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A New Approach to the Existence of Almost Everywhere Solutions of Nonlinear PDEs

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Abstract. We discuss the existence of almost everywhere solutions of nonlinear PDE's of first (in the scalar and vectorial cases) and second order.

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1 Introduction

This article presents some recent results obtained jointly with P. Marcellini (see [10], [11] and [12]). We propose a new approach for existence of almost everywhere solutions of nonlinear partial differential equations of the first and second order. This approach does not use the notion of viscosity solution since it is mainly intended for handling vectorial problems of non elliptic type. We also give an example (c.f. Theorem 3 and for more general results see [3]) where our method contrasts with the viscosity approach.

Our results establish only existence of solutions; it remains open, in general, to find a criterion of selection among the many solutions which are provided by our existence theorems. Of course when a Lipschitz viscosity solution exists and is unique, then this is, in general, the best criterion.

Our original motivation to study such problems comes from the calculus of variations and its applications to nonlinear elasticity and optimal design (see [9]).

2 First order PDE, the scalar case

Consider the Dirichlet problem

$$\begin{cases} F\left(Du\left(x\right)\right)=0, \text{ a.e. } x\in\Omega \\ u\left(x\right)=\varphi\left(x\right), \, x\in\partial\Omega \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded (or unbounded) open set, $F:\mathbb{R}^n\to\mathbb{R}$ and $\varphi\in W^{1,\infty}\left(\Omega\right)$. We then have

Theorem 1 (c.f. [10]). *Let* $E=\{\xi\in\mathbb{R}^n:F\left(\xi\right)=0\}$, *if*

$$D\varphi\left(x\right) \textit{ is compactly contained in } intcoE, \textit{ a.e. in } \Omega \tag{2}$$

where $intcoE$ *stands for the interior of the convex hull of* E, *then there exists (a dense set of)* $u\in W^{1,\infty}\left(\Omega\right)$ *that satisfies* (1). *If in addition* $\varphi\in C^1\left(\Omega\right)$ *and if* E *is closed then* (2) *can be replaced by*

$$D\varphi\left(x\right)\in E\cup intcoE \textit{ in } \Omega. \tag{3}$$

Remark 2. (i) One should note that no hypotheses of convexity or coercivity on F are made. The condition is close to the necessary condition which, in some sense, is

$$D\varphi\left(x\right)\in coE \text{ in } \Omega.$$

(ii) The condition (3) excludes, as it should do, the linear case since then $intcoE=\emptyset$.

(iii) The above theorem can be generalized to the case where $F=F(x,u,Du)$, c.f. [11], c.f. also [1] and [15].

(iv) It is interesting to compare the above result with the classical hypotheses (c.f. [17], [7], [18]) ensuring existence of Lipschitz viscosity solution to (1) i.e. F is convex, coercive ($\lim F\left(\xi\right)=+\infty$ if $|\xi|\to\infty$) then

$$E\cup intcoE=\{\xi\in\mathbb{R}^n:F\left(\xi\right)\leq 0\}$$

and we recover the usual compatibility condition $F\left(D\varphi\right)\leq 0$.

Proof. We very roughly outline the idea of the proof in the classical case i.e. when F is convex, coercive and $F\left(D\varphi\right)\leq 0$. We set

$$V=\left\{u\in\varphi+W_0^{1,\infty}\left(\Omega\right):F\left(Du\right)\leq 0\right\}.$$

Then $\varphi\in V$ and when endowed with the C^0 metric it becomes a complete metric space (this results from the convexity and coercivity of F). We then define

$$V^k=\left\{u\in V:\int_\Omega F\left(Du\right)>-\frac{1}{k}\right\}.$$

Then V^k is open and dense in V, the first property follows from the convexity of F while the second one is more difficult and is some kind of relaxation theorem used in the calculus of variations.

We then use Baire category theorem which ensures that

$$\bigcap V^k = \left\{ u \in \varphi + W_0^{1,\infty}(\Omega) : F(Du) = 0 \right\}$$

is dense (and hence non empty) in V. This achieves the outline of the proof.

The idea to use Baire theorem for Cauchy problem for ordinary differential inclusion is due to Cellina [5], c.f. also [14]. □

A natural question is then to ask if under the general assumption of the theorem one can always find among the many solutions a viscosity one (when F is convex and coercive this is the case). The answer is in general negative unless strong *geometric restrictions* are assumed. A necessary and sufficient condition is given in [3]. We give below such a result only in a particular example which sheds some light on the nature of these *geometric restrictions*. We will denote for $u = u(x,y)$ its partial derivatives by u_x, u_y.

