ARCHIVUM MATHEMATICUM (BRNO)
Tomus 33 (1997), 335 – 348

ON A CRITERION FOR THE EXISTENCE OF AT LEAST FOUR SOLUTIONS OF FUNCTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. A class of functional boundary conditions for the second order functional differential equation $x''(t) = (Fx)(t)$ is introduced. Here $F : C^1(J) \to L_1(J)$ is a nonlinear continuous unbounded operator. Sufficient conditions for the existence of at least four solutions are given. The proofs are based on the Bihari lemma, the topological method of homotopy, the Leray-Schauder degree and the Borsuk theorem.

1. Introduction, notation

Let $\mathbf{X}$ be the Banach space of continuous functions on a compact interval $J = [a,b]$ with the norm $\|x\|_0 = \max\{|x(t)| : a \le t \le b\}$ and L_1 (resp. $AC^1(J)$; $\mathbf{Y}$) be the Banach space of Lebesgue integrable functions on J (resp. functions with absolutely continuous derivative on J; C^1-functions on J) with the usual norm $\|x\|_{L_1} = \int_a^b |x(t)|\,dt$ (resp. $\|x\|_{AC^1} = \|x\|_0 + \|x'\|_0 + \|x''\|_{L_1}$; $\|x\|_1 = \|x\|_0 + \|x'\|_0$).

Denote by $\mathcal{A}$ the set of all functionals $\varrho : \mathbf{X} \to \mathbf{R}$ that are

(i) continuous,
(ii) $\varrho(x) = \varrho(|x|)$ for $x \in \mathbf{X}$,
(iii) $\lim\limits_{u \in R, u \to \infty} \varrho(u) = \infty$ (we identificate the subset of $\mathbf{X}$ of constant functions with $\mathbf{R}$)

and

(iv) $x, y \in \mathbf{X}$, $|x(t)| < |y(t)|$ for $t \in J \Rightarrow \varrho(x) < \varrho(y)$.

Set $\mathcal{A}_0 = \Big\{\varrho : \varrho \in \mathcal{A},\ \varrho(0) = 0\Big\}$. For any $\varphi : \mathbf{X} \to \mathbf{R}$, $\mathrm{Im}(\varphi)$ denotes the range of φ.

Remark 1. The sets $\mathcal{A}$ and $\mathcal{A}_0$ were stated on formulations of some functional boundary value conditions in [17] for the first time. Observe that properties (i), (iii) and (iv) of the set $\mathcal{A}$ do not imply property (ii) (see Example 3, [17]).

1991 *Mathematics Subject Classification*: 34K10.
Key words and phrases: functional boundary conditions, functional differential equation, existence, multiplicity, Bihari lemma, homotopy, Leray Schauder degree, Borsuk theorem.
Received October 24, 1996.

Example 1. Let $p \in C^0([0,\infty))$ be increasing on $[0,\infty)$ and $\lim_{u\to\infty} p(u) = \infty$. Set $\varrho(x) = \int_a^b p(|x(t)|)\,dt$ for $x \in \mathbf{X}$. Then $\varrho \in \mathcal{A}$. Equation $\varrho(x) = A$ was used by Brykalov [7] as a boundary condition. Next functionals belonging to the set $\mathcal{A}$ can be given like this:

$$\max\Big\{|x(t)| : t \in J_1\Big\}, \quad \int_{a^*}^{b^*} q(t) \max\{|x(s)| : t \le s \le b^*\}\,dt,$$

$$\min\Big\{|x(t)| : t \in J_1\Big\}, \quad \sum_{i=1}^{n} a_i |x(t_i)|,$$

where $J_1 \subset J$ is a compact interval, $a \le a^* < b^* \le b$, $q \in C^0([a^*, b^*])$ positive, $a_i > 0$ for $i = 1, 2, ..., n$ and $a \le t_1 < t_2 < \dots < t_n \le b$.

Let $F : \mathbf{Y} \to L_1(J)$ be a continuous operator, ω, $\gamma \in \mathcal{A}$. In the present paper we consider the functional boundary value problem (BVP for short)

$$x''(t) = (Fx)(t), \tag{1}$$

$$\omega(x) = A, \tag{2}$$

$$\gamma(x') = B, \tag{3}$$

where ω, $\gamma \in \mathcal{A}$ and $A, B \in \mathbf{R}$.

A function $x \in AC^1(J)$ is said to be a *solution of BVP* (1)–(3) if x satisfies boundary conditions (2), (3) and equation (1) is satisfied for a.e. $t \in J$.

The aim of this paper is to give sufficient conditions for the existence of at least four solutions x_i $(i = 1, 2, 3, 4)$ of BVP (1)–(3) satisfying the inequalities

$$\begin{aligned} &x_1(t) > 0, \quad x_1'(t) > 0; \quad x_2(t) > 0,\ x_2'(t) < 0;\\ &x_3(t) < 0, \quad x_3'(t) > 0; \quad x_4(t) < 0,\ x_4'(t) < 0 \end{aligned} \tag{4}$$

for $t \in J$. The results are proved by the homotopy, the Leray-Schauder degree theory and the Borsuk theorem (see, e.g., [8], [11]).

We refer that there are many papers devoted to the existence of multiplicity results for ordinary differential equations and functional differential equations that have started by Ambrosetti and Prodi multiplicity results in [1]. A lot of results have been obtained for ordinary differential equations in [1], [12], [13], [15] and references cited therein and others (usually with periodic or Neumann or Dirichlet boundary conditions) and for functional differential equations with functional nonlinear boundary conditions in [5]–[7], [14], [16]–[18] and the references therein. Interesting results for BVPs with finitely many solutions one can find for instance in [4], [9] and [19]. Recall that a nontraditional approach to functional differential equation is given in the remarkable monograph [2].

