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On Existence of Oscillatory Solutions of nth Order Differential Equations with Quasiderivatives

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Abstract. Sufficient conditions are given under which the nonlinear n-th order differential equation with quasiderivatives has oscillatory solutions.

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1 Introduction

Consider a nonlinear differential equation

$$y^{[n]} = f(t, y^{[0]}, \dots, y^{[n-1]}) \qquad \text{in} \quad D, \tag{1}$$

where $n \geq 3$, $R_+ = [0, \infty)$, $R = (-\infty, \infty)$, $D = R_+ \times R^n$, $y^{[i]}$ is the ith quasiderivative of y defined by

$$y^{[0]} = y,\ y^{[i]} = \frac{1}{a_i(t)} \left(y^{[i-1]} \right)',\ i = 1, 2, \dots, n-1,\ y^{[n]} = \left(y^{[n-1]} \right)', \tag{2}$$

the functions $a_i : R_+ \to (0, \infty)$ are continuous, $f : D \to R$ fulfills the local Carathéodory conditions and

$$f(t, x_1, \dots, x_n) x_1 \leq 0, \quad f(t, 0, x_2, \dots, x_n) = 0 \quad \text{in} \quad D\,. \tag{3}$$

Let $y : [0, b) \to R$, $b \le \infty$ be continuous, have the quasi-derivatives up to the order $n-1$ and let $y^{[n-1]}$ be absolutely continuous. Then y is called a *solution* of (1) if (1) is valid for almost all $t \in [0, b)$ and either $b = \infty$ or $b < \infty$ and $\limsup_{t \to b_-} \sum_{i=0}^{n-1} |y^{[i]}(t)| = \infty$. It is called *proper* if $b = \infty$ and $\sup_{\tau \le t < \infty} |y(t)| > 0$ holds for an arbitrary number $\tau \in R_+$. A proper solution is called *oscillatory* if there exists a sequence of its zeros tending to ∞.

Notation 1. Let $t_0 \in R_+, a_n, b \in C^0(R_+)$. Put

$$a_{n+i}(t) = a_i(t), i \in \{1, \dots, n - 1\}, I_0(t, t_0; a_s, b) \equiv 1,$$

$$I_k(t, t_0; a_s, b) = \int_{t_0}^{t} a_s(\tau_s) \int_{t_0}^{\tau_s} a_{s+1}(\tau_{s+1}) \cdots \int_{t_0}^{\tau_{s+k-3}} a_{s+k-2}(\tau_{s+k-2}) \times \\ \times \int_{t_0}^{\tau_{s+k-2}} b(\tau_{s+k-1}) \quad d_{\tau_{s+k-1}} \dots d_{\tau_s},$$

$$J(t, t_0; a_s) = \int_{t_0}^{t} a_s(\tau_s) \int_{\tau_s}^{\infty} a_{s+1}(\tau_{s+1}) I_{n-2}(\tau_{s+1}, \tau_s; a_{s+2}, a_{n+s-1}) d\tau_{s+1} d\tau_s.$$

We will assume the following hypotheses (not all simultaneously):

(H1): Let $\frac{a_1}{a_2} \in C^1(R_+)$ for $n = 3$; let $a_2 \in C^1(R_+)$, $a_j \in C^2(R_+)$, $j = 1, 3$ for $n = 4$; let an index $l \in \{1, 2, \dots, n-4\}$ exist such that $a'_{l+j} \in L_{\text{loc}}(R_+)$, $j = 1, 2$ are locally bounded from bellow a.e. on R_+ for $n > 4$.

