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ON PERIODIC IN THE PLANE SOLUTIONS OF SECOND ORDER LINEAR HYPERBOLIC SYSTEMS

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Abstract. Sufficient conditions for the problem

$$\frac{\partial^2 u}{\partial x \partial y} = \mathcal{P}_0(x,y)u + \mathcal{P}_1(x,y)\frac{\partial u}{\partial x} + \mathcal{P}_2(x,y)\frac{\partial u}{\partial y} + q(x,y),$$
$$u(x+\omega_1, y) = u(x,y), \quad u(x, y+\omega_2) = u(x,y)$$

to have the Fredholm property and to be uniquely solvable are established, where ω_1 and ω_2 are positive constants and $\mathcal{P}_j : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ $(j=0,1,2)$ and $q : \mathbb{R}^2 \to \mathbb{R}^n$ are continuous matrix and vector functions periodic in x and y.

Introduction

Let ω_1 and ω_2 be positive constants and $\mathcal{P}_j : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ $(j = 0,1,2)$ and $q : \mathbb{R}^2 \to \mathbb{R}^n$ be continuous matrix and vector functions, which are ω_1 periodic in the first and ω_2 periodic in the second argument. Consider the linear hyperbolic system

$$\frac{\partial^2 u}{\partial x \partial y} = \mathcal{P}_0(x,y)u + \mathcal{P}_1(x,y)\frac{\partial u}{\partial x} + \mathcal{P}_2(x,y)\frac{\partial u}{\partial y} + q(x,y) \tag{0.1}$$

with periodic conditions

$$u(x+\omega_1, y) = u(x,y), \quad u(x, y+\omega_2) = u(x,y)\,. \tag{0.2}$$

By a solution of (0.1), (0.2) we understand a continuous vector function $u : \mathbb{R}^2 \to \mathbb{R}^n$ which has continuous partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x \partial y}$ and satisfies system (0.1) and conditions (0.2) everywhere in $\mathbb{R}^2$.

Problem (0.1), (0.2) was previously considered in [1-5,13-15]. However, in contrast to the similar problem in a strip (see e. g. [7-10] and the references therein), the question of its solvability in many interesting cases remains open. The question

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when the problem (0.1), (0.2) has the Fredholm property is also practically open. The present paper deals with these questions.

Throughout the paper the following notation is used. $\mathbb{Z}$ is the set of integers; $\mathbb{N}$ is the set of natural numbers. $\mathbb{R}^{m\times n}$ is the space of $m\times n$ matrices $X=(x_{kl})$ with real components x_{kl} $(k=1,\dots,m,\ l=1,\dots,n)$ and the norm

$$\|x\|=\sum_{k=1}^{m}\sum_{l=1}^{n}|x_{kl}|\,.$$

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$; E is the unit matrix; Θ is the zero matrix.
i is the complex unit, i. e. $i^2=-1$.
$C^k_\omega(\mathbb{R};\mathbb{R}^{m\times n})$ is the space of k-times continuously differentiable ω-periodic functions $Z:\mathbb{R}\to\mathbb{R}^{m\times n}$ with the norm

$$\|Z\|_{C^k_\omega}=\max_{t\in\mathbb{R}}\sum_{l=0}^{k}\|Z^{(l)}(t)\|\,.$$

$C^k_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})$ $(k=0,1)$ is the space of k-times continuously differentiable matrix functions $Z:\mathbb{R}^2\to\mathbb{R}^{m\times n}$ satisfying the periodic conditions

$$Z(x+\omega_1,y)=Z(x,y),\quad Z(x,y+\omega_2)=Z(x,y)$$

with the norm

$$\|Z\|_{C^k_{\omega_1\,\omega_2}}=\max_{(x,y)\in\mathbb{R}^2}\sum_{l=0}^{k}\left(\left\|\frac{\partial^l}{\partial x^l}Z\right\|+\left\|\frac{\partial^l}{\partial y^l}Z\right\|\right)\,.$$

$$C_\omega(\mathbb{R};\mathbb{R}^{m\times n})=C^0_\omega(\mathbb{R};\mathbb{R}^{m\times n})\,,\quad C_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})=C^0_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})\,.$$

$C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})$ is the space of matrix functions $Z\in C_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})$ which have the continuous partial derivative in the first argument with the norm

$$\|Z\|_{C^{1,0}_{\omega_1\,\omega_2}}=\max_{(x,y)\in\mathbb{R}^2}\left(\|Z(x,y)\|+\left\|\frac{\partial}{\partial x}Z(x,y)\right\|\right)\,.$$

$C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})$ is the space of matrix functions $Z\in C_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})$ which have the continuous partial derivative in the second argument with the norm

$$\|Z\|_{C^{0,1}_{\omega_1\,\omega_2}}=\max_{(x,y)\in\mathbb{R}^2}\left(\|Z(x,y)\|+\left\|\frac{\partial}{\partial y}Z(x,y)\right\|\right)\,.$$

Definition 0.1. Let Λ be a subspace of the space $C_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})$. We say that problem (0.1), (0.2) has *property* F *in* Λ, if for its unique solvability for any $q\in\Lambda$ it is necessary and sufficient that the homogeneous system

$$\frac{\partial^2 u}{\partial x\partial y}=\mathcal{P}_0(x,y)u+\mathcal{P}_1(x,y)\frac{\partial u}{\partial x}+\mathcal{P}_2(x,y)\frac{\partial u}{\partial y}\tag{0.1$_0$}$$

has no nontrivial solution satisfying conditions (0.2).

In §1 of the present paper we obtain conditions for the problem (0.1), (0.2) to have property F in $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})$, $C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})$, $C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})$ and $C^{1}_{\omega_1\omega_2}(\mathbb{R}\mathbb{R}^{m\times n})$, respectively. On the basis of these results in §2 we prove the existence and uniqueness theorems in the case when $\mathcal{P}_j(x,y)\equiv\mathcal{P}_j(y)$, $(j=0,1,2)$, i. e. when the systems (0.1) and (0.1_0) have the form

$$\frac{\partial^2 u}{\partial x\partial y}=\mathcal{P}_0(y)u+\mathcal{P}_1(y)\frac{\partial u}{\partial x}+\mathcal{P}_2(y)\frac{\partial u}{\partial y}+q(x,y)\,, \tag{0.3}$$

$$\frac{\partial^2 u}{\partial x\partial y}=\mathcal{P}_0(y)u+\mathcal{P}_1(y)\frac{\partial u}{\partial x}+\mathcal{P}_2(y)\frac{\partial u}{\partial y}\,. \tag{0.3_0}$$

§1. Problem (0.1), (0.2) with property F

Every time when proving the existence of property F of the problem (0.1), (0.2) in $\Lambda\subset C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})$, we chose some Banach space depending on the subspace Λ, reduce the considered problem to some linear Fredholm equation in that space and apply the Fredholm alternative ([12, p. 275]).

For an arbitrary $x\in\mathbb{R}$ ($y\in\mathbb{R}$) by $Z_1(x,\cdot)$ ($Z_2(\cdot,y)$) we denote the fundamental matrix of the system of ordinary differential equations

$$\frac{dZ(x,y)}{dy}=\mathcal{P}_1(x,y)Z(x,y)\qquad\left(\frac{dZ(x,y)}{dx}=\mathcal{P}_2(x,y)Z(x,y)\right)$$

satisfying the initial condition

$$Z_1(x,0)=E\qquad (Z_2(0,y)=E)\,.$$

Introduce the following matrix functions

$$M_1(x)=Z_1^{-1}(x,\omega_2)-E,\quad M_2(y)=Z_2^{-1}(\omega_1,y)-E.$$

We consider three fundamental cases, when one can speak about property F of problem (0.1), (0.2) in the mentioned spaces. These cases will be formulated in terms of the matrix functions $M_1(x)$ and $M_2(y)$.