In connection with multiply solutions we refer to Brykalov [5]. His results concern the functional differential equation $x^{(n)}(t) = (F_1 x)(t)$ with functional nonlinear boundary conditions. Here $F_1 : C^{n-1}(J) \to L_1(J)$ is continuous and bounded. Results are proved by the Schauder fixed point theorem in cones. From the corollary in [5] it follows the following proposition.

Proposition 1. *Let f satisfy the Carathéodory conditions on $J \times \boldsymbol{R}^2$ and*

$$|f(t,u,v)| \le \alpha(t)$$

for a.e. $t \in J$ and each $u, v \in \boldsymbol{R}$, where $\alpha \in L_1(J)$. Then BVP

$$x'' = f(t,x,x'), \quad \|x\|_0 = A, \quad \|x'\|_0 = B$$

has at least four different solutions provided

$$\frac{1}{2}\int_a^b \alpha(t)\,dt < B, \quad \frac{B(b-a)}{2} < A.$$

In our paper we use the well known Bihari lemma (see, e.g., [3], [10]) in the following form.

Lemma 1. (Bihari lemma) *Let $q \in L_1(J)$, $f : [0,\infty) \to (0,\infty)$ be a nondecreasing function, $\int_0^\infty \frac{dt}{f(t)} = \infty$, $\xi \in J$, $k \in \boldsymbol{R}$, $k \ge 0$. Let $w \in C^0(J)$ satisfy the inequality*

$$|w(t)| \le k + \Big|\int_\xi^t |q(s)| f(|w(s)|)\,ds\Big|$$

for $t \in J$. Then

$$|w(t)| \le G^{-1}(G(k) + \|q\|_{L_1})$$

for $t \in J$, where G^{-1} means the inverse function to $G : [0,\infty) \to \boldsymbol{R}$,

$$G(u) = \int_0^u \frac{ds}{f(s)}. \tag{5}$$

2. Lemmas

Lemma 2. ([17]). *Let $\varrho \in \mathcal{A}$, $B \in Im(\varrho)$. Then*

(a) $x, y \in \mathbf{X}$, $|x(t)| \le |y(t)|$ for $t \in J \Rightarrow \varrho(x) \le \varrho(y)$,
(b) $\varrho(0) \le \varrho(x)$ for $x \in \mathbf{X}$

and

(c) there exists a unique nonnegative constant d such that $\varrho(d) = B$.

Lemma 3. ([17]) *Let $\varrho \in \mathcal{A}$ and $\varrho(x) = \varrho(y)$ for some $x, y \in \mathbf{X}$. Then there exists a $\xi \in J$ such that $|x(\xi)| = |y(\xi)|$.*

Corollary 1. *Let $\varrho \in \mathcal{A}_0$ and $\varrho(x) = 0$ for an $x \in \mathbf{X}$. Then there exists a $\xi \in J$ such that $x(\xi) = 0$.*

Lemma 4. *Let $\varrho \in \mathcal{A}$ and $\varrho(x) \leq \varrho(y)$ for some $x, y \in \mathbf{X}$. Then there exists a $\xi \in J$ such that $|x(\xi)| \leq |y(\xi)|$.*

Proof. Assume, on the contrary, that $|x(t)| > |y(t)|$ for $t \in J$. Then $\varrho(x) > \varrho(y)$, a contradiction. □

For each $\varrho \in \mathcal{A}$, define (cf. property (iii) of the set $\mathcal{A}$) $q_\varrho : [0, \infty) \to \mathbf{R}$ by the formula

$$q_\varrho(c) = \varrho(c). \tag{6}$$

Lemma 5. *For each $\varrho \in \mathcal{A}$, q_ϱ is a continuous increasing function mapping $[0, \infty)$ onto $[\varrho(0), \infty)$.*

Proof. By properties (i) and (iv) of the set $\mathcal{A}$, q_ϱ is continuous and increasing on $[0, \infty)$. From (iii) and Lemma 2 it follows that q_ϱ maps $[0, \infty)$ onto $[\varrho(0), \infty)$. □

For each $x \in \mathbf{X}$, define x_+, $x_- \in \mathbf{X}$ by the formulas

$$x_+(t) = \begin{cases} x(t) & \text{for } x(t) \geq 0 \\ 0 & \text{for } x(t) < 0, \end{cases} \qquad x_-(t) = \begin{cases} 0 & \text{for } x(t) \geq 0 \\ -x(t) & \text{for } x(t) < 0. \end{cases} \tag{7}$$

Then $x = x_+ - x_-$.

For each $\varphi : \mathbf{X} \to \mathbf{R}$, define $\varphi_+, \varphi_- : \mathbf{X} \to \mathbf{R}$ by

$$\varphi_+(x) = \varphi(x_+), \quad \varphi_-(x) = \varphi(x_-).$$

Lemma 6. *Let $\varrho \in \mathcal{A}$. Then ϱ_+ and ϱ_- are continuous functionals.*

Proof. Let $\{x_n\} \subset \mathbf{X}$ be a convergent sequence, $\lim_{n\to\infty} x_n = x$. Then

$$\lim_{n\to\infty} (x_n)_+ = x_+, \quad \lim_{n\to\infty} (x_n)_- = x_-.$$

As ϱ is continuous, we have

$$\lim_{n\to\infty} \varrho_+(x_n) = \lim_{n\to\infty} \varrho((x_n)_+) = \varrho(x_+) = \varrho_+(x),$$

and similarly $\lim_{n\to\infty} \varrho_-(x_n) = \varrho_-(x)$. □

We now state the following important lemma:

Lemma 7. *Let $\omega, \gamma \in \mathcal{A}_0$ and r, k, l, K be positive constants, $K > k$. Set*

$$\begin{aligned} \Omega \;\; = \;\; & \Big\{(x,\alpha,\beta): \; (x,\alpha,\beta) \in AC^1(J) \times \mathbf{R}^2, \, \|x\|_0 < r + K(b-a), \\ & \|x'\|_0 < K, \; \|x''\|_{L_1} < l, \; |\alpha| < r + K(b-a), \, |\beta| < K\Big\}. \end{aligned}$$

Let $\Gamma_i : \bar{\Omega} \to AC^1(J) \times \mathbf{R}^2$ $(i = 1, 2, 3, 4)$ be given by

$$\Gamma_1(x,\alpha,\beta) = \Big(\alpha + \beta(t-a), \, \alpha + \omega(x_+) - \omega(r), \, \beta + \gamma(x'_+) - \gamma(k)\Big),$$

$$\Gamma_2(x,\alpha,\beta) = \Big(\alpha + \beta(t-a), \, \alpha + \omega(x_+) - \omega(r), \, \beta - \gamma(x'_-) + \gamma(k)\Big),$$

$$\Gamma_3(x,\alpha,\beta) = \Big(\alpha + \beta(t-a), \, \alpha - \omega(x_-) + \omega(r), \, \beta + \gamma(x'_+) - \gamma(k)\Big)$$

and

$$\Gamma_4(x,\alpha,\beta) = \Big(\alpha + \beta(t-a), \, \alpha - \omega(x_-) + \omega(r), \, \beta - \gamma(x'_-) + \gamma(k)\Big).$$

Then

$$D(I - \Gamma_i, \Omega, 0) \neq 0 \quad \textit{for } i = 1, 2, 3, 4. \tag{8}$$

Here "D" denotes the Leray-Schauder degree and I is the identity operator on $AC^1(J) \times \mathbf{R}^2$.

Proof. We first see that Ω is an open bounded and symmetric with respect to $0 \in \Omega$ subset of the Banach space $AC^1(J) \times \mathbf{R}^2$ with the usual norm. Moreover, $\omega(r) > 0$ and $\gamma(k) > 0$ since $\gamma, \omega \in \mathcal{A}_0$ and $r > 0, \; k > 0$. Let (for $i = 1, 2, 3, 4$)

$$H_i : [0,1] \times \bar{\Omega} \to AC^1(J) \times \mathbf{R}^2$$

be defined by

$$\begin{aligned} H_1(\lambda, x, \alpha, \beta) \;\; &= \;\; \Big(\alpha + \beta(t-a), \alpha + \omega(x_+) - \omega((1-\lambda)x_-) - \lambda\omega(r), \\ & \qquad \beta + \gamma(x'_+) - \gamma((1-\lambda)x'_-) - \lambda\gamma(k)\Big), \\ H_2(\lambda, x, \alpha, \beta) \;\; &= \;\; \Big(\alpha + \beta(t-a), \alpha + \omega(x_+) - \omega((1-\lambda)x_-) - \lambda\omega(r), \\ & \qquad \beta - \gamma(x'_-) + \gamma((1-\lambda)x'_+) + \lambda\gamma(k)\Big), \\ H_3(\lambda, x, \alpha, \beta) \;\; &= \;\; \Big(\alpha + \beta(t-a), \alpha - \omega(x_-) + \omega((1-\lambda)x_+) + \lambda\omega(r), \\ & \qquad \beta + \gamma(x'_+) - \gamma((1-\lambda)x'_-) - \lambda\gamma(k)\Big), \\ H_4(\lambda, x, \alpha, \beta) \;\; &= \;\; \Big(\alpha + \beta(t-a), \alpha - \omega(x_-) + \omega((1-\lambda)x_+) + \lambda\omega(r), \\ & \qquad \beta - \gamma(x'_-) + \gamma((1-\lambda)x'_+) + \lambda\gamma(k)\Big). \end{aligned}$$

To prove (8) it is sufficient to show, by the homotopy theory and the Borsuk theorem (see, e.g., [8], [11]), that (for $i = 1, 2, 3, 4$)

(a) $H_i(0,\cdot,\cdot,\cdot)$ is an odd operator on $\bar{\Omega}$; that is,

$$H_i(0,-x,-\alpha,-\beta) = -H_i(0,x,\alpha,\beta)$$

for $(x,\alpha,\beta) \in \bar{\Omega}$,

(b) H_i is a compact operator, and

(c) $H_i(\lambda,x,\alpha,\beta) \neq (x,\alpha,\beta)$ for $(\lambda,x,\alpha,\beta) \in [0,1]\times\partial\Omega$.

We prove, for instance, $D(I-\Gamma_i,\Omega,0) \neq 0$ for $i=4$. The case where $i \in \{1,\,2,\,3\}$ treats similarly.

Fix $(x,\alpha,\beta) \in \bar{\Omega}$. Then

$$\begin{aligned}
&H_4(0,-x,-\alpha,-\beta)\\
&= \Big(-\alpha-\beta(t-a), -\alpha-\omega(x_+)+\omega(x_-), -\beta-\gamma(x'_+)+\gamma(x'_-)\Big)\\
&= -\Big(\alpha+\beta(t-a), \alpha+\omega(x_+)-\omega(x_-), \beta+\gamma(x'_+)-\gamma(x'_-)\Big)\\
&= -H_4(0,x,\alpha,\beta)
\end{aligned}$$

since $(-u)_+ = u_-$ and $(-u)_- = u_+$ for any $u \in \mathbf{X}$. It follows that $H_4(0,\cdot,\cdot,\cdot)$ is an odd operator on $\bar{\Omega}$.

