ARCHIVUM MATHEMATICUM (BRNO)
Tomus 33 (1997), 355 – 361

AN OSCILLATORY HALF-LINEAR DIFFERENTIAL EQUATION

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Abstract. A second-order half-linear ordinary differential equation of the type

$$(|y'|^{\alpha-1}y')' + \alpha q(t)|y|^{\alpha-1}y = 0 \tag{1}$$

is considered on an unbounded interval. A simple oscillation condition for (1) is given in such a way that an explicit asymptotic formula for the distribution of zeros of its solutions can also be established.

We consider the second-order half-linear differential equation

$$(|y'|^{\alpha-1}y')' + \alpha q(t)|y|^{\alpha-1}y = 0, \quad t \ge a, \tag{1}$$

where $\alpha > 0$ is a constant and $q : [a,\infty) \to (0,\infty)$ is a continuously differentiable function. Our attention is directed to the case where all solutions of (1) are oscillatory. In this case the equation (1) is said to be oscillatory.

Prototypes of oscillatory equations of the type (1) are the generalized harmonic oscillator

$$(|y'|^{\alpha-1}y')' + \alpha|y|^{\alpha-1}y = 0 \tag{2}$$

and the generalized Euler equation

$$(|y'|^{\alpha-1}y')' + \lambda\alpha t^{-\alpha-1}|y|^{\alpha-1}y = 0 \tag{3}$$

with $\lambda > \alpha^\alpha/(\alpha+1)^{\alpha+1}$.

From among various oscillation criteria for (1) we choose the one due to Elbert [2; Theorem 7]:

$$\int^\infty \frac{t^{\alpha+1}q(t) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}}{t}dt = \infty, \tag{4}$$

which gives rise to a simpler condition

$$\liminf_{t\to\infty} t^{\alpha+1}q(t) > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \tag{5}$$

1991 *Mathematics Subject Classification*: 34C10.
Key words and phrases: half-linear differential equations.
Received September 2, 1996.

guaranteeing the oscillation of (1). See also Li and Yeh [4].

The purpose of this paper is to give an oscillation criterion for (1) under which one can derive an asymptotic formula for the distribution of zeros of its solutions.

Theorem 1. *The equation* (1) *is oscillatory if*

$$\lim_{t\to\infty} q'(t)[q(t)]^{-\frac{2+\alpha}{1+\alpha}} = 0. \tag{6}$$

Proof. Set

$$Q(t) = q'(t)[q(t)]^{-\frac{2+\alpha}{1+\alpha}}$$

and define

$$Q^*(t) = \sup\{|Q(s)| : \ s \geq t\}, \ t \geq a. \tag{7}$$

Then $Q^*(t)$ is nonincreasing on $[a,\infty)$ and satisfies $\lim_{t\to\infty} Q^*(t) = 0$ by (6). We have

$$\left|[q(t+h)]^{-\frac{1}{1+\alpha}} - [q(t)]^{-\frac{1}{1+\alpha}}\right| = \frac{1}{1+\alpha}\left|\int_t^{t+h} Q(s)ds\right| \leq \frac{|h|}{1+\alpha}Q^*(t),$$

which implies that

$$\limsup_{h\to\infty}(t+h)^{-1}[q(t+h)]^{-\frac{1}{1+\alpha}} \leq \frac{1}{1+\alpha}Q^*(t), \ t \geq a.$$

It follows that

$$\lim_{t\to\infty} t^{-1}[q(t)]^{-\frac{1}{1+\alpha}} = 0,$$

or equivalently

$$\lim_{t\to\infty} t^{\alpha+1}q(t) = \infty. \tag{8}$$

Thus (5) holds, and so (1) is oscillatory. This completes the proof.

In order to study the distribution of zeros of the solutions of (1) we need the generalized sine function $S(\varphi)$ introduced by Elbert [1]. Let $S(\varphi)$ denote the solution of the generalized harmonic oscillator (2) now written as

$$(|\dot{S}|^{\alpha-1}\dot{S})^{\cdot} + \alpha|S|^{\alpha-1}S = 0$$

satisfying the initial condition: $S(0) = 0, \ \dot{S}(0) = 1$, where a dot means differentiation with respect to φ. It can be shown ([1]) that

$$S(\varphi + 2\pi_\alpha) = S(\varphi) \tag{9}$$

and

$$|S(\varphi)|^{\alpha+1} + |\dot{S}(\varphi)|^{\alpha+1} \equiv 1 \tag{10}$$

for all $\varphi \in \mathbf{R}$, where the constant π_α is given by

$$\pi_\alpha = \frac{2\pi}{\alpha+1}\Big/\sin\left(\frac{\pi}{\alpha+1}\right). \tag{11}$$

The main result of this paper will be stated and proved after the following simple lemma.