(H2): Let $b \in L_{loc}(R_+)$ and $g \in C_0(R_+)$ exist such that $g(x) > 0$ for $x > 0$, $\int_1^{\infty} \frac{dt}{g(t)} = \infty$ and

$$|f(t, x_1, \dots, x_n)| \le b(t) g\Big(\sum_{i=1}^{n} |x_i|\Big) \quad \text{on} \quad D\,.$$

(H3): Let constants $\bar{t} \in R_+$, $K \ge 0, 0 \le \lambda \le 1$ and functions $a_n \in L_{loc}(R_+)$ and $g \in C^0(R_+)$ exist such that $a_n \ge 0$, $g(x) > 0$ for $x > 0$, $g(x) = x^{\lambda}$ for $x \ge K$,

$$a_n(t) g(|\, x_1 \,|) \le |\, f(t, x_1, \dots, x_n) \,| \quad \text{on} \quad R_+ \times R^n, \tag{4}$$

$$\int_0^{\infty} a_1(t) dt = \infty, \tag{5}$$

and

$$I_{n-s}(\infty, \bar{t}; a_{s+1}, d_s) = \infty, \quad s = 1, 2, \dots, n-1, \tag{6}$$

where $\quad d_s(t) = a_n(t)\ [I_s(t, \bar{t}; a_1, a_s)]^{\lambda}$.
Further, let in case $\lambda = 1$ for $s = 1, 2, \ldots, n-1$ either

$$\liminf_{t\to\infty}\ e^{-J(t,\bar{t};a_s)} \int_{\bar{t}}^{t} a_s(\tau) e^{-I_n(\tau,\bar{t};a_{s+1},a_s)} d\tau = 0 \tag{7}$$

or

$$I_{n-1}(\infty, \bar{t}; a_{s+1}, a_{n+s-1}) = \infty \tag{8}$$

hold.

(H4): Let the hypothesis (H3) holds with $K = 0, \lambda \in [0, 1)$ and with the exception of (5) and let, moreover,

$$I_n(\infty, 0; a_1, a_n) = \infty. \tag{9}$$

A great effort has been devoted to the study of oscillatory solutions of Eq. (1) in the canonical form, i.e if

$$\int_0^{\infty} a_i(t)dt = \infty, \quad i = 1, 2, \ldots, n-1. \tag{10}$$

Definition 2. Eq. (1) is said to have Property A if every proper solution y is oscillatory for n even, and it is either oscillatory, or

$$\lim_{t\to\infty} y^{[i]}(t) = 0, \quad i = 0, \ldots, n-1$$

holds eventually on R_+ if n is odd.

Chanturia [5] proved the following theorem.

Theorem A ([5]). *Let* $f(t, t_1, \ldots, x_n) \equiv \bar{f}(t, x_1), \bar{f} \in C(R_+ \times R)$, (1) *have Property A. Let* (10) *and*

$$|\bar{f}(t, x_1)| \le b(t)|x_1| \qquad \text{on} \quad R_+ \times R$$

be valid where $b \in C^0(R_+)$. *Then* (1) *has an oscillatory solution.*

Sufficient conditions, under the validity of which, (1) has Property A were studied e.g. in [5], [7]. Generalizations of Th. A are stated in [3] and in [6] (for $n = 3$). Apart from other things

$$\int_0^{\infty} a_1(t)dt = \int_0^{\infty} a_2(t)dt = \infty \tag{11}$$

is supposed instead of (10).

In some applications of Eq. (1) the conditions (10) and (11) are not fulfilled. Although every Eq. (1) can be transformed into the canonical form by sequence of transformations preserving oscillations (see [8]) it is difficult to realize them. E.g. consider the third order differential equation

$$y''' + q(t)y' + r(t)g(y) = 0, \tag{12}$$

where $q \in C^0\ (R+)$, $\quad r \in L_{loc}\ (R_+)$, $\quad g \in C^0\ (R)$, $\quad r \le 0$ on $\quad R_+$, $g(x)x > 0 \quad$ for $x \ne 0$.

Let h be a positive solution on $[T, \infty), T \in R_+$ of the equation

$$h'' + q(t)h = 0 \tag{13}$$

Then (12) is equivalent with (see [4])

$$\left(h^2\left(\frac{1}{h}\ y'\right)'\right)' + rhg(y) = 0 \tag{14}$$

on $[T, \infty)$, where

$$y^{[1]} = \frac{y'}{h}, \qquad y^{[2]} = h^2\left(y^{[1]}\right)'.$$

If we define $h(t) \equiv h(T)$ on $[0, T]$, then (14) is defined on $R_+ \times R^3$ and it has the form (1) with

$$a_1 = h, \qquad a_2 = \frac{1}{h^2}, \quad f(t, x_1, x_2, x_3) \equiv -r(t)h(t)g(x_1) \tag{15}$$

and (3) holds.
If e.g. $q(t) \le const. < 0$, then it is clear that (10) and (11) for $n = 3$ are not valid.