CASE I.

$$\det M_1(x)\neq 0 \text{ for } x\in[0,\omega_1],\qquad \det M_2(y)\neq 0 \text{ for } y\in[0,\omega_2]\,. \tag{1.1}$$

By $H_1:\mathbb{R}^2\to\mathbb{R}^{n\times n}$ and $H_2:\mathbb{R}^2\to\mathbb{R}^{n\times n}$ we will denote the solutions of the matrix differential equations

$$\frac{\partial H_1(x,y)}{\partial x}=Z_1(x,y)\,(E-Z_1(x,\omega_2))^{-1}\,Z_1^{-1}(x,y)\int_0^{\omega_2}\mathcal{P}_0(x,t)\,dt\,H_1(x,y)$$

and

$$\frac{\partial H_2(x,y)}{\partial y}=Z_2(x,y)\,(E-Z_2(\omega_1,y))^{-1}\,Z_2^{-1}(x,y)\int_0^{\omega_1}\mathcal{P}_1(s,y)\,ds\,H_2(x,y)$$

satisfying the initial conditions

$$H_1(0,y)=E\quad\text{ for } y\in\mathbb{R}$$

and

$$H_2(x,0)=E\quad\text{ for } x\in\mathbb{R}\,.$$

Theorem 1.1. *Let $\mathcal{P}_j \in C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$ $(j=0,1,2)$, conditions (1.1) hold and let either*

$$\det(E - H_1(\omega_1, y)) \neq 0 \quad \text{for } y \in \mathbb{R} \tag{1.2}$$

or

$$\det(E - H_2(x, \omega_2)) \neq 0 \quad \text{for } x \in \mathbb{R}\,. \tag{1.3}$$

Then problem (0.1), (0.2) has property F in $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$.

Proof. Let $u(x,y)$ be a solution of problem (0.1), (0.2) and let

$$v(x,t) = \frac{\partial u(x,t)}{\partial x}\,.$$

Then

$$\frac{\partial v(x,t)}{\partial t} = \mathcal{P}_1(x,t)v(x,t) + q_1(x,t)\,,$$

where

$$q_1(x,t) = \mathcal{P}_0(x,t)u(x,t) + \mathcal{P}_2(x,t)\frac{\partial u(x,t)}{\partial t} + q(x,t).$$

From here by Cauchy formula for systems of linear ordinary differential equations (see, e.g. [6], p.66) we have

$$v(x,t) = Z_1(x,t)Z_1^{-1}(x,y)v(x,y) + \int_y^t Z_1(x,t)Z_1^{-1}(x,\tau)q_1(x,\tau)\,d\tau\,.$$

Consequently,

$$\begin{aligned}\frac{\partial u(x,t)}{\partial x} &= Z_1(x,t)Z_1^{-1}(x,y)\frac{\partial u(x,y)}{\partial x} + \\ &+ \int_y^t Z_1(x,t)Z_1^{-1}(x,\tau)\left(\mathcal{P}_0(x,\tau)u(x,\tau) + \mathcal{P}_2(x,\tau)\frac{\partial u(x,\tau)}{\partial \tau} + q(x,\tau)\right)d\tau\,.\end{aligned} \tag{1.4$_0$}$$

From (0.2), (1.1) and the identity $Z_1(x, y+\omega_2) \equiv Z_1(x,y)Z_1(x,\omega_2)$ it follows that

$$\begin{aligned}\frac{\partial u(x,y)}{\partial x} &= \\ &\int_y^{y+\omega_2}\left(Q_{11}(x,y,t)u(x,t) + Q_{12}(x,y,t)\frac{\partial u(x,t)}{\partial t}\right)dt + \varphi_1(x,y)\,,\end{aligned} \tag{1.4}$$

where

$$\begin{aligned}Q_{11}(x,y,t) &= Z_1(x,y)M_1^{-1}(x)Z_1^{-1}(x,t)\mathcal{P}_0(x,t),\\ Q_{12}(x,y,t) &= Z_1(x,y)M_1^{-1}(x)Z_1^{-1}(x,t)\mathcal{P}_2(x,t),\\ \varphi_1(x,y) &= Z_1(x,y)M_1^{-1}(x)\int_y^{y+\omega_2} Z_1^{-1}(x,t)q(x,t)\,dt\,.\end{aligned}$$

Similarly we get

$$(1.5)\quad \frac{\partial u(x,y)}{\partial y} = \int_x^{x+\omega_1} \left(Q_{21}(x,y,s)u(s,y) + Q_{22}(x,y,s)\frac{\partial u(s,y)}{\partial s} \right) ds + \varphi_2(x,y)\,,$$

where

$$Q_{21}(x,y,s) = Z_2(x,y)M_2^{-1}(y)Z_2^{-1}(s,y)\mathcal{P}_0(s,y),$$
$$Q_{22}(x,y,s) = Z_2(x,y)M_2^{-1}(y)Z_2^{-1}(s,y)\mathcal{P}_1(s,y),$$
$$\varphi_2(x,y) = Z_2(x,y)M_2^{-1}(y)\int_x^{x+\omega_1} Z_2^{-1}(s,y)q(s,y)\,ds\,.$$

Let us prove the theorem under the assumption that condition (1.2) holds (when the condition (1.3) holds the proof is similar). Let us transform (1.4) by means of integration by parts

$$\begin{aligned}\frac{\partial u(x,y)}{\partial x} &= Z_1(x,y)\left(E - Z_1(x,\omega_2)\right)^{-1} Z_1^{-1}(x,y)\int_0^{\omega_2}\mathcal{P}_0(x,t)\,dt\,u(x,y)+\\ &+\int_y^{y+\omega_2} Z_1(x,y)M_1^{-1}(x)Z_1^{-1}(x,t)\left(\mathcal{P}_1(x,t)\int_y^t \mathcal{P}_0(x,\tau)\,d\tau\,u(x,t)+\right.\\ &\left.+\left(\mathcal{P}_2(x,t) - \int_y^t \mathcal{P}_0(x,\tau)\,d\tau\right)\frac{\partial u(x,t)}{\partial t}\right) dt + \varphi_1(x,y)\end{aligned}$$

and with regard to (1.2) from (0.2) we obtain

$$(1.6)\quad u(x,y) = \int_x^{x+\omega_1}\int_y^{y+\omega_2}\left(K_{11}(x,y,s,t)u(s,t) + K_{12}(x,y,s,t)\frac{\partial u(s,t)}{\partial t}\right) ds\,dt + \psi_1(x,y)\,,$$

where

$$K_{11}(x,y,s,t) = H_1(x,y)\left(H_1^{-1}(\omega_1,y) - E\right)^{-1}\times$$
$$\times H_1^{-1}(s,y)Z_1(s,y)M_1^{-1}(s)Z_1^{-1}(s,t)\mathcal{P}_1(s,t)\int_y^t \mathcal{P}_0(s,\tau)\,d\tau\,,$$
$$K_{12}(x,y,s,t) = H_1(x,y)\left(H_1^{-1}(\omega_1,y) - E\right)^{-1}\times$$
$$\times H_1^{-1}(s,y)Z_1(s,y)M_1^{-1}(s)Z_1^{-1}(s,t)\left(\mathcal{P}_2(s,t) - \int_y^t \mathcal{P}_0(s,\tau)\,d\tau\right),$$
$$\psi_1(x,y) = H_1(x,y)\left(H_1^{-1}(\omega_1,y) - E\right)^{-1}\int_x^{x+\omega_1} H_1^{-1}(s,y)\varphi_1(s,y)\,ds\,.$$

