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SOME REMARKS ON QUATERNION–HERMITIAN MANIFOLDS

Andrew Swann

Abstract. Nearly-quaternionic Kähler manifolds of dimension at least 8 are shown to be quaternionic Kähler. Restrictions on the covariant derivative of the fundamental four-form of a semi-quaternionic Kähler are also found.

1. Introduction

A $4n$-dimensional manifold M is said to be *quaternion-Hermitian* if it has a reduction of its structure group to $Sp(n)\,Sp(1)$ and $n > 1$. Geometrically this means that M is equipped with a Riemannian metric g and a rank-three subbundle $\mathcal{G}$ of the endomorphism bundle $\operatorname{End} TM$ such that locally $\mathcal{G}$ has a basis I, J, K with $I^2 = J^2 = K^2 = -1$, $IJ = K = -JI$ and $g(AX, AY) = g(X, Y)$, for $A = I, J, K$. One thus has local 2-forms defined by $\omega_I(X, Y) = g(X, IY)$, etc., and this extends to a linear embedding of $\mathcal{G}$ into $\Lambda^2 T^*M$. One may define a global 4-form Ω, known as the fundamental 4-form, by the local formula

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K .$$

If Ω is parallel with respect to the Levi-Civita connection ∇ of g, then the holonomy group of M reduces to $Sp(n)\,Sp(1)$ and M is said to be *quaternionic Kähler*.

In [Sw1,3] the following result was proved:

Theorem 1.1.

(1) *A quaternion-Hermitian manifold of dimension* $4n \geqslant 12$ *is quaternionic Kähler if and only if* $d\Omega = 0$.

(2) *A quaternion-Hermitian* 8*-manifold is quaternionic Kähler if and only if* $d\Omega = 0$ *and* $d\mathcal{G} \subset \mathcal{G} \wedge T^*M$.

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The purpose of this note is to study two other conditions introduced in [I,M]. The first is the semi-quaternionic Kähler condition $d^*\Omega = 0$ and the second is the nearly-quaternionic Kähler condition $\nabla_X\Omega(X,Y,Z,W) = 0$, for all $X, Y, Z, W \in TM$. In dimension 8, [M] proved that nearly-quaternionic Kähler implies quaternionic Kähler. We extend this to show:

Theorem 1.2. *A nearly-quaternionic Kähler manifold of dimension $4n > 4$ is necessarily quaternionic Kähler.*

Our result for semi-quaternionic Kähler manifolds is a little more technical and will be found in the next section. The proof of Theorem 1.2 will be found in the last section.

Note that because our techniques are only based on complex representation theory, all of our results also apply to the case of an indefinite metric and structure group $Sp(p,q)\,Sp(1)$.

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2. Representation Theory

We briefly recall the notation of [Sa,Sw1–3] and refer the reader to [BtD] for general information on representation theory. Let E be the fundamental representation of $Sp(n)$ on $\mathbb{C}^{2n} \cong \mathbb{H}^n$ via left multiplication by quaternionic matrices and let H be the representation of $Sp(1)$ on $\mathbb{C}^2 \cong \mathbb{H}$ give by $q \cdot \xi = \xi\bar{q}$, for $q \in Sp(1)$ and $\xi \in \mathbb{H}$. An $Sp(n)\,Sp(1)$-structure on a manifold gives a decomposition

$$TM \otimes \mathbb{C} \cong E \otimes_{\mathbb{C}} H,$$

and this may be used to obtain decompositions of, for example, other parts of the exterior algebra [Sa,Sw2].

Let $\{e_1, \dots, e_n, \tilde{e}_1, \dots, \tilde{e}_n\}$ be a complex orthonormal basis for $E \cong \mathbb{C}^{2n}$ with $\tilde{e}_i = je_i$. We have an $Sp(n)$-invariant complex symplectic form ω_E on E given by

$$\omega_E = e_i \wedge \tilde{e}_i = e_i\tilde{e}_i - \tilde{e}_i e_i,$$

where we have used the summation convention and omitted tensor product signs. Similarly, H has a basis $\{h, \tilde{h}\}$ and symplectic form ω_H.

