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ON A GENERALIZED WIENER–HOPF INTEGRAL EQUATION

Malcolm T. McGregor

Abstract. Let α be such that $0 < \alpha < \frac{1}{2}$. In this note we use the Mittag-Leffler partial fractions expansion for $F_\alpha(\theta) = \Gamma\left(1-\alpha-\frac{\theta}{\pi}\right)\Gamma(\alpha)/\Gamma\left(\alpha-\frac{\theta}{\pi}\right)\Gamma(1-\alpha)$ to obtain a solution of a Wiener-Hopf integral equation.

1. Introduction

Wiener-Hopf equations, and the Wiener-Hopf technique for solving such equations, arose out of a study of the radiation equilibrium of the stars. Since its introduction in 1931, the Wiener-Hopf technique has been refined and applied to a variety of problems involving integral equations and partial differential equations. Application of the Fourier transform (or the Laplace transform) to such equations yields, in many cases, a Wiener-Hopf equation of the form

$$A(\theta)P_+(\theta) + B(\theta)Q_-(\theta) = C(\theta)$$

where $\theta = \sigma + i\tau$ belongs to a parallel-strip region $S : \tau_- < \mathrm{Im}\,\theta < \tau_+$ (or $\sigma_- < \mathrm{Re}\,\theta < \sigma_+$). Furthermore, $P_+(\theta)$ is regular in the upper half-plane $\tau > \tau_-$, and $Q_-(\theta)$ is regular in the lower half-plane $\tau < \tau_+$, whilst $A(\theta)$, $B(\theta)$, $C(\theta)$ are given functions of θ which are regular and non-zero in S. For an in-depth discussion of the Wiener-Hopf technique and its applications the reader is referred to [1] and [3].

Let $\widetilde{P}_\alpha(\theta)$ denote the Laplace transform of $P_\alpha(y)$, where α is such that $0 < \alpha < \frac{1}{2}$. We shall use complex analytic methods to solve the Wiener-Hopf equation

$$\sin(\alpha\pi + \theta)\widetilde{P}_\alpha(-\theta) + \sin(\alpha\pi - \theta)\widetilde{P}_\alpha(\theta) = 2\cos\alpha\pi\frac{\sin\theta}{\theta}$$

by showing that $\widetilde{P}_\alpha(\theta)$ is expressible in terms of the Gamma function. As a result, we obtain the solution $P_\alpha(y)$, as a series of exponentials, of a pair of associated integral equations. The case $\alpha = \frac{1}{4}$ was dealt with in an earlier paper.

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2. Posing the problem

In [2] we solved the integral equation

$$\int_0^\infty (\cosh\,\theta \cos\theta y - \sinh\,\theta \sin\theta y) P(y)\, dy = \frac{\sinh\,\theta}{\theta} \tag{1}$$

by assuming that $P(y)$ admits the series expansion

$$P(y) = \sum_{n=0}^{\infty} E_n e^{-\beta_n y}$$

so that its Laplace transform is

$$\widetilde{P}(\theta) = \mathcal{L}[P(y)](\theta) = \sum_{n=0}^{\infty} \frac{E_n}{\beta_n + \theta}\,.$$

In this case $\beta_n = \left(n + \frac{3}{4}\right)\pi$; $n = 0, 1, 2, \ldots$, and the coefficients $\{E_n\}$ are subject to the normalization

$$\sum_{n=0}^{\infty} E_n/\beta_n = 1\,.$$

By replacing θ by $i\theta$ in (1) we obtain the associated integral equation

$$\int_0^\infty \left(\sin\left(\frac{\pi}{4} + \theta\right) e^{\theta y} + \sin\left(\frac{\pi}{4} - \theta\right) e^{-\theta y}\right) P(y)\, dy = \sqrt{2}\frac{\sin\theta}{\theta}\,, \tag{2}$$

and (2) may be written as a Wiener-Hopf equation, namely:

$$\sin\left(\frac{\pi}{4} + \theta\right) \widetilde{P}(-\theta) + \sin\left(\frac{\pi}{4} - \theta\right) \widetilde{P}(\theta) = \sqrt{2}\frac{\sin\theta}{\theta}\,,$$

or

$$\sin\left(\frac{\pi}{4} + \theta\right) \sum_{n=0}^{\infty} \frac{E_n}{\beta_n - \theta} + \sin\left(\frac{\pi}{4} - \theta\right) \sum_{n=0}^{\infty} \frac{E_n}{\beta_n + \theta} = \sqrt{2}\frac{\sin\theta}{\theta}\,. \tag{3}$$

In [2] we obtained

$$\widetilde{P}(\theta) = \sum_{n=0}^{\infty} \frac{E_n}{\beta_n + \theta} = (F(-\theta) - 1)/\theta\,,$$

where

$$F(\theta) = \frac{\Gamma\left(\frac{3}{4} - \frac{\theta}{\pi}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4} - \frac{\theta}{\pi}\right)\Gamma\left(\frac{3}{4}\right)}\,,$$

so that

$$\widetilde{P}(\theta) = \left(\frac{\Gamma\left(\frac{3}{4} + \frac{\theta}{\pi}\right) \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4} + \frac{\theta}{\pi}\right) \Gamma\left(\frac{3}{4}\right)} - 1 \right) \Big/ \theta\,.$$

It was also shown in [2] that the coefficients $\{E_n\}$ in the series expansion for $P(y)$ are given by

$$E_n = \left(\Gamma(\frac{1}{4})\right)^2 \Gamma\left(n + \frac{3}{4}\right) \Big/ n!\pi^2\sqrt{2}\, \left(n + \frac{3}{4}\right) = K_n/\beta_n\,,$$

and that

$$\sum_{n=0}^{\infty} E_n/\beta_n = 1\,,$$

as required.