By substituting (1.6) and (1.4) into (1.5) we get

$$\frac{\partial u(x,y)}{\partial y} = \int_x^{x+\omega_1}\int_y^{y+\omega_2}\left(K_{21}(x,y,s,t)u(s,t)+K_{22}(x,y,s,t)\frac{\partial u(s,t)}{\partial t}\right)ds\,dt+\psi_2(x,y)\,,$$

where

$$K_{2j}(x,y,s,t)=\int_x^s Q_{21}(x,y,\xi)K_{1j}(\xi,y,s,t)\,d\xi+$$
$$+\int_s^{x+\omega_1} Q_{21}(x,y,\xi)K_{1j}(\xi,y,s+\omega_1,t)\,d\xi+Q_{22}(x,y,s)Q_{1j}(s,y,t)\quad (j=1,2)\,,$$
$$\psi_2(x,y)=\int_x^{x+\omega_1}\left(Q_{21}(x,y,s)\psi_1(s,y)+Q_{22}(x,y,s)\varphi_1(s,y)\right)\,ds\,.$$

In the space $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$ consider the operator equation

$$z(x,y)=\int_x^{x+\omega_1}\int_y^{y+\omega_2}K(x,y,s,t)z(s,t)\,ds\,dt+\varphi(x,y)\,, \tag{1.7}$$

where

$$K(x,y,s,t)=\begin{pmatrix}K_{11}(x,y,s,t) & K_{12}(x,y,s,t)\\ K_{21}(x,y,s,t) & K_{22}(x,y,s,t)\end{pmatrix},$$
$$\varphi(x,y)=\begin{pmatrix}\varphi_1(x,y)\\ \varphi_2(x,y)\end{pmatrix}.$$

A solution of (1.7) is a column vector function $z(x,y)=(z_j(x,y))_{j=1}^2$, where $z_j\in C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$ $(j=1,2)$.

It is not difficult to verify that the linear operator

$$\mathcal{A}(z)=\int_x^{x+\omega_1}\int_y^{y+\omega_2}K(x,y,s,t)z(s,t)\,ds\,dt$$

maps the space $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{2n})$ into itself and, consequently, by virtue of the continuity of the matrix function K, it is a completely continuous operator in $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{2n})$. Hence, equation (1.7) is the Fredholm equation in $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{2n})$.

On the other hand we have already shown that if $u(x,y)$ is a solution of problem (0.1), (0.2), then $(z_j(x,y))_{j=1}^2$, where $z_1(x,y)=u(x,y)$ and $z_2(x,y)=\frac{\partial u(x,y)}{\partial y}$ is a solution of equation (1.7). The converse statement can be easily verified, i. e. if $(z_j(x,y))_{j=1}^2$ is a solution of equation (1.7) from $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{2n})$, then $z_1(x,y)$ is a solution of problem (0.1), (0.2). Thus, problem (0.1), (0.2) is equivalent to equation (1.7). Therefore problem (0.1), (0.2) has the property F in $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$. $\square$

Theorem 1.2. *Let conditions (1.1) hold and let either $\mathcal{P}_1 \in C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$ or $\mathcal{P}_2 \in C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$. Then problem (0.1), (0.2) has property F in $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$.*

Proof. Let us prove the theorem under the assumption that $\mathcal{P}_1 \in C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$. The case $\mathcal{P}_2 \in C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$ is absolutely analogous. So, let $u(x,y)$ be a solution of problem (0.1), (0.2). Then the representation (1.5) is valid. By integration by parts and taking into account conditions (0.2) we arrive to the following equality

$$\begin{aligned}\frac{\partial u(x,y)}{\partial y} &= \mathcal{P}_1(x,y)u(x,y)+\\ &+\int_x^{x+\omega_1}\left(Q_{21}(x,y,s)-\frac{\partial}{\partial s}Q_{22}(x,y,s)\right)u(s,y)\,ds+\varphi_2(x,y),\end{aligned}\tag{1.8}$$

whence applying the same technique as when deriving (1.4), we obtain

$$u(x,y)=\int_x^{x+\omega_1}\int_y^{y+\omega_2}K(x,y,s,t)u(s,t)\,ds\,dt+\psi(x,y)\,,\tag{1.9}$$

where

$$\begin{aligned}K(x,y,s,t) &= Z_1(x,y)M_1^{-1}(x)Z_1^{-1}(x,t)Z_2(x,t)M_2^{-1}(t)Z_2^{-1}(s,t)\times\\ &\times\left(\mathcal{P}_0(s,t)+\mathcal{P}_2(s,t)\mathcal{P}_1(s,t)-\frac{\partial}{\partial s}\mathcal{P}_1(s,t)\right)\,,\\ \psi(x,y) &= Z_1(x,y)M_1^{-1}(x)\int_x^{x+\omega_1}\int_y^{y+\omega_2}Z_1^{-1}(x,t)Z_2(x,t)M_2^{-1}(t)Z_2^{-1}(s,t)q(s,t)ds\,dt\,.\end{aligned}$$

It is rather obvious fact that equation (1.9) is the Fredholm equation in the space $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$. It is also easy to verify that every solution of equation (1.9) from $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$ is also a solution of problem (0.1), (0.2). Thus we have shown that problem (0.1), (0.2) is equivalent to the Fredholm equation (1.9) in the space $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$. Therefore problem (0.1), (0.2) has property F in $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$.□

Remark 1.1. In Theorems 1.1 and 1.2 conditions (1.1) are essential and they cannot be weakened. Indeed, consider the problem

$$\frac{\partial^2 u}{\partial x\partial y}=\sin^2(y)\,u+\sin^4(y)\frac{\partial u}{\partial x}-\sin^2(y)\frac{\partial u}{\partial y}+|\sin y|\sin y\,,\tag{1.10}$$

$$u(x+2\pi,y)=u(x,y),\quad u(x,y+2\pi)=u(x,y)\,,\tag{1.11}$$

which satisfies all conditions of Theorems 1.1 and 1.2, except of conditions (1.1). In this case we have

$$\begin{aligned}M_1(x)&=\exp\left(-\int_0^{2\pi}\sin^2 t\,dt\right)-1=\exp(-\pi)-1<0\quad\text{for } x\in\mathbb{R}\,,\\ M_2(y)&=\exp\left(-2\pi\sin^2 y\right)-1=0\quad\text{for } y=\pi k,\ k\in\mathbb{Z}\,,\end{aligned}$$

i. e. only the one part of conditions (1.1) is violated and only at isolated points. Nevertheless, it follows from Remark 2.1 below that the homogeneous problem corresponding to (1.10), (1.11) has only trivial solution and that the solution of problem (1.10), (1.11) $u(x,y)$ has the form $u(x,y) = v(y)$. Therefore, problem (1.10), (1.11) is reduced to the periodic problem for the linear ordinary differential equation

$$\frac{dv}{dy} = v + \operatorname{sign}(\sin y)\,, \quad v(y+2\pi) = v(y)\,,$$

which has unique absolutely continuous but not continuously differentiable solution

$$v(y) = (\exp(-2\pi) - 1)^{-1} \int_y^{y+2\pi} \exp(-t) \operatorname{sign}(\sin t)\, dt\,.$$

Consequently, problem (1.10), (1.11) has no solution, in spite of the fact that the corresponding homogeneous problem has only trivial solution.