Our main goal is to prove the existence of oscillatory solutions of (1) without the validity of either (10) or (11) and to apply the results to Eq. (12).

2 Main results

In this section, a special set of oscillatory solutions will be investigated. Consider the Cauchy initial conditions:

$$\begin{aligned} & l \in \{0, 1, \ldots, n-1\}, \quad \sigma \in \{-1, 1\}, \\ \sigma\, y^{[i]}(0) &> 0 \quad \text{for } i = 0, 1, \ldots, l-1, \\ &\le 0 \quad \text{for } i = l, \\ &> 0 \quad \text{for } i = l+1, \ldots, n-1. \end{aligned} \tag{16}$$

We will show that a solution y of (1), fulfilling (16) is oscillatory under some assumptions posed on f and a_i.

Theorem 3. *Let* (H1) *and* (H2) *be valid. Then every solution y of* (1) *satisfying* (16) *is proper.*

Proof. See [2, Lemmas 4 and 9]. □

Theorem 4. *Let* (H3) *be valid. Then every proper solution* y *of* (1) *satisfying* (16) *is oscillatory.*

Proof. It follows from [2, Lemma 2] that every proper solution y satisfying (16) is either oscillatory or nonoscillatory, $s \in \{0, 1, \dots, n-1\}$ and T exists such that $T \geq \max(\bar{t}, 1)$,

$$\begin{aligned} y^{[j]}(t) y^{[s]}(t) &\geq 0 \quad \text{for } j = 0, 1, \dots, s, \\ &\leq 0 \quad \text{for } j = s+1, \dots, n, \\ y^{[m]}(t) \neq 0, \quad m = 0, 1, &\dots, n-2, \quad t \in [T, \infty). \end{aligned} \tag{17}$$

Let y fulfills (17). First, we prove that $s \neq 0$ and

$$\lim_{t \to \infty} |y\ (t)| = \infty. \tag{18}$$

Let, on the contrary, $s = 0$. Then (17) and (2) yield

$$y^{[0]}\ y^{[1]} < 0, \quad |y^{[1]}| \quad \text{is nondecreasing on} \quad [T, \infty]$$

and

$$\infty > |y(\infty) - y(T)| = \int_T^\infty a_1(t)\ |y^{[1]}(t)|\ dt\ \geq\ y^{[1]}(T)\ \int_{\bar{t}}^\infty a_1(t)\ dt = \infty.$$

Thus $s \in \{1, \dots, n-1\}$.

Let $s = 1$. Suppose, without loss of generality, that $y > 0$. Then (17) yields

$$\left.\begin{aligned} y > 0, &\qquad y \text{ increasing}, \\ y^{[1]} > 0, &\qquad y^{[1]} \text{ decreasing}, \\ y^{[i]} < 0, &\qquad |y^{[i]}| \text{ increasing for } i = 2, \dots, n-1. \end{aligned}\right\} \tag{19}$$

We prove that (18) holds. Thus, suppose, indirectly, that

$$\lim_{t \to \infty} y(t) = C < \infty. \tag{20}$$

If $y^{[1]}(\infty) > 0$, then

$$\infty > y(\infty) - y(T) = \int_T^\infty a_1(t) y^{[1]}(t) dt \geq y^{(1)}(\infty) \int_T^\infty a_1(t) dt = \infty.$$

The contradiction proves that

$$\lim_{t \to \infty} y^{[1]}(t) = 0. \tag{21}$$

It follows from (19), (2) and (4) that

$$\begin{aligned}
|y^{[i]}(t)| &= |\, y^{[i]}(T)| + \int_T^t a_{i+1}(\tau)|y^{[i+1]}(\tau)|d\tau \\
&\geq \int_T^\infty a_{i+1}(\tau)|y^{[i+1]}(\tau)|d\tau,\ i = 2, \dots, n-2, \\
y^{[n-1]}(t) &\geq \int_T^t |y^{[n]}(\tau)|d\tau \geq \int_T^t a_n(\tau)g(y(\tau))d\tau \\
&\geq C_1 \int_T^t a_n(\tau)d\tau, \quad C_1 = \max_{y(T)\leq\tau\leq C} g(\tau) > 0. \qquad (22)
\end{aligned}$$