Lemma. *If* (6) *holds, then*

$$\int_a^\infty [q(t)]^{\frac{1}{1+\alpha}}\,dt = \infty. \tag{12}$$

Proof of the Lemma. This follows immediately from relation (5).

Theorem 2. *Suppose that* (6) *holds. Let* $N[y;T]$ *denote the number of zeros of a solution* $y(t)$ *of* (1) *in the interval* $[a,T]$. *Then, we have*

$$N[y;T] = P[y;T] + R[y;T], \tag{13}$$

where $P[y;T]$ *is the principal term given by*

$$P[y;T] = \frac{1}{\pi_\alpha}\int_a^T [q(s)]^{\frac{1}{1+\alpha}}\,ds \tag{14}$$

and $R[y;T]$ *is the remainder which is of smaller order than* $P[y;T]$ *as* $T\to\infty$ *and satisfies*

$$|R[y;T]| \le \frac{1}{(1+\alpha)\pi_\alpha}\int_a^T \frac{|q'(s)|}{q(s)}ds + O(1). \tag{15}$$

Proof. Since by the Sturmian comparison theorem due to Elbert [1] $N[y_1;T]$ and $N[y_2;T]$ differ by one unit at most for any solutions $y_1(t)$ and $y_2(t)$ of (1), we may restrict our attention to the solution $y_0(t)$ of (1) determined by the initial condition: $y_0(a)=0,\ y_0'(a)=1$; $y_0(t)$ is oscillatory by Theorem 1.

We introduce the polar coordinates $\rho(t),\ \varphi(t)$ for $y_0(t)$ by setting

$$\begin{aligned} [q(t)]^{\frac{1}{1+\alpha}} y_0(t) &= \rho(t)S(\varphi(t)),\\ y_0'(t) &= \rho(t)\dot{S}(\varphi(t)). \end{aligned} \tag{16}$$

It can be shown without difficulty that $\rho(t)$ and $\varphi(t)$ are continuously differentiable on $[a,\infty)$ and satisfy the differential equations

$$\begin{aligned} \frac{\rho'}{\rho} &= \frac{1}{1+\alpha}\frac{q'(t)}{q(t)}|S(\varphi)|^{1+\alpha},\\ \varphi' &= [q(t)]^{\frac{1}{1+\alpha}} + \frac{1}{1+\alpha}\frac{q'(t)}{q(t)}S(\varphi)|\dot{S}(\varphi)|^{\alpha-1}\dot{S}(\varphi). \end{aligned} \tag{17}$$

We use the notation

$$g(\varphi) = S(\varphi)|\dot{S}(\varphi)|^{\alpha-1}\dot{S}(\varphi),$$

in terms of which the second equation in (17) is written as

$$\varphi' = [q(t)]^{\frac{1}{1+\alpha}} + \frac{1}{1+\alpha}\frac{q'(t)}{q(t)}g(\varphi). \tag{18}$$

From the first equation in (16) we see that $y_0(t) = 0$ if and only if $\varphi(t) = j\pi_\alpha,\ j \in \mathbf{Z}$. We may suppose that $\varphi(a) = 0$. In view of (6) there is no loss of generality in assuming that

$$Q^*(t) < (1+\alpha) \quad \text{for } t \geq a,$$

where $Q^*(t)$ is defined by (7). Since

$$|g(\varphi)| \leq 1 \ \text{ for all } \varphi, \tag{19}$$

we have

$$[q(t)]^{\frac{1}{1+\alpha}} + \frac{1}{1+\alpha}\frac{q'(t)}{q(t)} g(\varphi(t)) \geq [q(t)]^{\frac{1}{1+\alpha}} \left(1 - \frac{1}{1+\alpha} Q^*(t)\right) > 0,$$

which implies that $\varphi'(t) > 0$, so that $\varphi(t)$ is increasing for $t \geq a$.