CASE II.

Either

$$M_1(x) = \Theta \text{ for } x \in [0,\omega_1], \qquad \det M_2(y) \neq 0 \text{ for } y \in [0,\omega_2] \tag{1.12}$$

or

$$\det M_1(x) \neq 0 \text{ for } x \in [0,\omega_1], \quad M_2(y) = \Theta \text{ for } y \in [0,\omega_2]\,. \tag{1.13}$$

When $\mathcal{P}_1 \in C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$ and $\mathcal{P}_2 \in C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$ let us introduce the following matrix functions

$$\Gamma_1(x,y) = \mathcal{P}_0(x,y) + \mathcal{P}_2(x,y)\mathcal{P}_1(x,y) - \frac{\partial}{\partial x}\mathcal{P}_1(x,y)\,, \tag{1.14}$$

$$\bar\Gamma_1(x,y) = \mathcal{P}_0(x,y) + \mathcal{P}_2(x,y)\mathcal{P}_1(x,y)\,, \tag{1.15}$$

$$\Gamma_2(x,y) = \mathcal{P}_0(x,y) + \mathcal{P}_1(x,y)\mathcal{P}_2(x,y) - \frac{\partial}{\partial y}\mathcal{P}_2(x,y)\,, \tag{1.16}$$

$$\bar\Gamma_2(x,y) = \mathcal{P}_0(x,y) + \mathcal{P}_1(x,y)\mathcal{P}_2(x,y)\,. \tag{1.17}$$

Theorem 1.3. *Let conditions (1.12) hold,* $\mathcal{P}_j \in C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$ $(j=0,1,2)$ *and*

$$\det\left(\int_0^{\omega_2} Z_1^{-1}(x,t)\bar\Gamma_1(x,t)Z_1(x,t)\,dt\right) \neq 0 \text{ for } x \in [0,\omega_1]\,. \tag{1.18}$$

Then problem (0.1), (0.2) has property F *in* $C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$.

Proof. Let $q \in C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$ and let $u(x,y)$ be a solution of problem (0.1), (0.2). Then in view of the identity $Z_1(x,y+\omega_2) \equiv Z_1(x,y)Z_1(x,\omega_2)$ and condition (1.12) we have

$$Z_1(x,y+\omega_2)Z_1^{-1}(x,y) \equiv E\,.$$

Therefore if we substitute $t = y + \omega_2$ into (1.4_0), we obtain

$$\int_y^{y+\omega_2} Z_1^{-1}(x,t)\left(\mathcal{P}_0(x,t)u(x,t) + \mathcal{P}_2(x,t)\frac{\partial u(x,t)}{\partial t} + q(x,t)\right)dt = 0\,, \tag{1.19}$$

for $x \in \mathbb{R}$. By means of integration by parts and taking into account the identity

$$\mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y} = \mathcal{P}_2(x,y)\left(\frac{\partial u(x,y)}{\partial y} - \mathcal{P}_1(x,y)u(x,y)\right) + \mathcal{P}_2(x,y)\mathcal{P}_1(x,y)u(x,y) =$$
$$= \mathcal{P}_2(x,y)Z_1(x,y)\frac{\partial}{\partial y}\left(Z_1^{-1}(x,y)u(x,y)\right) + \mathcal{P}_2(x,y)\mathcal{P}_1(x,y)u(x,y),$$

from (1.19) we get

$$\begin{aligned}\int_y^{y+\omega_2} & Z_1^{-1}(x,t)\bar{\Gamma}_1(x,t)Z_1(x,t)\,dt\,Z_1^{-1}(x,y)u(x,y) - \\ - \int_y^{y+\omega_2} & \left(\int_y^t Z_1^{-1}(x,\tau)\bar{\Gamma}_1(x,\tau)Z_1(x,\tau)\,d\tau \cdot Z_1^{-1}(x,t) - Z_1^{-1}(x,t)\mathcal{P}_2(x,t)\right)\times \\ & \times\left(\frac{\partial u(x,t)}{\partial t} - \mathcal{P}_1(x,t)u(x,t)\right)dt + \int_y^{y+\omega_2} Z_1^{-1}(x,t)q(x,t)\,dt = 0\,.\end{aligned} \tag{1.20}$$

By virtue of conditions (1.12), (1.18) and the equality (1.8), finally we obtain

$$u(x,y) = \int_x^{x+\omega_1}\int_y^{y+\omega_2} K(x,y,s,t)u(s,t)\,ds\,dt + \psi(x,y)\,, \tag{1.21}$$

where

$$\begin{aligned}K(x,y,s,t) = Z_1(x,y) & \left(\int_0^{\omega_2} Z_1^{-1}(x,\tau)\bar{\Gamma}_1(x,\tau)Z_1(x,\tau)\,d\tau\right)^{-1}\times \\ & \times\left(\int_y^t Z_1^{-1}(x,\tau)\bar{\Gamma}_1(x,\tau)Z_1(x,\tau)\,d\tau - Z_1^{-1}(x,t)\mathcal{P}_2(x,t)\right)\times \\ & \times Z_2(x,t)M_2^{-1}(t)Z_2^{-1}(s,t)\Gamma_1(s,t)\,,\end{aligned}$$

$$\begin{aligned}\psi(x,y) = Z_1(x,y) & \left(\int_0^{\omega_2} Z_1^{-1}(x,\tau)\bar{\Gamma}_1(x,\tau)Z_1(x,\tau)\,d\tau\right)^{-1}\times \\ \times\Bigg(\int_x^{x+\omega_1}\int_y^{y+\omega_2} & \left(\int_y^t Z_1^{-1}(x,\tau)\bar{\Gamma}_1(x,\tau)Z_1(x,\tau)\,d\tau - Z_1^{-1}(x,t)\mathcal{P}_2(x,t)\right)\times \\ \times Z_2(x,t)M_2^{-1}(t) & Z_2^{-1}(s,t)q(s,t)\,ds\,dt - \int_0^{\omega_2} Z_1^{-1}(x,t)q(x,t)\,dt\Bigg)\,.\end{aligned}$$

The remaining reasoning is similar to that used in the proof of Theorem 1.2. □

In the same way we can prove the statement "symmetric" to Theorem 1.3.

Theorem 1.4. *Let conditions (1.13) hold, $\mathcal{P}_j \in C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$ $(j=0,1,2)$ and*

$$\det\left(\int_0^{\omega_1} Z_2^{-1}(s,y)\bar{\Gamma}_2(s,y)Z_2(s,y)\,ds\right)\neq 0 \quad \text{for } y\in[0,\omega_2]\,. \tag{1.22}$$

Then problem (0.1), (0.2) has property F in $C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$.

Remark 1.2. When proving Theorem 1.3, to derive equation (1.21), we used only the fact that $\mathcal{P}_1 \in C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$. The solvability of equation (1.21) in the space $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$ does not directly depend on smoothness of $K(x,y,s,t)$ and $\psi(x,y)$ and, hence, on smoothness of $\mathcal{P}_0(x,y)$, $\mathcal{P}_1(x,y)$ and $q(x,y)$. But in order that any solution of equation (1.21) to be a solution of problem (0.1), (0.2), the restrictions imposed on smoothness of $\mathcal{P}_j(x,y)$ $(j=0,1,2)$ and $q(x,y)$ are optimal in some sense and they cannot be weakened. In view of "symmetry" of Theorems 1.3 and 1.4, introduce some examples concerning Theorem 1.4.