To prove that H_4 is a compact operator, let $\Big\{\Big(\lambda_n,x_n,\alpha_n,\beta_n\Big)\Big\} \subset [0,1]\times\bar{\Omega}$ be a sequence. Then $0 \le \lambda_n \le 1$, $\|x_n\|_0 \le r+K(b-a)$, $\|x'_n\|_0 \le K$, $\|x''_n\|_{L_1} \le l$, $|\alpha_n| \le r+K(b-a)$ and $|\beta_n| \le K$ for $n \in \mathbf{N}$. By the Bolzano-Weierstrass theorem and the Arzelà-Ascoli theorem, there exist subsequences of $\{\lambda_n\}$, $\{x_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$, which for simplicity of notation we will write $\{\lambda_n\}$, $\{x_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ again, such that $\lim_{n\to\infty}\lambda_n = \lambda_0$, $\lim_{n\to\infty}\alpha_n = \alpha_0$, $\lim_{n\to\infty}\beta_n = \beta_0$ (in $\mathbf{R}$) and $\lim_{n\to\infty} x_n^{(i)} = x_0^{(i)}$ for $i=0,\,1$ (in $\mathbf{X}$) for some $\lambda_0,\,\alpha_0,\,\beta_0 \in \mathbf{R}$ and $x_0 \in \mathbf{Y}$. Clearly, $\lim\limits_{n\to\infty}(x_n^{(i)})_+ = (x_0^{(i)})_+$, $\lim\limits_{n\to\infty}(x_n^{(i)})_- = (x_0^{(i)})_-$ $(i=0,\,1)$ and therefore $\lim\limits_{n\to\infty}\omega((x_n)_-) = \omega((x_0)_-)$, $\lim\limits_{n\to\infty}\omega((1-\lambda_n)(x_n)_+) = \omega((1-\lambda_0)(x_0)_+)$, $\lim\limits_{n\to\infty}\gamma((x'_n)_-) = \gamma((x'_0)_-)$, $\lim\limits_{n\to\infty}\gamma((1-\lambda_n)(x'_n)_+) = \gamma((1-\lambda_0)(x'_0)_+)$. Then

$$\lim_{n\to\infty} H_4(\lambda_n,x_n,\alpha_n,\beta_n) = H_4(\lambda_0,x_0,\alpha_0,\beta_0)$$

in $AC^1(J)\times\mathbf{R}^2$. Moreover, from the continuity of ω and γ we deduce that H_4 is a continuous operator. Hence H_4 is a compact operator.

It remains to prove that $H_4(\lambda,x,\alpha,\beta) \neq (x,\alpha,\beta)$ for each $(\lambda,x,\alpha,\beta) \in [0,1]\times\partial\Omega$. Assume, on the contrary, that

$$H_4(\lambda_0,\,x_0,\,\alpha_0,\,\beta_0) = (x_0,\,\alpha_0,\,\beta_0)$$

for a $(\lambda_0,x_0,\alpha_0,\beta_0) \in [0,1]\times\partial\Omega$. Then

$$x_0(t) = \alpha_0+\beta_0(t-a) \quad \text{for } t \in J, \tag{9}$$

$$\omega((x_0)_-)-\omega((1-\lambda_0)(x_0)_+) = \lambda_0\omega(r) \tag{10}$$

and

$$\gamma((x_0')_-) - \gamma((1-\lambda_0)(x_0')_+) = \lambda_0\gamma(k). \tag{11}$$

The next part of the proof is divided into three steps by the sign of β_0.

Step 1. Let $\beta_0 = 0$. Then $x_0 = \alpha_0$ by (9), and (11) implies that $\lambda_0 = 0$ since $\gamma(k) > 0$ and $\gamma((x_0')_-) = \gamma((1-\lambda_0)(x_0')_+) = 0$. If $\alpha_0 \geq 0$, then (cf. (10)) $\omega(\alpha_0) = 0$, and so $\alpha_0 = 0$ by Corollary 1, which contradicts $(x_0, \alpha_0, \beta_0) = (0,0,0) \notin \partial\Omega$. If $\alpha_0 < 0$, then (cf. (10)) $\omega(-\alpha_0) = 0$, and consequently (cf. Corollary 1) $\alpha_0 = 0$. This again gives $(x_0, \alpha_0, \beta_0) = (0, 0, 0) \notin \partial\Omega$, a contradiction.

Step 2. Let $\beta_0 > 0$. Then (cf. (11))

$$-\gamma((1-\lambda_0)\beta_0) = \lambda_0\gamma(k).$$

(i) Assume $\lambda_0 = 0$. Then $\gamma(\beta_0) = 0$, and consequently $\beta_0 = 0$ by Corollary 1, and so $x_0 = \alpha_0$. By (10), $\omega(\alpha_0) = 0$ independent of the sign of α_0. So $\alpha_0 = 0$ and then $(x_0, \alpha_0, \beta_0) = (0, 0, 0)$, a contradiction.

(ii) Assume $\lambda_0 = 1$. Then $\gamma(k) = 0$, a contradiction.

(iii) Assume $\lambda_0 \in (0,1)$. Since $\lambda_0\gamma(k) > 0$, we have $\gamma((1-\lambda_0)\beta_0) < 0$, a contradiction.