We now integrate (18) over $[a, T]$, obtaining

$$\begin{aligned} \varphi(T) &= \int_a^T [q(s)]^{\frac{1}{1+\alpha}} ds + \frac{1}{1+\alpha} \int_a^T \frac{q'(s)}{q(s)} g(\varphi(s)) ds \\ &= F(T) + G(T)\,. \end{aligned} \tag{20}$$

From (19) it is clear that

$$|G(T)| \leq \frac{1}{1+\alpha} \int_a^T \frac{|q'(s)|}{q(s)} ds. \tag{21}$$

Noting that the number of zeros of $y_0(t)$ in $[a, T]$ is given by

$$N[y_0; T] = \left[\frac{\varphi(T)}{\pi_\alpha}\right] + 1,$$

where $[u]$ denotes the greatest integer not exceeding u, we see from (20) and (21) that the conclusion of the theorem holds with the choice

$$P[y_0; T] = \frac{1}{\pi_\alpha} F(T) = \frac{1}{\pi_\alpha} \int_a^T [q(s)]^{\frac{1}{1+\alpha}} ds.$$

That the term $R[y_0; T] = N[y_0; T] - P[y_0; T]$ is of smaller order than $P[y_0; T]$ follows from the observation that

$$\begin{aligned} \int_a^T \frac{|q'(s)|}{q(s)} ds &= \int_a^T |Q(s)| [q(s)]^{\frac{1}{1+\alpha}} ds \\ &\leq \int_a^T Q^*(s) [q(s)]^{\frac{1}{1+\alpha}} ds = o\left(\int_a^T [q(s)]^{\frac{1}{1+\alpha}} ds\right) \quad \text{as } T \to \infty\,. \end{aligned}$$

This completes the proof.

Remark 1. Theorems 1 and 2 generalize a result for the linear equation $y'' + q(t)y = 0$ found in Hille's book [3; Theorem 9.5.1].

Example 1. Consider the equation

$$(|y'|^{\alpha-1}y')' + \alpha t^{\beta}|y|^{\alpha-1}y = 0,\ t \geq 1, \tag{22}$$

where β is a constant with $1+\alpha+\beta > 0$. The function $q(t) = t^{\beta}$ satisfies

$$\begin{aligned} \int_1^T [q(s)]^{\frac{1}{1+\alpha}} ds &= \frac{1+\alpha}{1+\alpha+\beta}\left(T^{\frac{1+\alpha+\beta}{1+\alpha}} - 1\right), \\ \int_1^T \frac{|q'(s)|}{q(s)} ds &= |\beta| \log T, \end{aligned}$$

and so we conclude from Theorem 2 that the quantity $P[y;T]$ can be taken to be

$$P[y;T] = \frac{1+\alpha}{(1+\alpha+\beta)\pi_\alpha} T^{\frac{1+\alpha+\beta}{1+\alpha}}$$

and (13) holds with this $P[y;T]$ and $R[y;T]$ satisfying

$$R[y;T] = \frac{|\beta|}{(1+\alpha)\pi_\alpha}\log T + O(1).$$

Remark 2. Theorems 1 and 2 cannot be applied to the generalized Euler equation (3), since the function $q(t) = \lambda\alpha t^{-\alpha-1}$ does not satisfy (6). A calculation of $P[y;T]$ and $R[y;T]$ for (3) shows that both of them are of the same logarithmic order as $T \to \infty$.

Remark 3. In [5], M. Piros has investigated a similar problem under a more stringent restriction on $q(t)$, namely he supposed that $q^{\nu}(t)$ is a concave function of t for some $\nu > 0$, fixed. Then he proved that the error term $R[y;T]$ in (13) is $O(1)$. Exactly, the differential equation (22) with $\beta = 1/\nu$ plays the exceptional role in determining the precise value of $R[y;T]$.

We conclude this paper with a remark that Theorems 1 and 2 for (1) can be generalized to the equation

$$(p(t)|y'|^{\alpha-1}y')' + \alpha q(t)|y|^{\alpha-1}y = 0,\ t \geq a, \tag{23}$$

where α and $q(t)$ are as in (1) and $p : [a,\infty) \to (0,\infty)$ is a continuously differentiable function such that

$$\int_a^\infty \frac{dt}{(p(t))^{1/\alpha}} = \infty. \tag{24}$$

Put $P(t) = \int_a^t \frac{ds}{(p(s))^{1/\alpha}}$ and perform the change of variables $(t,y) \to (\tau,\eta)$ defined by $\tau = P(t),\ \eta(\tau) = y(t)$. The equation (23) then reduces to

$$(|\dot\eta|^{\alpha-1}\dot\eta)^{\cdot} + \alpha Q(\tau)|\eta|^{\alpha-1}\eta = 0,\ \tau \geq 0, \tag{25}$$

where $Q(\tau) = (p(t))^{\frac{1}{\alpha}} q(t)$ and $\cdot = d/d\tau$. Since (25) is of the form (1), we can apply Theorems 1 and 2 to (25). Translating the results thus obtained in the original variables, we have the corresponding theorems for (23).