For equations

$$\frac{\partial^2 u}{\partial x\partial y} = p_0(y)u + \frac{\partial u}{\partial x} - 1 \tag{1.23_1}$$

$$\frac{\partial^2 u}{\partial x\partial y} = u + p_1(y)\frac{\partial u}{\partial x} - \sin x - p_1(y)(\cos x + 1)\,, \tag{1.23_2}$$

$$\frac{\partial^2 u}{\partial x\partial y} = u + \frac{\partial u}{\partial x} - q(y)\,, \tag{1.23_3}$$

where $p_0(y)$, $p(y)$ and $q(y)$ are positive 2π-periodic and continuous but not differentiable functions, consider the periodic problem

$$u(x+2\pi,y)=u(x,y), \quad u(x,y+2\pi)=u(x,y). \tag{1.24}$$

It follows from Remark 2.1 below that each of the problems (1.23_j), (1.24) $(j=1,2,3)$ may have at most one solution and these solutions, respectively, must have the following forms

$$u_1(x,y)=v(y), \quad u_2(x,y)=c_0(y)+c_1(y)\sin x + c_2(y)\cos x, \quad u_3(x,y)=w(y)\,.$$

If we substitute them, respectively, into the equations (1.23_j) $(j=1,2,3)$, then we get

$$u_1(x,y)=\frac{1}{p_0(y)}, \quad u_2(x,y)=p_1(y)+\sin x, \quad u_3(x,y)=q(y)\,.$$

But it is impossible due to the fact that $p_0(y)$, $p_1(y)$ and $q(y)$ are not differentiable. Consequently, problems (1.23_j), (1.24) $(j=1,2,3)$ have no solution, although all conditions of Theorem 1.4, except the conditions of smoothness of $\mathcal{P}_j$ $(j=0,1,2)$ and q, are fulfilled.

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$$M_1(x)=\Theta \text{ for } x\in[0,\omega_1], \quad M_2(y)=\Theta \text{ for } y\in[0,\omega_2]. \tag{1.25}$$

Theorem 1.5. *Let conditions (1.25) hold, $\mathcal{P}_j \in C^1_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$ $(j=0,1,2)$ and inequalities (1.18), (1.22) take place. Then problem (0.1), (0.2) has property F in $C^1_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$.*

Proof. Let $q \in C^1_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$, $u(x,y)$ again be an arbitrary solution of problem (0.1), (0.2) and let

$$v(x,y) = \frac{\partial}{\partial x}\left(Z_2^{-1}(x,y)u(x,y)\right)\,,$$
$$w(x,y) = \frac{\partial}{\partial y}\left(Z_1^{-1}(x,y)u(x,y)\right)\,.$$

We have shown above the validity of equality (1.20). Therefore taking into the account (1.18), from (1.20) we obtain

$$u(x,y) = \int_y^{y+\omega_2} Q_1(x,y,t)w(x,t)\,dt + \varphi_1(x,y), \tag{1.26}$$

where

$$Q_1(x,y,t) = Z_1(x,y)\left(\int_0^{\omega_2} Z_1^{-1}(x,\tau)\bar{\Gamma}_1(x,\tau)Z_1(x,\tau)\,d\tau\right)^{-1}\times$$
$$\times\left(\int_y^t Z_1^{-1}(x,\tau)\bar{\Gamma}_1(x,\tau)Z_1(x,\tau)\,d\tau - Z_1^{-1}(x,t)\mathcal{P}_2(x,t)Z_1(x,t)\right)\,,$$
$$\varphi_1(x,y) = -Z_1(x,y)\left(\int_0^{\omega_2} Z_1^{-1}(x,\tau)\bar{\Gamma}_1(x,\tau)Z_1(x,\tau)\,d\tau\right)^{-1}\times$$
$$\times\int_0^{\omega_2} Z_1^{-1}(x,\tau)q(x,\tau)\,d\tau\,.$$

Similarly we obtain

$$u(x,y) = \int_x^{x+\omega_1} Q_2(x,y,s)v(s,y)ds + \varphi_2(x,y)\,, \tag{1.27}$$

where

$$Q_2(x,y,s) = Z_2(x,y)\left(\int_0^{\omega_1} Z_2^{-1}(\xi,y)\bar{\Gamma}_2(\xi,y)Z_2(\xi,y)\,d\xi\right)^{-1}\times$$
$$\times\left(\int_x^s Z_2^{-1}(\xi,y)\bar{\Gamma}_2(\xi,y)Z_2(\xi,y)\,d\xi - Z_2^{-1}(s,y)\mathcal{P}_1(s,y)Z_2(s,y)\right)\,,$$
$$\varphi_2(x,y) = -Z_2(x,y)\left(\int_0^{\omega_1} Z_2^{-1}(\xi,y)\bar{\Gamma}_2(\xi,y)Z_2(\xi,y)\,d\xi\right)^{-1}\times$$
$$\times\int_0^{\omega_1} Z_2^{-1}(\xi,y)q(\xi,y)\,d\xi\,.$$

Note that

$$\frac{\partial v(x,y)}{\partial y}=\frac{\partial}{\partial y}\left(Z_2^{-1}(x,y)\left(\frac{\partial u(x,y)}{\partial x}-\mathcal{P}_2(x,y)u(x,y)\right)\right)=$$
$$=Z_2^{-1}(x,y)\left(\mathcal{P}_1(x,y)Z_2(x,y)-\frac{\partial}{\partial y}Z_2(x,y)\right)v(x,y)+$$
$$+Z_2^{-1}(x,y)\left(\Gamma_2(x,y)u(x,y)+q(x,y)\right)$$

and, analogously,

$$\frac{\partial w(x,y)}{\partial x}=Z_1^{-1}(x,y)\left(\mathcal{P}_2(x,y)Z_1(x,y)-\frac{\partial}{\partial x}Z_1(x,y)\right)w(x,y)+$$
$$+Z_1^{-1}(x,y)\left(\Gamma_1(x,y)u(x,y)+q(x,y)\right).$$

Therefore from (1.26) and (1.27) we have

$$v(x,y)=\frac{\partial}{\partial x}\left(\int_y^{y+\omega_2}Z_2^{-1}(x,y)Q_1(x,y,t)w(x,t)\,dt+Z_2^{-1}(x,y)\varphi_1(x,y)\right)=$$
$$=\int_y^{y+\omega_2}\left(K_{11}(x,y,t)u(x,t)+K_{12}(x,y,t)w(x,t)\right)\,dt+\psi_1(x,y)\,, \tag{1.28}$$

$$w(x,y)=\frac{\partial}{\partial y}\left(\int_x^{x+\omega_1}Z_1^{-1}(x,y)Q_2(x,y,s)v(s,y)\,ds+Z_1^{-1}(x,y)\varphi_2(x,y)\right)=$$
$$=\int_x^{x+\omega_1}\left(K_{21}(x,y,s)u(s,y)+K_{22}(x,y,s)v(s,y)\right)\,ds+\psi_2(x,y)\,, \tag{1.29}$$