Step 3. Let $\beta_0 < 0$. Then (cf. (11))

$$\gamma(|\beta_0|) = \lambda_0\gamma(k)$$

and therefore $\gamma(|\beta_0|) \leq \gamma(k)$. By Lemma 4 (with $\varrho = \gamma$, $x = |\beta_0|$, $y = k$),

$$|\beta_0| \leq k.$$

(i) Assume $x_0(t) < 0$ on J. By (10),

$$\omega(-x_0) = \lambda_0\omega(r) \leq \omega(r)$$

and therefore (cf. Lemma 4 with $\varrho = \omega$, $x = -x_0$, $y = r$) $-x_0(\xi) \leq r$ for a $\xi \in J$. Hence $\alpha_0 \geq -\beta_0(\xi - a) - r$ and $x_0(t) \geq \beta_0(t-\xi) - r$ on J. So $|x_0(t)| \leq |\beta_0||t-\xi| + r \leq k(b-a)+r < K(b-a)+r$, $|x_0'(t)| = |\beta_0| \leq k < K$ for $t \in J$, $|\alpha_0| = |x_0(a)| < K(b-a)+r$, $|\beta_0| < K$, which contradicts $(x_0, \alpha_0, \beta_0) \in \partial\Omega$.

(ii) Assume $x_0(t) > 0$ on J. By (10),

$$-\omega((1-\lambda_0)x_0) = \lambda_0\omega(r). \tag{12}$$

If $\lambda_0 = 0$, then $\omega(x_0) = 0$, a contradiction. If $\lambda_0 = 1$, then $\omega(r) = 0$, a contradiction. So $\lambda_0 \in (0,1)$, and consequently $\omega((1-\lambda_0)x_0) > 0$, $\lambda_0\omega(r) > 0$ which contradicts (12).

(iii) Assume $x_0(\varepsilon) = 0$ for an $\varepsilon \in J$. Then $x_0(t) = \beta_0(t-\varepsilon)$ and therefore $|x_0(t)| \leq |\beta_0|(b-a) \leq k(b-a) < K(b-a)$, $|x_0'(t)| = |\beta_0| < K$ for $t \in J$, $|\alpha_0| = |x_0(a)| < K(b-a)$, $|\beta_0| < K$ which contradicts $(x_0, \alpha_0, \beta_0) \in \partial\Omega$.

Hence our lemma is proved. □

In this paper we assume that the operator F satisfies the following assumption:

(H) There exist a nonnegative function $q \in L_1(J)$ and a nondecreasing function $f : [0,\infty) \to (0,\infty)$ such that

$$\int_0^\infty \frac{dt}{f(t)} = \infty$$

and

$$|(Fx)(t)| \le q(t) f(|x'(t)|) \quad \text{for a.e. } t \in J \text{ and each } x \in \mathbf{Y}.$$

Consider the functional differential equation

$$x''(t) = \lambda (Fx)(t), \quad \lambda \in [0,1] \tag{13_λ}$$

depending on the parameter λ.

Lemma 8. *Let assumption (H) be satisfied and $m \in \mathbf{R}$, $m \ge 0$. Let $u(t)$ be a solution of (13_λ) on J with a $\lambda \in [0,1]$ and $|u'(\nu)| = m$ for a $\nu \in J$. Then*

$$\|u'\|_0 \le G^{-1}(G(m) + \|q\|_{L_1}).$$

If, moreover, $u(\tau) = 0$ for a $\tau \in J$, then

$$\|u\|_0 \le (b-a) G^{-1}(G(m) + \|q\|_{L_1}).$$

Here $G : [0,\infty) \to [0,\infty)$ is defined by (5) and G^{-1} means the inverse function to G.

Proof. From the inequalities (for a.e. $t \in J$)

$$|u''(t)| = \lambda |(Fu)(t)| \le q(t) f(|u'(t)|)$$

and the assumption $|u'(\nu)| = m$ we obtain

$$|u'(t)| \le m + \left| \int_\nu^t q(s) f(|u'(s)|)\, ds \right| \quad \text{for } t \in J.$$

By Lemma 1 (with $w = u'$ and $k = m$),

$$|u'(t)| \le G^{-1}(G(m) + \|q\|_{L_1}) \quad \text{for } t \in J.$$

If $u(\tau) = 0$ for a $\tau \in J$, then

$$|u(t)| \le \left| \int_\tau^t |u'(s)|\, ds \right| \le (b-a) G^{-1}(G(m) + \|q\|_{L_1})$$

for $t \in J$. □

Corollary 2. *Let assumption (H) be satisfied, ξ, $\varrho \in J$ and let $u(t)$ be a solution of (13_λ) on J with a $\lambda \in [0,1]$ such that*

$$|u'(\xi)| > G^{-1}(\|q\|_{L_1}) \tag{14}$$

and

$$|u(\varrho)| > (b-a)G^{-1}(G(|u'(\xi)|) + \|q\|_{L_1}). \tag{15}$$

Then $|u(t)| > 0$, $|u'(t)| > 0$ *on* J.

Proof. Assume $u'(\nu) = 0$ for a $\nu \in J$. Then, by Lemma 8 (with $m = 0$), $\|u'\|_0 \le G^{-1}(\|q\|_{L_1})$ which contradicts (14).