Theorem 3 ([3]). *Let $\Omega \subset \mathbb{R}^2$ be convex. Then*

$$\begin{cases} F(Du) = \left(u_x^2 - 1\right)^2 + \left(u_y^2 - 1\right)^2 = 0, \ \text{a.e. in } \Omega \\ u = 0, \ \text{on } \partial\Omega \end{cases} \tag{4}$$

has a $W^{1,\infty}$ viscosity solution if and only if Ω is a rectangle whose faces are orthogonal to the vectors $(1,1)$ and $(1,-1)$.

Remark 4. Note that by Theorem 1 the problem (4) has a $W^{1,\infty}$ solution since

$$0 \in intcoE = \left\{ \xi \in \mathbb{R}^2 : |\xi_1|, |\xi_2| < 1 \right\}.$$

3 First order PDE, the vectorial case

We now want to discuss the analogue of Theorem 1 in the vectorial case. The problem is then

$$\begin{cases} F_1(Du) = \cdots = F_N(Du) = 0, \ \text{a.e. in } \Omega \\ u = \varphi, \ \text{on } \partial\Omega \end{cases} \tag{5}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $n, m > 1$, and $F_i : \mathbb{R}^{m \times n} \to \mathbb{R}$, $i = 1, ..., N$.

We then let

$$E = \left\{ \xi \in \mathbb{R}^{m \times n} : F_i(\xi) = 0, i = 1, ..., N \right\}.$$

A natural conjecture (c.f. [11]) is then

Conjecture 5. *The system* (5) *has a* $W^{1,\infty}$ *solution provided* $\varphi \in C^1\left(\overline{\Omega};\mathbb{R}^m\right)$ *is such that*

$$D\varphi(x) \in E \cup intQcoE,\ \text{ in } \Omega$$

where $QcoE$ *denotes the quasiconvex (in the sense of Morrey) hull of* E.

This conjecture is a theorem under some extra technical conditions which are discussed in [11]. In the scalar case the notions of convexity and quasiconvexity are equivalent, therefore $QcoE = coE$. As in the scalar case the conjecture is close to the necessary condition which is, in some sense,

$$D\varphi(x) \in QcoE,\ \text{ in } \Omega.$$

These types of problems are important in the calculus of variations (see [9]) and in nonlinear elasticity (phase transitions, problem of potential wells, c.f. also in this case [20]) or in optimal design.

We now give one typical case that can be handled by our method (c.f. [11] and [13], c.f. also [4]).

Let $\xi \in \mathbb{R}^{n\times n}$ and denote by $0 \le \lambda_1(\xi) \le ... \le \lambda_n(\xi)$ the singular values of the matrix ξ (i.e. the eigenvalues of $(\xi^t\xi)^{1/2}$). This implies in particular that

$$|\xi|^2 = \sum_{i,j=1}^{n}\xi_{ij}^2 = \sum_{i=1}^{n}(\lambda_i(\xi))^2,\ |\det\xi| = \prod_{i=1}^{n}\lambda_i(\xi).$$

Theorem 6. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set,* $a_i : \overline{\Omega}\times\mathbb{R}^n \to \mathbb{R}$, $i = 1,...,n$ *be continuous functions satisfying*

$$0 < c \le a_1(x,s) \le ... \le a_n(x,s)$$

for some constant c *and for every* $(x,s) \in \overline{\Omega}\times\mathbb{R}^n$. *Let* $\varphi \in C^1\left(\overline{\Omega};\mathbb{R}^n\right)$ *satisfy*

$$\prod_{i=\nu}^{n}\lambda_i(D\varphi(x)) < \prod_{i=\nu}^{n}a_i(x,\varphi(x)),\ x\in\Omega,\ \nu = 1,...,n \tag{6}$$

(in particular $\varphi \equiv 0$*), then there exists (a dense set of)* $u \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ *such that*

$$\begin{cases} \lambda_i(Du(x)) = a_i(x,u(x)),\ a.e.\ x\in\Omega,\ i=1,...,n \\ u(x) = \varphi(x),\ x\in\partial\Omega. \end{cases} \tag{7}$$

Remark 7. If $a_i \equiv 1$, for every $i = 1,...,n$, then (6) becomes

$$\lambda_n(D\varphi(x)) < 1,\ x\in\Omega.$$

The problem (7) can then equivalently be rewritten as

$$Du(x) \in O(n),\ \text{a.e. in } \Omega.$$

The case $n = 3$, $a_i \equiv 1$ and $\varphi \equiv 0$ has also been studied in [6].

4 Second order case

Since second order equations can be rewritten as first order systems, this section seems to fall in the preceding one; however some of the equations are then linear and hence this corresponds to the case where

$$intQcoE = \emptyset.$$

We here present two types of results of more general ones, see [12].