From this and from (19), (20) and (21)

$$\begin{aligned}
&\infty > y(\infty) - y(T) = \int_T^\infty a_1(\tau_1)y^{[1]}(\tau_1)d\tau_1 \\
&= \int_T^\infty a_1(\tau_1)\int_{\tau_1}^\infty a_2(\tau_2)|y^{[2]}(\tau_2)|d\tau_2 d\tau_1 \\
&\geq C_1 \int_T^\infty a_1(\tau_1)\int_{\tau_1}^\infty a_2(\tau_2)I_{n-2}(\tau_2, T; a_3, a_n)d\tau_2 d\tau_1 \\
&= C_1 \int_T^\infty a_2(\tau_2)I_{n-2}(\tau_2, T; a_3, a_n)\ \int_T^{\tau_2} a_1(\tau_1)d\tau_1 d\tau_2 \\
&\geq C_1 I_n(\infty, T; a_2, a_1) = \infty
\end{aligned}$$

as according to (6), $i = 1$

$$I_{n-1}(\infty, \bar{t}; a_2, d_1) = \infty \Longrightarrow I_{n-1}(\infty, T; a_2, d_1) = \infty$$

and thus

$$I_n(\infty, T; a_2, a_1) \geq I_{n-1}(\infty, T; a_2, d_1) = \infty.$$

The contradiction proves that (18) is valid for $s = 1$.
Let $s > 1$. Then (17) and (2) yield

$$y(t)y^{[1]}(t) > 0, \quad |y^{[1]}| \text{ is nondecreasing on } [T, \infty),$$

$$|y(t) - y(T)| = \int_T^\infty a_1(\tau)|y^{[1]}(\tau)|d\tau \geq |y^{[1]}(\tau)| \int_T^t a_1(\tau)d\tau \quad \underset{t\to\infty}{\longrightarrow} \quad \infty.$$

Thus (18) is valid for all $s \in \{1, \dots, n-1\}$.

Let $0 \leq \lambda < 1$. The statement of the theorem was proved in [3, Ths 1-3] if the more restrictive assumption (H4) is supposed instead of (H3). In this case the inequality (4) was used only for $x_1 = y(t),\ t \in [T, \infty]$ where y fulfills (17). From this, using (18), the statement is valid under the validity of (H3), too (note, that (9) follows from (5)).

Finally, suppose $\lambda = 1$.

Let $s \in \{1, \ldots, n-1\}$. We prove that the solution y, fulfilling (17) does not exist.

First, we estimate $y^{[s]}$. Let, for the simplicity, $y > 0$ for large t. According to (18) there exists $T_1 \geq T$ such that

$$y(t) \geq K, \qquad t \in [T_1, \infty) \tag{23}$$

and (17) yields

$$\left.\begin{array}{lll} y^{[j]}(t) > 0, & y^{[j]} \text{ is increasing}, & j = 0, 1, \ldots, s-1, \\ y^{[s]}(t) > 0, & y^{[s]} \text{ is decreasing}, & \\ y^{[m]}(t) < 0, & |y^{[m]}| \text{ is nondecreasing}, & m = s+1, \ldots, n-1, \\ & & t \in [T_1, \infty). \end{array}\right\} \tag{24}$$