In this paper we shall solve a more general Wiener-Hopf equation than (3), and consequently solve a more general integral equation than (2); the two equations will now contain a parameter α with $0 < \alpha < \frac{1}{2}$. We shall show that

$$\sin(\alpha\pi + \theta) \sum_{n=0}^{\infty} \frac{K_{\alpha,n}}{a_{\alpha,n}(a_{\alpha,n} - \theta)} + \sin(\alpha\pi - \theta) \sum_{n=0}^{\infty} \frac{K_{\alpha,n}}{a_{\alpha,n}(a_{\alpha,n} + \theta)} = 2\cos\alpha\pi \frac{\sin\theta}{\theta}\,, \tag{4}$$

where $a_{\alpha,n} = (n + 1 - \alpha)\pi$; $n = 0, 1, 2, \ldots$, and the coefficients $\{K_{\alpha,n}\}$ are given by

$$K_{\alpha,n} = \pi(-1)^{n+1}\Gamma(\alpha)/n!\,\Gamma(1-\alpha)\Gamma(2\alpha - n - 1)\,.$$

The case $\alpha = \frac{1}{4}$ yields (3). In the α-case the analogue of (1) is the integral equation

$$\int_0^{\infty} (\tan\alpha\pi \cosh\,\theta \cos\theta y - \sinh\,\theta \sin\theta y) P_\alpha(y)\, dy = \frac{\sinh\,\theta}{\theta}\,, \tag{5}$$

and when we replace θ by $i\theta$ (5) becomes

$$\int_0^{\infty} \left(\sin(\alpha\pi + \theta)e^{\theta y} + \sin(\alpha\pi - \theta)e^{-\theta y}\right) P_\alpha(y)\, dy = 2\cos\alpha\pi \frac{\sin\theta}{\theta}\,. \tag{6}$$

Clearly, (6) reduces to (2) when we set $\alpha = \frac{1}{4}$.

Let α be such that $0 < \alpha < \frac{1}{2}$. We shall assume that $P_\alpha(y)$ admits the series expansion

$$P_\alpha(y) = \sum_{n=0}^{\infty} E_{\alpha,n} e^{-a_{\alpha,n} y} ,$$

so that its Laplace transform is

$$\widetilde{P}_\alpha(\theta) = \mathcal{L}[P_\alpha(y)](\theta) = \sum_{n=0}^{\infty} \frac{E_{\alpha,n}}{a_{\alpha,n} + \theta} ,$$

and (6) takes the form of a Wiener-Hopf equation

$$\sin(\alpha\pi + \theta)\widetilde{P}_\alpha(-\theta) + \sin(\alpha\pi - \theta)\widetilde{P}_\alpha(\theta) = 2\cos\alpha\pi \frac{\sin\theta}{\theta} .$$

This latter equation is, of course, (4) with

$$E_{\alpha,n} = K_{\alpha,n}/a_{\alpha,n} .$$

3. Finding the coefficients $E_{\alpha,n}$ and solving the problem

With $0 < \alpha < \frac{1}{2}$ and $a_{\alpha,n} = (n + 1 - \alpha)\pi$, $n = 0, 1, 2, \ldots$, we shall show that (4) holds with

$$K_{\alpha,n} = \pi(-1)^{n+1}\Gamma(\alpha)/n!\Gamma(1-\alpha)\Gamma(2\alpha - n - 1)$$

by considering the meromorphic function

$$F_\alpha(\theta) = \frac{\Gamma\left(1 - \alpha - \frac{\theta}{\pi}\right)\Gamma(\alpha)}{\Gamma\left(\alpha - \frac{\theta}{\pi}\right)\Gamma(1-\alpha)} . \tag{7}$$