where

$$K_{11}(x,y,t)=Z_2^{-1}(x,y)Q_1(x,y,t)Z_1^{-1}(x,t)\Gamma_1(x,t)\,,$$
$$K_{12}(x,y,t)=\frac{\partial}{\partial x}\left(Z_2^{-1}(x,y)Q_1(x,y,t)\right)+Z_2^{-1}(x,y)Q_1(x,y,t)Z_1^{-1}(x,t)\times$$
$$\times\left(\mathcal{P}_2(x,t)Z_1(x,t)-\frac{\partial}{\partial x}Z_1(x,t)\right)\,,$$
$$K_{21}(x,y,s)=Z_1^{-1}(x,y)Q_2(x,y,s)Z_2^{-1}(s,y)\Gamma_2(s,y)\,,$$
$$K_{22}(x,y,s)=\frac{\partial}{\partial y}\left(Z_1^{-1}(x,y)Q_2(x,y,s)\right)+Z_1^{-1}(x,y)Q_2(x,y,s)Z_2^{-1}(s,y)\times$$
$$\times\left(\mathcal{P}_1(s,y)Z_2(s,y)-\frac{\partial}{\partial y}Z_2(s,y)\right)\,,$$
$$\psi_1(x,y)=Z_2^{-1}(x,y)\int_y^{y+\omega_2}Q_1(x,y,t)Z_1^{-1}(x,t)q(x,t)\,dt+\frac{\partial}{\partial x}\left(Z_2^{-1}(x,y)\varphi_1(x,y)\right)\,,$$
$$\psi_2(x,y)=Z_1^{-1}(x,y)\int_x^{x+\omega_1}Q_2(x,y,s)Z_2^{-1}(s,y)q(s,y)\,ds+\frac{\partial}{\partial y}\left(Z_1(x,y)\varphi_2(x,y)\right)\,.$$

If we now substitute (1.27) and (1.29) into (1.28) then we shall get

$$v(x,y) = \int_x^{x+\omega_1} \int_y^{y+\omega_2} K(x,y,s,t)v(s,t)\,ds\,dt + \psi(x,y)\,, \tag{1.30}$$

where

$$\begin{aligned} K(x,y,s,t) &= K_{11}(x,y,t)Q_2(x,t,s) + K_{12}(x,y,t)K_{22}(x,t,s) + \\ &+ \int_x^s K_{12}(x,y,t)K_{21}(x,t,\xi)Q_2(\xi,t,s)\,d\xi + \\ &+ \int_s^{x+\omega_1} K_{12}(x,y,t)K_{21}(x,t,\xi)Q_2(\xi,t,s+\omega_1)\,d\xi\,, \\ \psi(x,y) &= \int_y^{y+\omega_2} K_{11}(x,y,t)\varphi_2(x,t)\,dt + \int_y^{y+\omega_2} K_{12}(x,y,t)\psi_2(x,t)\,dt + \\ &+ \int_x^{x+\omega_1} \int_y^{y+\omega_2} K_{12}(x,y,t)K_{21}(x,t,s)\varphi_2(s,t)\,ds\,dt\,. \end{aligned}$$

Thus we have shown that if $u(x,y)$ is a solution of problem (0.1), (0.2), then $v(x,y) = \frac{\partial}{\partial x}\left(Z_2^{-1}(x,y)u(x,y)\right)$ is a solution of (1.30). The fact that (1.30) is the Fredholm equation in $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$ is rather obvious. It is easy to verify that if $v(x,y)$ is a solution of (1.30) then $u(x,y)$, defined by equality (1.27), is a solution of problem (0.1), (0.2). Consequently, problem (0.1), (0.2) is equivalent to (1.30). Therefore it has property F in $C^1_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$. □

Remark 1.3. The restriction, imposed on smoothness of $q(x,y)$ in Theorem 1.5 is optimal and it cannot be weakened. Indeed, for the equation

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}u - (q_1(x) + q_2(y))\,, \tag{1.31}$$

where $q_1(x)$ and $q_2(y)$ are 2π-periodic continuous, but not differentiable functions, consider the periodic problem

$$u(x+2\pi,y) = u(x,y), \quad u(x,y+2\pi) = u(x,y)\,. \tag{1.32}$$

It follows from Remark 2.1 below that problem (1.31), (1.32) can have at most one solution and its solution must be of the form

$$u(x,y) = u_1(x) + u_2(y)\,,$$

where $u_1(x)$ and $u_2(y)$ are some 2π-periodic continuously differentiable functions. But then we get that

$$u(x,y) = 2(q_1(x) + q_2(y))\,,$$

what is impossible due to the fact that $q_1(x)$ and $q_2(y)$ are not differentiable functions.

Thus we have shown that problem (1.31), (1.32) has no solution although the corresponding homogeneous problem has only the trivial solution.

§2. Existence and uniqueness theorems for problem (0.3), (0.2)

In Section 1 when proving the existence of property F of (0.1), (0.2), every time we were reducing problem (0.1), (0.2) to some Fredholm integral system and thus, we were connecting the unique solvability of our problem with the unique solvability of the above mentioned integral system. Therefore, of course, we can obtain various sufficient conditions of the unique solvability of problem (0.1), (0.2) imposing the well-known smallness conditions on the kernel of the integral system. But in this section we shall consider only the special case of problem (0.1), (0.2): problem (0.3), (0.2), since, the conditions of the unique solvability of problem (0.3), (0.2) have rather transparent form and they are unimprovable in a certain sense.

If for some $m \in \mathbb{Z}$

$$\det\left(i\frac{2\pi m}{\omega_1}E - \mathcal{P}_2(y)\right) \neq 0 \quad \text{for } y \in [0,\omega_2],$$

then by $W_m(y)$ denote a solution of the matrix differential equation

$$\frac{dW}{dy} = \left(i\frac{2\pi m}{\omega_1}E - \mathcal{P}_2(y)\right)^{-1}\left(\mathcal{P}_0(y) + i\frac{2\pi m}{\omega_1}\mathcal{P}_1(y)\right)W$$

satisfying the initial condition

$$W(0) = E\,.$$

Theorem 2.1. *Let for every $m \in \mathbb{Z}$*

$$\det\left(i\frac{2\pi m}{\omega_1}E - \mathcal{P}_2(y)\right) \neq 0 \text{ for } y \in [0,\omega_2] \tag{2.1}$$

and

$$\det(E - W_m(\omega_2)) \neq 0\,. \tag{2.2}$$

Then problem (0.3_0), (0.2) has only the trivial solution.

Proof. Let $u(x,y)$ be an arbitrary solution of problem (0.3_0), (0.2). Consider its associated Fourier series

$$\sum_{m\in\mathbb{Z}} c_m(y)\exp\left(i\frac{2\pi m}{\omega_1}x\right)$$

with continuous complex-valued vector coefficient $c_m(y)$ such that for every $y \in \mathbb{R}$

$$\lim_{k\to\infty}\int_0^{\omega_1}\left\| u(s,y) - \sum_{m=-k}^{k} c_m(y)\exp\left(i\frac{2\pi m}{\omega_1}s\right)\right\| ds = 0\,. \tag{2.3}$$

It is clear that (2.3) holds if and only if

$$c_m(y) = \frac{1}{\omega_1}\int_0^{\omega_1} u(s,y)\exp\left(-i\frac{2\pi m}{\omega_1}s\right) ds$$

for any $m \in \mathbb{Z}$. Let us denote this correspondence in the following way

$$u(x,y) \approx \sum_{m\in\mathbb{Z}} c_m(y)\exp\left(i\frac{2\pi m}{\omega_1}x\right).$$

Then it is clear that

$$\frac{\partial u(x,y)}{\partial x} \approx \sum_{m\in\mathbb{Z}} i\frac{2\pi m}{\omega_1}c_m(y)\exp\left(i\frac{2\pi m}{\omega_1}x\right),$$

$$\frac{\partial u(x,y)}{\partial y} \approx \sum_{m\in\mathbb{Z}} c'_m(y)\exp\left(i\frac{2\pi m}{\omega_1}x\right),$$

$$\frac{\partial^2 u(x,y)}{\partial x\partial y} \approx \sum_{m\in\mathbb{Z}} i\frac{2\pi m}{\omega_1}c'_m(y)\exp\left(i\frac{2\pi m}{\omega_1}x\right).$$