From this, from (24), (2) and (4) we have

$$\begin{aligned} |y^{[i]}(t)| &\geq \int_{T_1}^{t} a_{i+1}(\tau)|y^{[i+1]}(\tau)|d\tau, \ i = 0, \ldots, n-2, \ i \neq s, \\ |y^{[n-1]}(t)| &\geq \int_{T_1}^{t} |y^{[n]}(\tau)|d\tau \geq \int_{T_1}^{t} a_n(\tau)y(\tau)d\tau \ \text{ if } \ s \neq n-1 \end{aligned} \tag{25}$$

and thus, using (24),

$$\begin{aligned} |y^{[s+1]}(t)| &\geq I_{n-1}(t, T_1; a_{s+2}, a_s y^{[s]}) \\ &\geq y^{[s]}(t) I_{n-1}(t, T_1; a_{s+2}, a_s), \quad s \in \{1, \ldots, n-2\}, \\ |y(t)| &\geq y^{[n-1]}(t) I_{n-1}(t, T_1; a_1, a_{n-1}) \ \text{ for } \ s = n-1. \end{aligned}$$

Further, using (2) and (24), it follows from this that

$$\begin{aligned} \left(y^{[s]}(t)\right)' &= a_{s+1}(t) y^{[s+1]}(t) = -a_{s+1}(t)|y^{[s+1]}(t)| \\ &\leq -a_{s+1}(t) I_{n-1}(t, T_1; a_{s+2}, a_s) y^{[s]}(t) \\ &\quad \text{for } s \in \{1, \ldots, n-2\}, \\ \left(y^{[n-1]}(t)\right)' &= -|y^{[n]}(t)| \leq -a_n(t)y(t) \leq -a_n(t) I_{n-1}(t, T_1; a_1, a_{n-1}) \\ &\quad \times y^{[n-1]}(t) \quad \text{for } \ s = n-1, \ t \geq T_1. \end{aligned}$$

Thus

$$y^{[s]}(t) \leq y^{[s]}(T_1) e^{-I_n(t, T_1; a_{s+1}, a_s)}. \tag{26}$$

Especially, using (6),

$$\lim_{t \to \infty} y^{[s]}(t) = 0. \tag{27}$$

Let the assumption (7) be valid. Using (24), (25) and (27)

$$\begin{aligned}
y^{[s-1]}(t) &= y^{[s-1]}(T_1) + \int_{T_1}^{t} a_s(\tau_s) y^{[s]}(\tau_s) d\tau_s \\
&= y^{[s-1)}(T_1) + \int_{T_1}^{t} a_s(\tau_s) \int_{\tau_s}^{\infty} a_{s+1}(\tau_{s+1}) |y^{[s+1]}(\tau_{s+1})| d\tau_{s+1} d\tau_s
\end{aligned}$$

$$\geq y^{[s-1]}(T_1) + \int_{T_1}^{t} a_s(\tau_s) \int_{\tau_s}^{\infty} a_{s+1}(\tau_{s+1}) I_{n-2}(\tau_{s+1}, T_1; a_{s+2}, a_{s-1}\ y^{[s-1]}) d\tau_{s+1} d\tau_s$$

$$\geq y^{[s-1]}(T_1) + \int_{T_1}^{t} a_s(\tau_s) \int_{\tau_s}^{\infty} a_{s+1}(\tau_{s+1}) I_{n-2}(\tau_{s+1}, \tau_s; a_{s+2}, a_{s-1}\ y^{[s-1]}) d\tau_{s+1} d\tau_s$$

$$\geq y^{[s-1]}(T_1) + \int_{T_1}^{t} y^{[s-1]}(\tau_s) a_s(\tau_s) \int_{\tau_s}^{\infty} a_{s+1}(\tau_{s+1}) I_{n-2}(\tau_{s+1}, \tau_s; a_{s+2}, a_{s-1}) d\tau_{s+1} d\tau_s,$$
$$t \geq T_1.$$

Thus Gronwall's inequality yields

$$y^{[s-1]}(t) \geq y^{[s-1]}(T_1) e^{J\ (t, T_1; a_s)}, \quad t \geq T_1. \tag{28}$$

On the other side, using (26), we have

$$y^{[s-1]}(t) \leq y^{[s-1]}(T_1) + y^{[s]}(T_1) \int_{T_1}^{t} a_s(\tau) e^{-I_n(\tau, T_1; a_{s+1}, a_s)}\ d\tau.$$

From this and from (28)

$$\begin{aligned}
1 \leq e^{-J(t,T_1;a_s)} + \frac{y^{[s]}(T_1)}{y^{[s-1]}(T_1)}\, e^{-J(t,T_1;a_s)} \int_{T_1}^{t} a_s(\tau) \\
\times \quad e^{-I_n(\tau,T_1;a_{s+1},a_s)}\ d\tau, \quad t \geq T_1
\end{aligned}$$

that contradicts to (7).

