation tests for delay equations, *Technical Report No 265,* University of Ioannina, Department of Mathematics, June 1996.
24. A. Kneser, Untersuchungen über die reelen Nullstellen der Integrale linearen Differentialgleichungen, *Math. Ann.,* **42** (1893), 409–503; *J. Reine Angew. Math.,* **116** (1896), 178–212.
25. R. G. Koplatadze, T. A. Chanturija, On the oscillatory and monotonic solutions of first order differential equations with deviating arguments, *Differencial'nyje Uravnenija,* **18** (1982), 1463–1465. (In Russian)
26. E. Kozakiewicz, Conditions for the absence of positive solutions of a first order differential inequality with a single delay, *Archivum Mathematicum* (Brno), **31** (1995), 291–297.
27. E. Kozakiewicz, Conditions for the absence of positive solutions of a first order differential inequality with one continuously retarded argument, *Berlin, preprint,* (1997), 1–6.
28. E. Kozakiewicz, Über das asymptotische Verhalten der nichtschwingenden Lösungen einer linearen Differentialgleichung mit nacheilendem Argument, *Wiss. Z. Humboldt Univ. Berlin, Math. Nat. R.,* **13**:4 (1964), 577–589.
29. E. Kozakiewicz, Zur Abschätzung des Abklingens der nichtschwingenden Lösungen einer linearen Differentialgleichung mit nacheilendem Argument, *Wiss. Z. Humboldt Univ. Berlin, Math. Nat. R.,* **15**:5 (1966), 675–676.
30. E. Kozakiewicz, Über die nichtschwingenden Lösungen einer linearen Differentialgleichung mit nacheilendem Argument, *Math. Nachr.,* **32**: 1/2 (1966), 107–113.
31. T. Krisztin, A note on the convergence of the solutions of a linear functional differential equation, *J. Math. Anal. Appl.,* **145** (1990), 17–25, (1990).
32. A. D. Myshkis, Linear Differential Equations with Retarded Arguments, (2^{nd} Ed.), *Nauka,* 1972. [In Russian]
33. F. Neuman, On equivalence of linear functional-differential equations, *Results in Mathematics,* **26** (1994), 354–359.

34. F. Neuman, On transformations of differential equations and systems with deviating argument, *Czechoslovak Mathematical Journal,* **31** (106) (1981), 87–90.
35. M. Pituk, Asymptotic characterization of solutions of functional differential equations, *Bolletino U. M. I.,* **7**, 7-B, (1993), 653–683.
36. K. P. Rybakowski, Ważewski's principle for retarded functional differential equations, *Journal of Differential Equations,* **36** (1980), 117–138.
37. C. A. Swanson, Comparison and Oscillating Theory of Linear Differential Equations, *Academic Press,* 1968.
38. T. Ważewski, Sur un principle topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles, *Ann. Soc. Polon. Math.,* **20** (1947), 279–313.
39. J. Werbowski, Oscillations of first order linear differential equations with delay, *Proceedings of the Conference on Ordinary Differential Equations, Poprad* (*Slovak Republic*), 87–94, 1996.
40. Yu Jiang, Yan Jurang, Positive solutions and asymptotic behavior of delay differential equations with nonlinear impulses, *J. Math. Anal. Appl.,* **207** (1997), 388–396.
41. S. N. Zhang, Asymptotic behaviour and structure of solutions for equation $\dot{x}(t) = p(t)[x(t) - x(t-1)]$, *J. Anhui University (Natural Science Edition),* **2** (1981), 11–21. (In Chinese)
42. D. Zhou, On a problem of I. Györi, *J. Math. Anal. Appl.,* **183** (1994), 620–623.