The first one deals with one single equation. For this purpose we introduce the following notations and terminology

$$\mathbb{R}_s^{n\times n} = \left\{\xi \in \mathbb{R}^{n\times n} : \xi = \xi^t\right\}.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set, $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_s^{n\times n} \to \mathbb{R}$, $F = F(x, s, p, \xi)$, we say that F is *coercive* with respect to the last variable ξ *in the rank one direction* λ, if $\lambda \in \mathbb{R}_s^{n\times n}$ with $rank\{\lambda\} = 1$, and for every bounded set of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_s^{n\times n}$ there exist constants $m, q > 0$, such that

$$F(x, s, p, \xi + t\lambda) \geq m|t| - q$$

for every $t \in \mathbb{R}$ and every (x, s, p, ξ) that vary in the bounded set of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_s^{n\times n}$. Examples of such functions are

$$F(\xi) = |\xi|^2 - 1 = \sum_{i,j=1}^{n} \left(\xi_{ij}^2\right) - 1 \text{ or } F(\xi) = |\text{trace } \xi| - 1.$$

Theorem 8. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_s^{n\times n} \to \mathbb{R}$ be a continuous function, convex with respect to the last variable and coercive in a rank one direction λ. Let $\varphi \in C^2(\mathbb{R}^n)$ satisfy*

$$F\left(x, \varphi(x), D\varphi(x), D^2\varphi(x)\right) \leq 0,\ x \in \overline{\Omega}. \tag{8}$$

Then there exists (a dense set of) $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} F\left(x, u(x), Du(x), D^2u(x)\right) = 0,\ a.e.\ x \in \Omega \\ u(x) = \varphi(x),\ Du(x) = D\varphi(x),\ x \in \partial\Omega. \end{cases}$$

Remark 9. (i) The theorem remains valid if convexity is replaced by quasiconvexity in the sense of Morrey (for this notion see [19] or [8]).

(ii) The coercivity condition in a rank one direction excludes from our analysis linear equations as well as the so called *fully non linear elliptic equations* (in the sense of [2], [7], [16] or [21]).

(iii) Note that if u and φ are smooth functions and $\partial\Omega$ is smooth, then to write $u = \varphi$, $Du = D\varphi$, on $\partial\Omega$ is equivalent as simultaneously prescribing the normal and tangential derivatives. Therefore the boundary conditions are at the same time of Dirichlet and Neumann type.

Examples of applications of this result are

Example 10. (i) The following Dirichlet-Neumann problem admits a $W^{2,\infty}$ solution

$$\begin{cases} |\Delta u| = a\left(x, u\left(x\right), Du\left(x\right)\right), \text{ a.e. in } \Omega \\ u = \varphi,\ Du = D\varphi, \text{ on } \partial\Omega \end{cases}$$

provided the compatibility condition is satisfied, namely

$$|\Delta\varphi| \leq a\left(x, \varphi\left(x\right), D\varphi\left(x\right)\right).$$

(ii) Similarly the problem

$$\begin{cases} \left|D^2 u\right| = a\left(x, u\left(x\right), Du\left(x\right)\right), \text{ a.e. in } \Omega \\ u = \varphi,\ Du = D\varphi, \text{ on } \partial\Omega \end{cases}$$

has a $W^{2,\infty}$ solution provided

$$\left|D^2\varphi\right| \leq a\left(x, \varphi\left(x\right), D\varphi\left(x\right)\right).$$

Similar results can be established for systems of equations (c.f. [12]). We only quote here the following second order version of Theorem 6 that we get by our method.

Theorem 11. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $\varphi \in C^2\left(\mathbb{R}^n\right)$ satisfy*

$$\lambda_n\left(D^2\varphi\left(x\right)\right) < 1,\ x \in \Omega \tag{9}$$

(in particular $\varphi \equiv 0$), then there exists (a dense set of) $u \in W^{2,\infty}\left(\Omega\right)$ such that

$$\begin{cases} \lambda_i\left(D^2 u\left(x\right)\right) = 1,\ a.e.\ x \in \Omega,\ i = 1, ..., n \\ u\left(x\right) = \varphi\left(x\right),\ Du\left(x\right) = D\varphi\left(x\right),\ x \in \partial\Omega. \end{cases} \tag{10}$$

Remark 12. (i) Observe that since in this theorem the matrices are symmetric then the singular values are the absolute values of the eigenvalues of the matrices.

(ii) Note that as a consequence of the above theorem we have that if (9) holds, then the following Dirichlet-Neumann problem admits a solution

$$\begin{cases} \left|\det D^2 u\right| = \prod_{i=1}^{n} \lambda_i\left(D^2 u\right) = 1, \text{ a.e. in } \Omega \\ u = \varphi,\ Du = D\varphi, \text{ on } \partial\Omega. \end{cases}$$

Observe that because of the Dirichlet-Neumann boundary data the above problem cannot be handled as a corollary of the results on Monge-Ampère equation.

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