Let the assumption (8) be valid. Then (24) and (25) yield

$$\begin{aligned}
\infty > |y^{[s]}(\infty) - y^{[s]}(T_1)| = \\
= \int_{T_1}^{\infty} a_{s+1}(\tau) |y^{[s+1]}(\tau)| d\tau \geq I_{n-1}(\infty, T_1; a_{s+1}, a_{s-1} y^{[s-1]}) \geq \\
\geq y^{[s-1]}(T_1) I_{n-1}(\infty, T_1; a_{s+1}, a_{s-1}) = \infty.
\end{aligned}$$

Thus, the solution y, fulfilling (17), does not exist. □

Remark 5. (i) Theorem 4 generalizes results of [3], [6] and Theorem A.
(ii) The statements of Theorems 3 and 4 are valid for a solution y on $[\alpha, \infty)$ if the Cauchy conditions (16) are taken in $t = \alpha$ and $\bar{t} \geq \alpha$ (see (H3)).

3 Applications

We apply the previous results to Eq. (12)

$$y''' + q(t)y' + r(t)g(y) = 0 \tag{12}$$

under the validity of the assumption

$$\lambda \in [0,1], \qquad |x|^{\lambda} \le |g(x)| \quad \text{for large } |x|. \tag{29}$$

Let

$$q^{+}(t) = \max(q(t),0),\ \bar{q}(t) = \min(q(t),0),\ t \in R_{+}.$$

Cecchi and Marini [6] studied Eq. (12) under the following hypothesis:

(H5): Let $\int_0^\infty tq^{-}(t)dt = -K > -\infty$, and let the equation

$$h'' + e^{-2K}q^{+}(t)h = 0$$

be disconjugate on R_+ (i.e. every its solution has at most one zero on R_+).

They proved the following theorem.

Theorem B ([6]). *Let* (H5) *and* g *be nondecreasing for large* $|y|$*. Let*

$$\int_0^\infty |g(kt)|r(t)dt = \infty \quad \textit{for every } k \in (0,1). \tag{30}$$

Then every proper solution of Eq. (12) *with a zero is oscillatory.*

Note, that if the estimation (29) holds, then (30) has the form

$$\int_0^\infty t^{\lambda} r(t)dt = \infty. \tag{31}$$

In case

$$\int_0^\infty tq^{+}(t)dt < \infty, \tag{32}$$

using our previous results, the statement of Th. B can be proved under weaker assumption than (31).

Theorem 6. *Let* (H5), (32) *and* (29) *be valid. Further, let*

$$\int_0^\infty t^{2\lambda} r(t)dt = \infty \quad \textit{if} \quad \lambda \in [0,1) \tag{33}$$

and let

$$r(t) \ge \frac{\sigma}{t^3} \qquad \textit{for large } t \qquad \textit{if} \quad \lambda = 1, \tag{34}$$

where $\sigma > 1$ *is a constant. Then every proper solution with a zero is oscillatory.*

Proof. Let y be a proper solution of (12) with a zero $T \in R_+, y(T) = 0$. If $\sum_{i=0}^{2} |y^{[i]}(T)| = 0$, then according to [1] there exists $t_0 > T$ such that the Cauchy initial conditions at t_0 fulfill (16). In the opposite case it is evident that (16) holds in some right neighbourhood of $t = T$. Thus, in all cases, there exists $t_0 > T$ such that (16) is valid in $t = t_0$.