Note that $Sp(1)$ is isomorphic to $SU(2)$, so the irreducible representations are precisely the symmetric powers $S^kH \cong \mathbb{C}^{k+1}$. An irreducible representation of $Sp(n)$ is determined by its dominant weight $(\lambda_1, \dots, \lambda_n)$, where λ_i are integers with $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n \geqslant 0$. This representation will be denoted $V^{(\lambda_1,\dots,\lambda_r)}$, where r is the largest integer such that $\lambda_r > 0$. Certain of these modules are well-known and we will use familiar notation for these modules. Thus, $V^{(k)}$ is the symmetric power S^kE and $V^{(1\dots1)}$ (r ones) is Λ^r_0E, the largest irreducible summand of Λ^rE, which decomposes as

$$\begin{aligned}\Lambda^rE &= \Lambda^r_0E + \omega_E\wedge\Lambda^{r-2}_0E + \omega_E^2\wedge\Lambda^{r-4}_0E + \cdots\\ &\cong \Lambda^r_0E + \Lambda^{r-2}_0E + \Lambda^{r-4}_0E + \cdots .\end{aligned}$$

Also K will denote the module $V^{(21)}$, which arises in the decomposition

$$E\otimes\Lambda^2_0E \cong \Lambda^3_0E + K + E.$$

As a further convention, we will regard any module with dominant weight of length greater than n as the zero-module $\{0\}$ and often will omit tensor product signs between representations.

We will need the following facts.

Proposition 2.1. [Sa,Sw1] *If M has dimension at least 8, then*

$$\Lambda^3_{\mathbb{C}}T^*M \cong (\Lambda^3_0E + E)S^3H + (K+E)H.$$

When M has dimension 8, then $\Lambda^5T^*M \cong \Lambda^3T^*M$, whereas if M has dimension at least 12, we have

$$\begin{aligned}\Lambda^5_{\mathbb{C}}T^*M \cong{}& (\Lambda^5_0E + \Lambda^3_0E + E)S^5H + (V^{(2111)} + K + \Lambda^3_0E + E)S^3H\\ &+ (V^{(221)} + K + \Lambda^3_0E + E)H.\end{aligned}$$

Proposition 2.2. [Sw1] *The covariant derivative of the fundamental 4-form has the property that*

$$\nabla\Omega \in EH\otimes(\Lambda^2_0ES^2H) \cong (K + \Lambda^3_0E + E)(S^3H + H).$$

The first part of Theorem 1.1 is now proved by noting that all the above summands occur in the decomposition of Λ^5T^*M, and then by showing that the alternation map $T^*\otimes\Lambda^4T^*\to\Lambda^5T^*$ is non-zero on each summand. This is sufficient, since Schur's Lemma states that a non-zero equivariant map between irreducible modules is an isomorphism. In dimension 8, one has $\nabla\Omega\in(K+E)(S^3H+H)$, but the summand KS^3H does not occur in the decomposition of Λ^5T^*M. In this case one merely concludes that $d\Omega=0$ implies $\nabla\Omega\in KS^3H$.

Theorem 2.3. *On a quaternion-Hermitian manifold of dimension $4n\geqslant 8$, the semi-quaternionic Kähler condition $d^*\Omega=0$ implies*

$$\nabla\Omega\in KS^3H + \Lambda^3_0EH.$$

Proof. By definition $d^*\Omega = -*d*\Omega$. Since Ω^n is a non-zero constant multiple of the volume, one has that $*\Omega$ is a constant times Ω^{n-1}. Thus $d*\Omega$ is a non-zero constant multiple of $\Omega^{n-2}\wedge d\Omega$. Now Bonan [B] showed that the map

$$\Lambda^5T^* \xrightarrow{\wedge\Omega^{n-2}} \Lambda^{4n-3}T^* \xrightarrow{\quad *\quad} \Lambda^3T^*$$

is surjective. However this map is also $Sp(n)\,Sp(1)$-equivariant, so the vanishing of $d*\Omega$ is equivalent to the vanishing of all components of $d\Omega$ lying in summands of Λ^3T^*M. However, by Theorem 1.1 and the above discussion, these are precisely the summands of $\nabla\Omega$ which lie in Λ^3T^*M. The result follows by comparing the decompositions in Propositions 2.1 and 2.2. $\square$

3. Nearly-quaternionic Kähler Manifolds

The aim of this section is to prove Theorem 1.2. The tensor $\nabla_X\Omega(X,Y,Z,W)$ is just the image of $\nabla\Omega$ under the symmetrisation map

$$\mathbf{s}\colon T^*\otimes\Lambda^4T^*\longrightarrow S^2T^*\otimes\Lambda^3T^*.$$

The decomposition of $S^2T^*\otimes\Lambda^3T^*$ as a sum of $Sp(n)\,Sp(1)$-modules is not really required for the proof, only the observation that it contains several copies of each of the summands occurring in the module in which $\nabla\Omega$ lies, however for completeness we state:

Proposition 3.1. *The module $S^2T^*\otimes\Lambda^3T^*$ decomposes as follows:*
Case 1: *if* $\dim M\geqslant 16$, *then*

$$\begin{aligned}S^2T^*\otimes\Lambda^3T^* &= \bigl(V^{(311)}+V^{(2111)}+S^3E+2K+\Lambda^3_0E+E\bigr)S^5H\\ &= \bigl(V^{(41)}+V^{(32)}+2V^{(311)}+2V^{(221)}+2V^{(2111)}\\ &\qquad +\Lambda^5_0E+3S^3E+7K+5\Lambda^3_0E+6E\bigr)S^3H\\ &= \bigl(V^{(41)}+2V^{(32)}+3V^{(311)}+2V^{(221)}+2V^{(2111)}\\ &\qquad +4S^3E+9K+4\Lambda^3_0E+6E\bigr)H;\end{aligned}$$

Case 2: *if* $\dim M=12$, *then*

$$\begin{aligned}S^2T^*\otimes\Lambda^3T^* &= \bigl(V^{(311)}+S^3E+2K+\Lambda^3_0E+E\bigr)S^5H\\ &= \bigl(V^{(41)}+V^{(32)}+2V^{(311)}+2V^{(221)}\\ &\qquad +3S^3E+7K+4\Lambda^3_0E+6E\bigr)S^3H\\ &= \bigl(V^{(41)}+2V^{(32)}+3V^{(311)}+2V^{(221)}\\ &\qquad +4S^3E+9K+4\Lambda^3_0E+6E\bigr)H;\end{aligned}$$

Case 3: *if* $\dim M=8$, *then*

$$\begin{aligned}S^2T^*\otimes\Lambda^3T^* &= \bigl(S^3E+K+E\bigr)S^5H\\ &= \bigl(V^{(41)}+V^{(32)}+3S^3E+5K+5E\bigr)S^3H\\ &= \bigl(V^{(41)}+V^{(32)}+4S^3E+7K+6E\bigr)H.\end{aligned}$$

To complete the proof of Theorem 1.2 we exhibit an element β of $T^*\otimes\Lambda^2_0ES^H$ with the following property: for each irreducible summand W of $T^*\otimes\Lambda^2_0ES^2H$, the image $\mathbf{s}(\beta)$ has a non-zero component in some summand of $S^2T^*\otimes\Lambda^3T^*$ isomorphic to W.

Define

$$\alpha_{12}=e_1h\wedge e_2h\wedge e_i\tilde h\wedge\tilde e_i\tilde h-e_1\tilde h\wedge e_2\tilde h\wedge e_ih\wedge\tilde e_ih\in\Lambda^4T^*.$$

Since the symplectic form ω_E is invariant, the map obtained by evaluating ω_E on the last two components of an element of $\Lambda^4 T^*$ is equivariant. Applying this map to α_{12} gives

$$\begin{aligned}
&2ne_1h \wedge e_2h \otimes \tilde{h}\tilde{h} - e_1h \wedge e_2\tilde{h} \otimes (h\tilde{h} + \tilde{h}h) + e_1\tilde{h} \wedge e_2h \otimes (h\tilde{h} + \tilde{h}h)\\
&- 2ne_1\tilde{h} \wedge e_2\tilde{h} \otimes hh - e_1\tilde{h} \wedge e_2h \otimes (\tilde{h}h + h\tilde{h}) + e_1h \wedge e_2\tilde{h} \otimes (\tilde{h}h + h\tilde{h})\\
&= 2ne_1 \wedge e_2(hh\tilde{h}\tilde{h} - \tilde{h}\tilde{h}hh)\\
&\in \Lambda^2_0 E \otimes \Lambda^2(S^2H) \cong \Lambda^2_0 ES^2H,
\end{aligned}$$

and shows that $\alpha_{12} \in \Lambda^2_0 ES^2H$.

We define $\beta = \tilde{e}_1h \otimes \alpha_{12}$. The image $\mathbf{s}(\beta) \in S^2T^* \otimes \Lambda^3 T^*$ is then given by

$$\begin{aligned}
\mathbf{s}(\beta) = {}&(\tilde{e}_1h \vee e_1h) \otimes (e_2h \wedge e_i\tilde{h} \wedge \tilde{e}_i\tilde{h}) - (\tilde{e}_1h \vee e_1\tilde{h}) \otimes (e_2\tilde{h} \wedge e_ih \wedge \tilde{e}_ih)\\
&- (\tilde{e}_1h \vee e_2h) \otimes (e_1h \wedge e_i\tilde{h} \wedge \tilde{e}_i\tilde{h}) + (\tilde{e}_1h \vee e_2\tilde{h}) \otimes (e_1\tilde{h} \wedge e_ih \wedge \tilde{e}_ih)\\
&+ (\tilde{e}_1h \vee e_i\tilde{h}) \otimes (e_1h \wedge e_2h \wedge \tilde{e}_i\tilde{h}) - (\tilde{e}_1h \vee e_ih) \otimes (e_1\tilde{h} \wedge e_2\tilde{h} \wedge \tilde{e}_ih)\\
&- (\tilde{e}_1h \vee \tilde{e}_i\tilde{h}) \otimes (e_1h \wedge e_2h \wedge e_i\tilde{h}) + (\tilde{e}_1h \vee \tilde{e}_ih) \otimes (e_1\tilde{h} \wedge e_2\tilde{h} \wedge e_ih).
\end{aligned} \tag{3.1}$$