Assume $u(\delta) = 0$ for a $\delta \in J$. By Lemma 8 (with $m = |u'(\xi)|$), $\|u\|_0 \le (b-a)G^{-1}(G(|u'(\xi)|) + \|q\|_{L_1})$ which contradicts (15). □

3. Existence results

Our existence results are given in two theorems. BVP (1)–(3) with ω, $\gamma \in \mathcal{A}_0$ is considered in Theorem 1. For any ω, $\gamma \in \mathcal{A}$, a multiplicity result for BVP (1)–(3) is obtained in Theorem 2. Recall that $G : [0,\infty) \to [0,\infty)$ is defined by (5) and $q_\varrho : [0,\infty) \to \mathbf{R}$ by (6). Let $q_\varrho^{-1} : [\varrho(0),\infty) \to [0,\infty)$ be the inverse function to q_ϱ (see Lemma 5).

Theorem 1. *Let assumption* (H) *be satisfied and* ω, $\gamma \in \mathcal{A}_0$. *Let*

$$B > q_\gamma(G^{-1}(\|q\|_{L_1})) \text{ and } A > q_\omega\Big((b-a)G^{-1}(G(q_\gamma^{-1}(B)) + \|q\|_{L_1})\Big).$$

Then any solution of BVP (1)–(3) and its derivative do not vanish on J, *and there exist at least four different solutions* x_1, x_2, x_3, x_4 *satisfying* (4) *for* $t \in J$.

Proof. Fix A, $B \in \mathbf{R}$,

$$B > q_\gamma(G^{-1}(\|q\|_{L_1})), \quad A > q_\omega\Big((b-a)G^{-1}(G(q_\gamma^{-1}(B)) + \|q\|_{L_1})\Big).$$

Set $k = q_\gamma^{-1}(B)$, $r = q_\omega^{-1}(A)$. Then

$$k > G^{-1}(\|q\|_{L_1}), \quad r > (b-a)G^{-1}(G(k) + \|q\|_{L_1}). \tag{16}$$

Let $u(t)$ be a solution of BVP (1)–(3). Then $\omega(u) = A$, $\gamma(u') = B$, and consequently (cf. Lemma 3) $|u(\varrho)| = r$, $|u'(\xi)| = k$ for some ϱ, $\xi \in J$. Thus (cf. (16))

$$|u(\varrho)| > (b-a)G^{-1}(G(|u'(\xi)|) + \|q\|_{L_1}), \quad |u'(\xi)| > G^{-1}(\|q\|_{L_1}),$$

and so $|u(t)| > 0$, $|u'(t)| > 0$ for $t \in J$ by Corollary 2 (with $\lambda = 1$). Hence any solution of BVP (1)–(3) and its derivative do not vanish on J.

We now show that there exists a solution $x_1(t)$ of BVP (1)–(3) satisfying the inequalities

$$x_1(t) > 0, \quad x_1'(t) > 0 \quad \text{for } t \in J. \tag{17}$$

Set

$$K = G^{-1}(G(k) + \|q\|_{L_1}) + 1 \ (> k+1)$$

and

$$\Omega = \Big\{(x,\alpha,\beta):\ (x,\alpha,\beta)\in AC^1(J)\times \mathbf{R}^2,\ \|x\|_0 < r + K(b-a),$$
$$\|x'\|_0 < K,\ \|x''\|_{L_1} < f(K)\|q\|_{L_1} + 1,\ |\alpha| < r + K(b-a),\ |\beta| < K\Big\}.$$

Then Ω is an open bounded subset of $AC^1(J)\times \mathbf{R}^2$. Let the operator

$$V_1 : [0,1]\times\bar\Omega \to AC^1(J)\times\mathbf{R}^2$$

be given by the formula

$$V_1(\lambda,x,\alpha,\beta) = \Big(\alpha+\beta(t-a)+\lambda\int_a^t\int_a^s (Fx)(\tau)\,d\tau\,ds,$$
$$\alpha+\omega(x_+)-A,\ \beta+\gamma(x'_+)-B\Big).$$

Of course, $V_1(0,x,\alpha,\beta) = \Gamma_1(x,\alpha,\beta)$ for $(x,\alpha,\beta)\in\bar\Omega$, where Γ_1 is defined in Lemma 7 (with $r = q_\omega^{-1}(A)$, $k = q_\gamma^{-1}(B)$ and $l = f(K)\|q\|_{L_1}+1$).

Consider the operator equation

$$(18_\lambda)\qquad V_1(\lambda,x,\alpha,\beta) = (x,\alpha,\beta),\quad \lambda\in[0,1],$$

depending on the parameter λ. We now show that (18_1) has a solution. As $D(I-\Gamma_1,\Omega,0)\neq 0$ by Lemma 7, it is sufficient to verify that (cf., e.g., [8])

(a) $V_1(\lambda,x,\alpha,\beta)$ is a compact operator

and

(b) $V_1(\lambda,x,\alpha,\beta)\neq(x,\alpha,\beta)$ for each $(\lambda,x,\alpha,\beta)\in[0,1]\times\partial\Omega$.

From the continuity F, ω, γ and Lemma 6 we deduce that V_1 is a continuous operator. Let $\{(\lambda_n,x_n,\alpha_n,\beta_n)\}\subset[0,1]\times\bar\Omega$ be a sequence. We can now proceed analogously to the proof of Lemma 7. Without restriction of generality we may assume that the sequences $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are convergent in $\mathbf{R}$ and $\{x_n\}$ is convergent in $\mathbf{Y}$, say $\lim_{n\to\infty}\lambda_n = \lambda_0$, $\lim_{n\to\infty}\alpha_n = \alpha_0$, $\lim_{n\to\infty}\beta_n = \beta_0$ and $\lim_{n\to\infty}x_n = x$. Then $\lim_{n\to\infty}Fx_n = Fx$ in $L_1(J)$ and $\lim_{n\to\infty}\omega((x_n)_+) = \omega(x_+)$, $\lim_{n\to\infty}\gamma((x'_n)_+) = \gamma(x'_+)$. Hence $\{V_1(\lambda_n,x_n,\alpha_n,\beta_n)\}$ is convergent and

$$\lim_{n\to\infty}V_1(\lambda_n,x_n,\alpha_n,\beta_n) = \Big(\alpha_0+\beta_0(t-a)+\lambda_0\int_a^t\int_a^s (Fx)(\tau)\,d\tau\,ds,$$
$$\alpha_0+\omega(x_+)-A,\ \beta_0+\gamma(x'_+)-B\Big).$$

So V_1 is a compact operator.