Therefore, again by virtue of uniqueness of the Fourier series with regard to conditions (2.1) we get

$$c'_m(y) = \left(i\frac{2\pi m}{\omega_1}E - \mathcal{P}_2(y)\right)^{-1}\left(\mathcal{P}_0(y) + i\frac{2\pi m}{\omega_1}\mathcal{P}_1(y)\right)c_m(y),$$

$$c_m(y+\omega_2) = c_m(y).$$

But conditions (2.2) yield that

$$c_m(y) \equiv 0 \text{ for } m \in \mathbb{Z}.$$

This means that $u(x,y) \equiv 0$. □

In the similar way we prove

Theorem 2.2. *Let $\mathcal{P}_2(y) \equiv \Theta$,*

$$\det\mathcal{P}_0(y) \neq 0 \quad \text{for } y \in [0,\omega_2]$$

and let conditions (2.2) hold for every $m \in \mathbb{Z}\backslash\{0\}$. Then problem ($0.3_0$), (0.2) has only the trivial solution.

Note that for system (0.3) we have

$$Z_1(x,y) \equiv Z_1(y), \quad M_1(x) \equiv M_1 = \text{ const},$$

$$Z_2(x,y) = \exp\left(x\mathcal{P}_2(y)\right), \quad M_2(y) = \exp\left(-\omega_1\mathcal{P}_2(y)\right) - E.$$

Corollary 2.1. *Let*

$$\det M_1 \neq 0 \tag{2.4}$$

and conditions (2.1) and (2.2) hold for any $m \in \mathbb{Z}$. Then problem (0.3), (0.2) is uniquely solvable for any $q \in C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$.

Proof. In fact, we have nothing to prove, since all conditions of Theorem 2.1 hold and inequality (2.4) and inequality (2.1) for $m = 0$ ensure that the conditions of Theorem 2.1 are satisfied. □

Analogously it can be verified that Corollaries 2.2, 2.3 and 2.4 below follow directly from Theorems 1.3 and 2.1, 1.4 and 2.2, 1.5 and 2.2, respectively.

Corollary 2.2. *Let* $M_1 = \Theta$,

$$\det \int_0^{\omega_2} Z_1^{-1}(t)\left(\mathcal{P}_0(t) + \mathcal{P}_2(t)\mathcal{P}_1(t)\right) Z_1(t)\, dt \neq 0 \tag{2.5}$$

and conditions (2.1) and (2.2) hold for any $m \in \mathbb{Z}$. *Then problem (0.3), (0.2) is uniquely solvable for any* $q \in C^{1,0}_{\omega_1 \omega_2}(\mathbb{R}^2;\mathbb{R}^n)$.

Corollary 2.3. *Let* $\det M_1 \neq 0$, $\mathcal{P}_0(y)$ *and* $\mathcal{P}_1(y)$ *be continuously differentiable and let all conditions of Theorem 2.2 hold. Then problem (0.3), (0.2) is uniquely solvable for any* $q \in C^{0,1}_{\omega_1 \omega_2}(\mathbb{R}^2;\mathbb{R}^n)$.

Corollary 2.4. *Let* $M_1 = \Theta$,

$$\det \int_0^{\omega_2} Z_1^{-1}(t)\mathcal{P}_0(t) Z_1(t)\, dt \neq 0\,,$$

$\mathcal{P}_0(y)$ *and* $\mathcal{P}_1(y)$ *be continuously differentiable and let all conditions of Theorem 2.2 hold. Then problem (0.3), (0.2) is uniquely solvable for any* $q \in C^{1}_{\omega_1 \omega_2}(\mathbb{R}^2;\mathbb{R}^n)$.

Remark 2.1. Note that if $u(x,y)$ is a solution of problem (0.3), (0.2) then:

a) for every $m \in \mathbb{Z}$ its Fourier coefficient $c_m(y)$ is a solution of the periodic problem

$$\left(i\frac{2\pi m}{\omega_1}E - \mathcal{P}_2(y)\right) c_m'(y) = \left(\mathcal{P}_0(y) + i\frac{2\pi m}{\omega_1}\mathcal{P}_1(y)\right) c_m(y) + q_{0m}(y)\,, \tag{2.6$_m$}$$

$$c_m(y+\omega_2) = c_m(y), \tag{2.7$_m$}$$

where

$$q_{0m}(y) = \frac{1}{\omega_2}\int_0^{\omega_2} q_0(s,y)\exp\left(-i\frac{2\pi m}{\omega_1}s\right)\, ds\,;$$

b) in view of the fact that $u(x,y)$ has continuous partial derivatives $\frac{\partial u(x,y)}{\partial x}$, $\frac{\partial u(x,y)}{\partial y}$, $\frac{\partial^2 u(x,y)}{\partial x \partial y}$ the Fourier series

$$\sum_{m\in\mathbb{Z}} c_m(y)\exp\left(i\frac{2\pi m}{\omega_1}x\right)$$

converge to $u(x,y)$ absolutely and uniformly in $\mathbb{R}^2$ (see [11]).

Therefore if the conditions of Corollary 2.k ($k = 1,2,3,4$) hold then the solution of problem (0.3), (0,2) has the form

$$u(x,y) = \sum_{m\in\mathbb{Z}} c_m(y)\exp\left(i\frac{2\pi m}{\omega_1}x\right)\,,$$

where for every $m \in \mathbb{Z}$ $c_m(y)$ is the solution of (2.6$_m$), (2.7$_m$).

Finally, let us study, separately, the case $n = 1$, i. e. when (0.3) is an equation. For the equation

$$\frac{\partial^2 u}{\partial x \partial y} = p_0(y)u + p_1(y)\frac{\partial u}{\partial x} + p_2(y)\frac{\partial u}{\partial y} + q(x,y)\,, \tag{2.8}$$

as well as for the homogeneous one

$$\frac{\partial^2 u}{\partial x \partial y} = p_0(y)u + p_1(y)\frac{\partial u}{\partial x} + p_2(y)\frac{\partial u}{\partial y} \tag{2.8$_0$}$$

consider the periodic problem

$$u(x+\omega_1, y) = u(x,y), \quad u(x, y+\omega_2) = u(x,y)\,, \tag{2.9}$$

where $p_j \in C_{\omega_2}(\mathbb{R};\mathbb{R})$ $(j=0,1,2)$ and $q \in C_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R})$.

Let $\eta \in C_{\omega_2}(\mathbb{R};\mathbb{R})$ be an arbitrary function. We shall make use of the following notation and definition:

$$I_\eta = \{y \in [0,\omega_2]:\ \eta(y) = 0\}, \quad J_\eta = [0,\omega_2]\backslash I_\eta\,,$$

$\bar{J}_\eta$ is the closure of J_η.

We say that a function $p \in C_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R})$ is *η-continuous* if it admits the representation

$$p(x,y) = \eta(y)\tilde{p}(x,y)\,, \tag{2.10}$$

where $\tilde{p} \in C_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R})$. I_η is a closed set (in view of continuity of $\eta(y)$) and therefore it is clear that the representation (2.10) is unique if and only if I_η is a nowhere dense set or, what is the same, $\bar{J}_\eta = [0,\omega_2]$.