Apply ω_E to the last two places to get

$$\begin{aligned}
&(\tilde{e}_1h \vee e_1h)[2ne_2h\tilde{h}^2 - e_2\tilde{h}(h\tilde{h} + \tilde{h}h)] - (\tilde{e}_1h \vee e_1\tilde{h})[2ne_2\tilde{h}h^2 - e_2h(h\tilde{h} + \tilde{h}h)]\\
&- (\tilde{e}_1h \vee e_2h)[2ne_1h\tilde{h}^2 - e_1\tilde{h}(h\tilde{h} + \tilde{h}h)] + (\tilde{e}_1h \vee e_2\tilde{h})[2ne_1\tilde{h}h^2 - e_1h(h\tilde{h} + \tilde{h}h)]\\
&+ (\tilde{e}_1h \vee e_2\tilde{h})e_1h(h\tilde{h} + \tilde{h}h) - (\tilde{e}_1h \vee e_1\tilde{h})e_2h(h\tilde{h} + \tilde{h}h)\\
&- (\tilde{e}_1h \vee e_2h)e_1\tilde{h}(\tilde{h}h + h\tilde{h}) + (\tilde{e}_1h \vee e_1h)e_2\tilde{h}(\tilde{h}h + h\tilde{h}).
\end{aligned}$$

Now contract with ω_H on the third and fifth terms, collect terms and write out the symmetric products in full:

$$\begin{aligned}
2n\big\{&hh\tilde{h}(\tilde{e}_1e_1e_2 + e_1\tilde{e}_1e_2 - \tilde{e}_1e_2e_1 - e_2\tilde{e}_1e_1)\\
&+ h\tilde{h}h(\tilde{e}_1e_1e_2 - \tilde{e}_1e_2e_1)\\
&+ \tilde{h}hh(e_1\tilde{e}_1e_2 - e_2\tilde{e}_1e_1)\big\}.
\end{aligned}$$

Applying ω_H to the first two places gives

$$2nh(\tilde{e}_1e_1e_2 - \tilde{e}_1e_2e_1 - e_1\tilde{e}_1e_2 + e_2\tilde{e}_1e_1). \tag{3.2}$$

When we map this element to $\Lambda^3 EH$ we obtain $2nh(\tilde{e}_1 \wedge e_1 \wedge e_2 - e_1 \wedge \tilde{e}_1 \wedge e_2)$, which has non-zero contraction with ω_E on the first two components, so gives a non-zero element in EH. However, it is not a multiple of ω_E, so it is also non-zero in $\Lambda^3_0 EH$. Furthermore, (3.2) is not in $\Lambda^3 E$, so it has a non-zero projection to KH. Similarly, by symmetrising the H's we obtain non-zero elements in KS^3H and ES^3H. The element (3.2) happens to be zero in $\Lambda^3_0 ES^3H$, but we may apply other contractions to $\mathbf{s}(\beta)$ to obtain this component: in (3.1), contracting with ω_H on the first two indices and symmetrising the last three H's we obtain

$$\begin{aligned}(h^2 \vee \tilde{h})(-\tilde{e}_1 \wedge e_1 \otimes e_2 \wedge e_i \wedge \tilde{e}_i + \tilde{e}_i \wedge e_2 \otimes e_1 \wedge e_i \wedge \tilde{e}_i \\ + \tilde{e}_1 \wedge e_i \otimes e_1 \wedge e_2 \wedge \tilde{e}_i - \tilde{e}_1 \wedge \tilde{e}_i \otimes e_1 \wedge e_2 \wedge e_i)\end{aligned}$$

Contracting with ω_E in the second and third places and mapping to $\Lambda^3 ES^3 H$ gives

$$-(\tilde{e}_1 \wedge e_2 \wedge e_1 + e_2 \wedge e_i \wedge \tilde{e}_i)(h^2 \vee \tilde{h}),$$

which is non-zero in both $\Lambda^3_0 ES^3 H$ and $ES^3 H$. This completes the proof.

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School of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY, UNITED KINGDOM
E-email: A.F.swann@maths.bath.ac.uk