To prove property (b) of V_1 we assume, on the contrary, that

$$(19)\qquad V_1(\lambda_0,x_0,\alpha_0,\beta_0) = (x_0,\alpha_0,\beta_0)$$

for a $(\lambda_0,x_0,\alpha_0,\beta_0)\in[0,1]\times\partial\Omega$. Then

$$(20)\qquad x_0(t) = \alpha_0+\beta_0(t-a)+\lambda_0\int_a^t\int_a^s (Fx_0)(\tau)\,d\tau\,ds\quad\text{for } t\in J,$$

$$\omega((x_0)_+) = \omega(r)\ (= A) \tag{21}$$

and

$$\gamma((x_0')_+) = \gamma(k)\ (= B). \tag{22}$$

By (20), $x_0(t)$ is a solution of (13_{λ_0}) and $x_0(\xi) = (x_0)_+(\xi) = r$, $x_0'(\tau) = (x_0')_+(\tau) = k$ for some ξ, $\tau \in J$ by (21), (22) and Lemma 3. Hence, (cf. (16))

$$x_0'(\tau) > G^{-1}(\|q\|_{L_1}), \quad x_0(\xi) > (b-a)G^{-1}(G(x_0'(\tau)) + \|q\|_{L_1}),$$

and consequently $x_0(t) > 0$, $x_0'(t) > 0$ for $t \in J$ by Corollary 2 (with $\lambda = \lambda_0$). Moreover,

$$x_0'(t) \le k + \lambda_0 \Big| \int_\tau^t (Fx_0)(s)\, ds \Big| \le k + \Big| \int_\tau^t q(s) f(x_0'(s))\, ds \Big| \quad \text{for } t \in J.$$

Lemma 1 shows that

$$x_0'(t) \le G^{-1}(G(k) + \|q\|_{L_1}) < K \quad \text{for } t \in J.$$

From the last inequality we deduce that

$$x_0(t) \le r + \Big| \int_\xi^t x_0'(s)\, ds \Big| < r + K(b-a)$$

for $t \in J$. Moreover,

$$\|x_0''\|_{L_1} = \lambda_0 \int_a^b |(Fx_0)(t)|\, dt \le \int_a^b q(t) f(x_0'(t))\, dt \le f(K)\|q\|_{L_1} < f(K)\|q\|_{L_1} + 1.$$

Since $\alpha_0 = x_0(a)$, $\beta_0 = x_0'(a)$, we have

$$0 < \alpha_0 < r + K(b-a), \quad 0 < \beta_0 < K.$$

Thus $(x_0, \alpha_0, \beta_0) \notin \partial\Omega$, a contradiction.

We have proved that the operator equation (18_1) has a solution, say (x_1, α_1, β_1). Then x_1 is a solution of (1) satisfying boundary conditions

$$\omega((x_1)_+) = A, \quad \gamma((x_1')_+) = B.$$

Since $(x_1)_+(\xi) = r$ and $(x_1')_+(\tau) = k$ for some $\xi, \tau \in J$ by Lemma 3, Corollary 2 shows that $x_1(t) > 0$ and $x_1'(t) > 0$ on J; hence $\omega(x_1) = A$, $\gamma(x_1') = B$.

If the operators

$$\begin{aligned} V_2(\lambda, x, \alpha, \beta) \;=\; & \Big(\alpha + \beta(t-a) + \lambda \int_a^t \int_a^s (Fx)(\tau)\, d\tau\, ds, \\ & \alpha + \omega(x_+) - A,\ \beta - \gamma(x_-') + B\Big), \end{aligned}$$

$$\begin{aligned} V_3(\lambda,x,\alpha,\beta) &= \Big(\alpha+\beta(t-a)+\lambda\int_a^t\int_a^s(Fx)(\tau)\,d\tau\,ds,\\ &\qquad \alpha-\omega(x_-)+A,\ \beta+\gamma(x'_+)-B\Big),\\ V_4(\lambda,x,\alpha,\beta) &= \Big(\alpha+\beta(t-a)+\lambda\int_a^t\int_a^s(Fx)(\tau)\,d\tau\,ds,\\ &\qquad \alpha-\omega(x_-)+A,\ \beta-\gamma(x'_-)+B\Big) \end{aligned}$$

are considered on the set $[0,1]\times\bar{\Omega}$ instead of $V_1(\lambda,x,\alpha,\beta)$, one can prove the existence of solutions $x_2(t)$, $x_3(t)$ and $x_4(t)$ of BVP (1)–(3) satisfying on J the inequalities

$$x_2(t)>0,\ x_2'(t)<0;\quad x_3(t)<0,\ x_3'(t)>0;\quad x_4(t)<0,\ x_4'(t)<0. \qquad \square$$

Theorem 2. *Let assumption (H) be satisfied and ω, $\gamma\in\mathcal{A}$. Let*

$$B>q_\gamma(G^{-1}(\|q\|_{L_1}))\ \textit{and}\ \ A>q_\omega\Big((b-a)G^{-1}(G(q_\gamma^{-1}(B))+\|q\|_{L_1})\Big).$$

Then the conclusion of Theorem 1 holds.