The function F_α is such that $F_\alpha(0) = 1$, and F_α has simple poles at $\theta = (n+1-\alpha)\pi$, $n = 0, 1, 2, \ldots$, due to $\Gamma\left(1 - \alpha - \frac{\theta}{\pi}\right)$, and simple zeros at $\theta = (n + \alpha)\pi$, $n = 0, 1, 2, \ldots$, due to $1/\Gamma\left(\alpha - \frac{\theta}{\pi}\right)$. With $a_{\alpha,n}$ as above, the Mittag-Leffler expansion for $F_\alpha(\theta)$ gives

$$\begin{aligned} F_\alpha(\theta) &= 1 + \sum_{n=0}^{\infty} \left(\frac{K_{\alpha,n}}{\theta - a_{\alpha,n}} + \frac{K_{\alpha,n}}{a_{\alpha,n}} \right) \\ &= 1 + \theta \sum_{n=0}^{\infty} \frac{K_{\alpha,n}}{a_{\alpha,n}(\theta - a_{\alpha,n})} \end{aligned}$$

with a corresponding expression for $F_\alpha(-\theta)$. Next, we form the sum

$$\begin{aligned}\sin(\alpha\pi+\theta)\sum_{n=0}^{\infty}\frac{K_{\alpha,n}}{a_{\alpha,n}(a_{\alpha,n}-\theta)}&+\sin(\alpha\pi-\theta)\sum_{n=0}^{\infty}\frac{K_{\alpha,n}}{a_{\alpha,n}(a_{\alpha,n}+\theta)}\\ &=\sin(\alpha\pi+\theta)(1-F_\alpha(\theta))/\theta+\sin(\alpha\pi-\theta)(F_\alpha(-\theta)-1)/\theta\\ &=2\cos\alpha\pi\frac{\sin\theta}{\theta},\end{aligned}$$

provided

$$\sin(\alpha\pi-\theta)F_\alpha(-\theta)=\sin(\alpha\pi+\theta)F_\alpha(\theta)\,. \tag{8}$$

Using (7), we see that (8) is equivalent to

$$\Gamma\left(\alpha-\frac{\theta}{\pi}\right)\Gamma\left(1-\alpha+\frac{\theta}{\pi}\right)\sin(\alpha\pi-\theta)=\Gamma\left(\alpha+\frac{\theta}{\pi}\right)\Gamma\left(1-\alpha-\frac{\theta}{\pi}\right)\sin(\alpha\pi+\theta)\,,$$

and each side of this equation reduces to π when we use the well-known formula

$$\Gamma(z)\,\Gamma(1-z)=\pi/\sin\pi z$$

with $z=\alpha-\frac{\theta}{\pi}$ and $z=\alpha+\frac{\theta}{\pi}$ respectively. Our proof of (4) will be complete when we determine the numbers $K_{\alpha,n}$.

From the Mittag-Leffler expansion for $F_\alpha(\theta)$ we have

$$\begin{aligned}K_{\alpha,n}&=\lim_{\theta\to a_{\alpha,n}}(\theta-a_{\alpha,n})\,F_\alpha(\theta)\\ &=\frac{\Gamma(\alpha)}{\Gamma(1-\alpha)\Gamma(2\alpha-n-1)}\lim_{\theta\to a_{\alpha,n}}(\theta-a_{\alpha,n})\,\Gamma\left(1-\alpha-\frac{\theta}{\pi}\right),\end{aligned}$$

and with $z=1-\alpha-\frac{\theta}{\pi}$ in

$$\Gamma(z)=\Gamma(z+n+1)/z(z+1)\dots(z+n)$$

we deduce that

$$K_{\alpha,n}=\pi(-1)^{n+1}\Gamma(\alpha)/n!\Gamma(1-\alpha)\Gamma(2\alpha-n-1)$$

as required. Clearly, if we set $\alpha=\frac{1}{4}$ in (4) we obtain (3) with

$$E_n=K_{\frac{1}{4},n}/a_{\frac{1}{4},n}\,,$$

where $a_{\frac{1}{4},n} = \beta_n = \left(n + \frac{3}{4}\right)\pi$; $n = 0, 1, 2, \ldots$, and

$$K_{\frac{1}{4},n} = \pi(-1)^{n+1}\Gamma\left(\frac{1}{4}\right) \Big/ n!\Gamma\left(\frac{3}{4}\right)\Gamma\left(-n-\frac{1}{2}\right)$$
$$= \left(\Gamma\left(\frac{1}{4}\right)\right)^2 \Gamma\left(n+\frac{3}{2}\right) \Big/ n!\pi\sqrt{2}.$$

Finally, we show that

$$\sum_{n=0}^{\infty} \frac{E_{\alpha,n}}{a_{\alpha,n}} = \cot \alpha\pi\,,$$

where

$$E_{\alpha,n} = K_{\alpha,n}/a_{\alpha,n}\,.$$

Clearly,

$$\sum_{n=0}^{\infty} \frac{K_{\alpha,n}}{(a_{\alpha,n})^2} = -\lim_{\theta\to 0} \frac{F_\alpha(\theta)-1}{\theta} = -F'_\alpha(0)\,,$$

and by (7)

$$-F'_\alpha(0) = \frac{1}{\pi}\left(\frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right)$$

and since $\Gamma(\alpha)\Gamma(1-\alpha) = \pi/\sin\pi\alpha$ implies

$$\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} = -\pi\cot\pi\alpha\,,$$

we have immediately $-F'_\alpha(0) = \cot\alpha\pi$, as required.

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Department of Mathematics
University of Wales Swansea
Singleton Park
Swansea SA2 8PP Wales
United Kingdom