By analogy with the case of system introduce the function

$$\mu(y) = \ \exp\left(-\omega_1 p_2(y)\right) - 1\,.$$

It is clear that $I_\mu = I_{p_2}$, $\mu(y)$ is p_2-continuous and vice versa, $p_2(y)$ is μ-continuous.

Theorem 2.3. *Let $\int_0^{\omega_2} p_1(t)\,dt \neq 0$, $p_0(y)$, $p_1(y)$ and $q(x,y)$ be μ-continuous,*

$$\bar{J}_\mu = [0,\omega_2] \tag{2.11}$$

and let for every $m \in \mathbb{Z}$ either

$$\int_0^{\omega_2} \frac{p_0(t)p_2(t) - \frac{4\pi^2 m^2}{\omega_1^2}p_1(t)}{p_2^2(t) + \frac{4\pi^2 m^2}{\omega_1^2}}\,dt \neq 0 \tag{2.12$_m$}$$

or

$$\frac{m}{\omega_1}\int_0^{\omega_2} \frac{p_0(t) + p_1(t)p_2(t)}{p_2^2(t) + \frac{4\pi^2 m^2}{\omega_1^2}}\,dt \notin \mathbb{Z}\,.$$

Then problem (2.8), (2.9) is uniquely solvable.

Proof. Let $u(x,y)$ be a solution of problem (2.8), (2.9). By virtue of μ-continuity of $p_0(y)$, $p_1(y)$ and $q(x,y)$ the following representation

$$\begin{aligned} u(x,y) = &\int_x^{x+\omega_1}\int_y^{y+\omega_2} K(x,y,s,t)\left(\tilde{p}_0(t) + \tilde{p}_1(t)p_2(t)\right)u(s,t)\,ds\,dt + \\ &+ \int_x^{x+\omega_1}\int_y^{y+\omega_2} K(x,y,s,t)\tilde{q}(s,t)\,ds\,dt \end{aligned} \tag{2.13}$$

is valid, where $p_j(y) = \mu(y)\tilde{p}_j(y)$, $(j = 0, 1)$, $q(x, y) = \mu(y)\tilde{q}(x, y)$ and

$$K(x, y, s, t) = \frac{\exp\left(\int_t^y p_1(\tau)\,d\tau + p_2(t)(x - s)\right)}{\exp\left(-\int_0^{\omega_2} p_1(\tau)\,d\tau\right) - 1}\,.$$

In view of condition (2.11), such representation is unique. It is clear also that every solution of integral equation (2.13) from $C_{\omega_1\omega_2}(\mathbb{R}^2; \mathbb{R})$ is a solution of problem (2.8), (2.9) too. Consequently, we have only to verify that the homogeneous equation

$$(2.13_0) \qquad u(x, y) = \int_x^{x+\omega_1} \int_y^{y+\omega_2} K(x, y, s, t)\,(\tilde{p}_0(t) + \tilde{p}_1(t)p_2(t))\,u(s, t)\,ds\,dt$$

has only the trivial solution in $C_{\omega_1\omega_2}(\mathbb{R}^2; \mathbb{R})$. But this is equivalent to the fact that for any $m \in \mathbb{Z}$ problem (2.6_m), (2.7_m) has only the trivial solution what follows immediately from conditions (2.11), (2.12_m). □

Theorem 2.4. *Let* $\int_0^{\omega_2} p_1(t)\,dt \neq 0$, $p_0(y)$, $p_1(y)$ *and* $q(x, y)$ *be* μ*-continuous,*

$$(2.14) \qquad \bar{J}_\mu \neq [0, \omega_2]$$

and let for every $m \in \mathbb{Z}\backslash\{0\}$ *condition* (2.12_m) *hold. Then problem (2.8), (2.9) is solvable; moreover, the corresponding homogeneous problem has the infinite dimensional set of solutions.*

Proof. By virtue of condition (2.14) there exist positive constants $\alpha < \beta$ such that $[\alpha, \beta] \subset I_\mu$. Introduce a ω_2-periodic function $\gamma(y)$ such that

$$\gamma(y) = \begin{cases} (y - \alpha)(\beta - y) & \text{for } y \in [\alpha, \beta] \\ 0 \text{ for} & y \in [0, \omega_2]\backslash[\alpha, \beta] \end{cases}.$$

Let $\tilde{p}_0(y)$ be a continuous ω_2-periodic function such that

$$(2.15) \qquad p_0(y) = \mu(y)\tilde{p}_0(y)\,.$$

Then by $\tilde{\tilde{p}}_0(y)$ denote a continuous ω_2-periodic function such that

$$\tilde{\tilde{p}}_0(y) = \begin{cases} \tilde{p}_0(y) & \text{for } y \in I_\mu \\ \tilde{p}_0(y)\frac{\mu(y)}{p_2(y)} & \text{for } y \in J_\mu \end{cases}.$$

It is obvious that if some $\tilde{p}_0(y)$ satisfies (2.15), then $\tilde{p}_0(y) + \lambda\gamma(y)$ satisfies also (2.15) for any $\lambda \in \mathbb{R}$. Therefore, we may chose $\tilde{p}_0(t)$ such that

$$(2.16) \qquad \int_0^{\omega_2} \tilde{\tilde{p}}_0(t)\,dt \neq 0\,.$$

Let us consider the integral equation (2.13), where $\tilde{p}_0(y)$ satisfies (2.15) and (2.16) and $\tilde{p}_1 \in C_{\omega_2}(\mathbb{R}; \mathbb{R})$ and $\tilde{q} \in C_{\omega_1\omega_2}(\mathbb{R}^2; \mathbb{R})$ are arbitrary functions such that

$$p_1(y) = \mu(y)\tilde{p}_1(y), \quad q(x, y) = \mu(y)\tilde{q}(x, y)\,.$$

In view of that every solution of (2.13) from $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R})$ is also a solution of the problem (2.8), (2.9), it is sufficient to verify that (2.13_0) has only trivial solution in $C_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R})$. But this is equivalent to the fact that the problem

$$c_0'(y) = \tilde{\tilde{p}}_0(y)c_0(y), \quad c_0(y+\omega_2) = c_0(y),$$

as well as the problem (2.6_m), (2.7_m), for every $m \in \mathbb{Z}\backslash\{0\}$, have only the trivial solution, what follows immediately from conditions (2.16) and (2.12_m) for $\mathbb{Z}\backslash\{0\}$.

Finally, note that for any $k \in \mathbb{N}$ the solution of the equation

$$\begin{aligned} u(x,y) = \int_x^{x+\omega_1}\int_y^{y+\omega_2} K(x,y,s,t)\left(\tilde{p}_0(t) + \tilde{p}_1(t)p_2(t)\right)u(s,t)\,ds\,dt + \\ + \int_x^{x+\omega_1}\int_y^{y+\omega_2} K(x,y,s,t)\gamma^k(t)\,ds\,dt \end{aligned}$$

is at the same time a solution of problem (2.8_0), (2.9). Consequently, problem (2.8_0), (2.9) has the infinitedimensional set of solutions. □

Theorems 2.3 and 2.4 yield the following

Corollary 2.5. *Let $p_0(y)$, $p_1(y)$ and $q(x,y)$ be μ-continuous,*

$$p_0(y) \geq 0, \quad p_1(y)p_2(y) \leq 0 \quad \text{for } y \in [0,\omega_2]$$

and

$$\int_0^{\omega_2} p_j(t)\,dt \neq 0 \quad (j = 0,1).$$

Then problem (2.8), (2.9) is solvable and its solution is unique if and only if

$$\bar{J}_\mu = [0,\omega_2]\,.$$

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