Proof. Fix $B>q_\gamma(G^{-1}(\|q\|_{L_1}))$ *and* $A>q_\omega\Big((b-a)G^{-1}(G(q_\gamma^{-1}(B))+\|q\|_{L_1})\Big)$ and set $\bar\omega(x)=\omega(x)-\omega(0)$, $\bar\gamma(x)=\gamma(x)-\gamma(0)$ for $x\in\mathbf{X}$. Then $\bar\omega$, $\bar\gamma\in\mathcal{A}_0$. Consider equation (1) subject to the boundary conditions

$$\bar\omega(x)=A-\omega(0),\quad \bar\gamma(x)=B-\gamma(0). \tag{23}$$

Obviously,

$$B-\gamma(0)>q_{\bar\gamma}(G^{-1}(\|q\|_{L_1})),$$

$$A-\omega(0)>q_{\bar\omega}\Big((b-a)G^{-1}(G(q_{\bar\gamma}^{-1}(B-\gamma(0)))+\|q\|_{L_1})\Big).$$

Applying Theorem 1 to BVP (1), (23), any solution of BVP (1), (23) and its derivative do not vanish on J and there exist at least four solutions x_i ($i=1,2,3,4$) satisfying inequalities (4). Since $x(t)$ is a solution of BVP (1)–(3) if and only if that is a solution of BVP (1), (23), our theorem is proved. $\square$

Example 2. Consider the functional differential equation

$$x''(t)=(F_1x)(t)+(F_2x)(t)g(x'(t)), \tag{24}$$

where F_1, $F_2:\mathbf{Y}\to L_1(J)$ are continuous, $|g(v)|\le|v|$ for $v\in\mathbf{R}$, $|(F_1x)(t)|\le\alpha q(t)$, $|(F_2x)(t)|\le\beta q(t)$ for each $x\in\mathbf{Y}$ and a.e. $t\in J$, where α, β are positive constants, $q\in L_1(J)$ and $\|q\|_{L_1}=b-a$. Then (24) satisfies assumption (H) with $f(u)=\alpha+\beta u$, $u\in[0,\infty)$. Clearly, $G(u)=\frac{1}{\beta}\ln(1+\frac{\beta u}{\alpha})$, $G^{-1}(u)=\frac{\alpha}{\beta}(e^{\beta u}-1)$ for $u\in[0,\infty)$. Consider, for instance, the boundary conditions

$$\|x\|_0=A,\quad \|x'\|_0=B \tag{25}$$

or

$$\int_a^b |x(t)|\,dt = A, \quad \min\{|x'(t)| : t \in J\} = B \tag{26}$$

or

$$|x(\xi)| = A, \quad \int_a^b \sqrt{1+(x'(t))^2}\,dt = B, \tag{27}$$

where $\xi \in J$. Set

$$\omega_1(x) = \gamma_1(x) = \|x\|_0,\ \omega_2(x) = \int_a^b |x(t)|\,dt,\ \omega_3(x) = |x(\xi)|,$$

$$\gamma_2(x) = \min\{|x(t)| : t \in J\},\ \gamma_3(x) = \int_a^b \sqrt{1+(x(t))^2}\,dt$$

for $x \in \mathbf{X}$. Then $\omega_i, \gamma_i \in \mathcal{A}$ for $i = 1, 2, 3$ and $q_{\omega_1}(c) = q_{\omega_3}(c) = q_{\gamma_1}(c) = q_{\gamma_2}(c) = c$, $q_{\omega_2}(c) = (b-a)c$, $q_{\gamma_3}(c) = (b-a)\sqrt{1+c^2}$ for $c \in [0,\infty)$. Of course, boundary conditions (25) or (26) or (27) we can write in the form $\omega_1(x) = A$, $\gamma_1(x') = B$ or $\omega_2(x) = A$, $\gamma_2(x') = B$ or $\omega_3(x) = A$, $\gamma_3(x') = B$. By Theorem 2 (with $Fx = F_1x + g(x')F_2x$, $\omega = \omega_i$ and $\gamma = \gamma_i$, $i = 1, 2, 3$), any solution of BVP (24), (j) ($j = 25, 26, 27$) and its derivative do not vanish on J, and there exist at least four solutions x_1, x_2, x_3, x_4 satisfying (4) provided

$$B > \tfrac{\alpha}{\beta}\Big(e^{\beta(b-a)} - 1\Big), \quad A > \tfrac{\alpha(b-a)}{\beta}\Big(\Big(1 + \tfrac{\beta B}{\alpha}\Big)e^{\beta(b-a)} - 1\Big)$$
$$\text{for BVP (24), (25),}$$

$$B > \tfrac{\alpha}{\beta}\Big(e^{\beta(b-a)} - 1\Big), \quad A > \tfrac{\alpha(b-a)^2}{\beta}\Big(\Big(1 + \tfrac{\beta B}{\alpha}\Big)e^{\beta(b-a)} - 1\Big)$$
$$\text{for BVP (24), (26),}$$

and

$$B > (b-a)\sqrt{1 + \Big(\tfrac{\alpha}{\beta}\Big(e^{\beta(b-a)} - 1\Big)\Big)^2},$$
$$A > \tfrac{\alpha(b-a)}{\beta}\Big(\Big(1 + \tfrac{\beta}{\alpha}\sqrt{(\tfrac{B}{b-a})^2 - 1}\Big)e^{\beta(b-a)} - 1\Big)$$
$$\text{for BVP (24), (27).}$$

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