Theorem 3. *The equation* (23) *is oscillatory if*

$$\lim_{t\to\infty}(p(t))^{\frac{1}{\alpha}}((p(t))^{\frac{1}{\alpha}}q(t))'((p(t))^{\frac{1}{\alpha}}q(t))^{-\frac{2+\alpha}{1+\alpha}}=0. \tag{26}$$

Theorem 4. *Suppose that* (26) *holds. Let* $N[y;T]$ *be the number of zeros of a solution* $y(t)$ *of* (23) *in* $[a,T]$. *Then,*

$$N[y;T]=P[y;T]+R[y;T], \tag{27}$$

where $P[y;T]$ *is the principal term given by*

$$P[y;T]=\frac{1}{\pi_\alpha}\int_a^T\left(\frac{q(s)}{p(s)}\right)^{\frac{1}{1+\alpha}}ds \tag{28}$$

and $R[y;T]$ *is of smaller order than* $P[y;T]$ *as* $T\to\infty$ *and satisfies*

$$|R[y;T]|\le\frac{1}{(1+\alpha)\pi_\alpha}\int_a^T\frac{|((p(s))^{1/\alpha}q(s))'|}{(p(s))^{1/\alpha}q(s)}ds+O(1). \tag{29}$$

The above theorems are illustrated by the following examples.

Example 2. Consider the equation

$$(t^\gamma|y'|^{\alpha-1}y')'+\alpha t^\beta|y|^{\alpha-1}y=0,\ t\ge1, \tag{30}$$

where β and γ are constants such that

$$\gamma\le\alpha\ \text{ and }\gamma<1+\alpha+\beta. \tag{31}$$

The functions $p(t)=t^\gamma$ and $q(t)=t^\beta$ satisfy (24) and

$$(p(t))^{\frac{1}{\alpha}}\left((p(t))^{\frac{1}{\alpha}}q(t)\right)'\left((p(t))^{\frac{1}{\alpha}}q(t)\right)^{-\frac{2+\alpha}{1+\alpha}}=\left(\frac{\gamma}{\alpha}+\beta\right)t^{-\frac{1+\alpha+\beta-\gamma}{1+\alpha}}.$$

Hence (30) is oscillatory by Theorem 3. Theorem 4 applied to (30) shows that (27) holds with $P[y;T]$ and $R[y;T]$ satisfying

$$P[y;T]=\frac{1+\alpha}{(1+\alpha+\beta-\gamma)\pi_\alpha}T^{\frac{1+\alpha+\beta-\gamma}{1+\alpha}}$$

and

$$|R[y;T]|\le\frac{\alpha^{-1}\gamma+\beta}{(1+\alpha)\pi_\alpha}\log T+O(1)\ \textit{as}\ T\to\infty,$$

respectively.

Example 3. Consider the partial differential equation

$$div(|Du|^{m-2}Du)+(m-1)|x|^n|u|^{m-2}u=0,\ x\in E, \tag{32}$$

where $m>1$ and n are constants, $Du=(\partial u/\partial x_1,\cdots,\partial u/\partial x_N)$, $|x|$ is the Euclidean length of $x\in\mathbf{R}^N$, $N\ge2$, and E is an exterior domain in $\mathbf{R}^N$. We assume that E is the complement of the unit ball. Our attention will be restricted to radial solutions of (32), that is, those solutions of (32) which depend only on $|x|$. It is easy to see that a radial function $u = y(|x|)$ is a solution of (32) if and only if $y(t)$ satisfies the ordinary differential equation

$$(t^{N-1}|y'|^{m-2}y')' + (m-1)t^{N+n-1}|y|^{m-2}y = 0, \ t \geq 1, \tag{33}$$

which is a special case of (30) with $\alpha = m-1$, $\beta = N+n-1$ and $\gamma = N-1$. The condition (31) for (33) reads:

$$N \leq m \ \text{ and } m+n > 0. \tag{34}$$

Applying the results of Example 2 to (33) and noting that a zero t_0 of $y(t)$ corresponds to a spherical node $|x| = t_0$ of $u = y(|x|)$, we have the following statements:

(i) If (34) holds, then all radial solutions of (32) are oscillatory.

(ii) Suppose that (34) holds. Let $u(x)$ be a radial solution of (32) defined in E and let $N[u;T]$ denote the number of spherical nodes of $u(x)$ contained in the annular domain $\{1 \leq |x| \leq T\}$. Then, $N[u;T]$ is the sum of the principal part $P[u;T]$ and the remainder $R[u;T]$ satisfying, respectively,

$$P[u;T] = \frac{m}{(m+n)\pi_\alpha} T^{\frac{m+n}{m}}$$

and

$$|R[u;T]| \leq \frac{m(N-1)+n(m-1)}{m(m-1)\pi_\alpha} \log T + O(1) \quad \text{as} \quad T \to \infty.$$

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