In [6, Proposition 1] it is proved that (H5) and (32) yield the existence of a solution $h : R_+ \to R$ of Eq. (13) which is positive on $(0, \infty)$, increasing and

$$\lim_{t\to\infty} h(t) = h_0 \in (0, \infty). \tag{35}$$

Thus, (12) is equivalent to (14) on $(0, \infty)$ and (15) yields

$$a_1 = h, \quad a_2 = \frac{1}{h^2}, \quad a_3 = rh \quad \text{on} \quad (0, \infty) \tag{36}$$

and

$$\int_{t_0}^{\infty} a_1(s)ds = \int_{t_0}^{\infty} a_2(s)ds = \infty. \tag{37}$$

Let $\varepsilon > \sqrt[4]{\sigma}$ and let $\tau > t_0$ be such that

$$\frac{h_0}{\varepsilon} \le h(t) \le \varepsilon h_0, \qquad t \ge \tau. \tag{38}$$

We will verify hypothesis (H3) with $\bar{t} = \tau$ (see Remark 5 (ii)). According to (37), (5), (6) for $i = 1$ and (8) for $i = 1$ (in case $\lambda = 1$) are valid. Thus it is necessary to verify (6) for $i = 2$ and, in case $\lambda = 1$, the condition (7) for $i = 2$.

Condition (6), $i = 2$: Using (38) we have

$$\begin{aligned} I_1(\infty, \tau; a_3) &= \int_{\tau}^{\infty} r(t)h(t) \left[\int_{\tau}^{t} h(\alpha) \int_{\tau}^{\alpha} \frac{d\beta}{h^2(\beta)}\, d\alpha\right]^{\lambda} dt \\ &\ge \varepsilon^{-1-3\lambda}\, h_0^{1-\lambda}\, 2^{-\lambda} \int_{\tau}^{\infty} r(t)\, (t-\tau)^{2\lambda} dt = \infty. \end{aligned}$$

Condition (7), $i = 2$, $\lambda = 1$:

$$\begin{aligned} J\,(t, \tau; a_2) &= \int_{\tau}^{t} \frac{1}{h^2(s)} \int_{s}^{\infty} h(s_1)r(s_1) \int_{s}^{s_1} h(s_2)ds_2 ds_1 ds \\ &\ge \varepsilon^{-4} \int_{\tau}^{t} \int_{s}^{\infty} (s_1 - s)r(s_1)ds_1 ds \ge \sigma_1 \ \ln\ \frac{t}{\tau}, \quad \sigma_1 = \frac{\sigma}{2}\, \varepsilon^{-4} > \frac{1}{2}\,, \end{aligned}$$

$$\begin{aligned} I_3(t, \tau; a_3, a_2) &= \int_{\tau}^{t} r(s)h(s) \int_{\tau}^{s} h(s_1) \int_{\tau}^{s_1} \frac{ds_2}{h^2(s_2)} ds_1 ds \\ &\ge \sigma_1 \int_{\tau}^{t} \frac{(s-\tau)^2}{s^3} ds \ge \sigma_1 \ \left[\ln \frac{t}{\tau} - 2\right]\ . \end{aligned}$$

From this, according to (36), (37) and (38)

$$0 \le \liminf_{t\to\infty} e^{-J(t,\tau;a_2)} \int_\tau^t a_2(s) e^{-I_3(s,\tau;a_3,a_2)}\, ds$$
$$\le \liminf_{t\to\infty} \left(\frac{\tau}{t}\right)^{\sigma_1} \int_\tau^t \frac{\varepsilon^2}{h_0^2}\, e^{2\sigma_1} \left(\frac{\tau}{s}\right)^{\sigma_1} ds = 0\,.$$

□

Remark 7. Let the assumptions of Th. 6 and hypotheses (H1) and (H2) hold. Then, using Th. 3, it is evident that (12) has an oscillatory solution.

The following example shows that (33) is not sufficient condition for the existence of oscillatory solutions in case $\lambda = 1$ and it shows how far is condition (34) from necessary one.

Example 8. Consider the equation

$$y''' + \frac{\sigma}{t^3}\, y = 0, \qquad \sigma \ge 0. \tag{39}$$

Lemma 9. *Eq.* (39) *has an oscillatory solution if, and only if*

$$\sigma > \frac{2\sqrt{3}}{9} \sim 0,385.$$

Proof. (sketch) Eq. (39) can be transformed into the equation with constant coefficients $\dddot{Y} - 3\ddot{Y} + 2\dot{Y} + \sigma Y = 0$ by $t = e^x$, $\quad y(t) = Y(x)$. □

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