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OSCILLATION OF A SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the oscillatory behavior of the solutions of the delay differential equation of the form

$$\left(\frac{1}{r(t)}y'(t)\right)' + p(t)y(\tau(t)) = 0.$$

The obtained results are applied to *n-th* order delay differential equation with *quasi-derivatives* of the form

$$L_n u(t) + p(t)u(\tau(t)) = 0.$$

We consider the second order delay differential equation

$$\left(\frac{1}{r(t)}y'(t)\right)' + p(t)y(\tau(t)) = 0\,, \tag{1}$$

where $p,\ r,\ \tau \in C([t_0,\infty))$, $r(t)$, $p(t)$ are positive, $\tau(t) \leqslant t$ and $\lim\limits_{t\to\infty} \tau(t) = \infty$. In the sequel we will assume that Eq.(1) is in canonical form, that is

$$R(t) = \int_{t_0}^{t} r(s)\,\mathrm{d}s \to \infty \quad \text{as} \quad t \to \infty\,. \tag{2}$$

There are many papers devoted to oscillation of all solutions of (1) (see enclosed references). By oscillatory solution of (1) we mean any nontrivial solution that has arbitrarily large zeros for $t \geqslant t_0$, that is there exists a sequence of zeros $\{t_n\}$ $(y(t_n) = 0)$ of $y(t)$ such that $\lim\limits_{n\to\infty} t_n = \infty$. Otherwise $y(t)$ is said to be nonoscillatory. Eq.(1) itself is said to be oscillatory if all its solutions are oscillatory.

Oscillation of a special case of Eq.(1), namely, the linear ordinary equation

$$y''(t) + p(t)y(t) = 0 \tag{3}$$

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has been discussed in the papers of Hille, Wintner, Hartmann and many others (see books [5], [7], [12]). On the other hand Ladde, Lakshmikhantam and Zhang [7] presented the following oscillatory criterion for (1) (see also Shevelo and Varech [13]).

$$\int^{\infty} R^{1-\epsilon}(\tau(t))p(t)\,\mathrm{d}t = \infty\,, \qquad 0<\epsilon<1\,. \tag{4}$$

We note that ϵ cannot be equal to zero. In fact, this criterion cannot cover an important class of Euler's equations of the form

$$\left(\frac{1}{t^{\alpha}}y'\right)' + \frac{c}{t^{\alpha+2}}y = 0\,, \tag{5}$$

where α and c are constants with $\alpha \leqslant 1$.

The first purpose of this paper is to provide an integral criterion for oscillation of Eq.(1), which is similar to (4) but is applicable also to Eq.(5). The second aim of this paper is to show that many others known criteria are included in the obtained result (cf. Theorem 1). The third intention of the paper is to apply obtained results for investigation of oscillation and other asymptotic properties of the *n-th* order differential equation of the form

$$L_n u(t) + p(t)u(\tau(t)) = 0\,.$$

Theorem 1. *Assume that (2) holds and*

$$\tau'(t) \geqslant 0\,, \tag{6}$$

$$\int^{\infty}\left(R(\tau(t))p(t) - \frac{r(\tau(t))\tau'(t)}{4R(\tau(t))}\right)\mathrm{d}t = \infty\,. \tag{7}$$

Then Eq.(1) is oscillatory.

Proof. Let $y(t)$ be a nonoscillatory solution of (1). Without loss of generality, we may suppose that $y(t)>0$, $y(\tau(t))>0$ for $t \geqslant t_1$. From (1) we have $\left(\frac{1}{r(t)}y'(t)\right)' < 0$. This implies in view of (2) (see e.g. [2], [7]) that $\frac{1}{r(t)}y'(t) > 0$. Define

$$z(t) = \frac{R(\tau(t))}{y(\tau(t))}\left(\frac{y'(t)}{r(t)}\right)\,. \tag{8}$$

Then $z(t)>0$. Observe that

$$z'(t) = \frac{r(\tau(t))\tau'(t)}{y(\tau(t))}\left(\frac{y'(t)}{r(t)}\right) + \frac{R(\tau(t))}{y(\tau(t))}\left(\frac{y'(t)}{r(t)}\right)' - \frac{R(\tau(t))}{y(\tau(t))}\left(\frac{y'(t)}{r(t)}\right)\frac{y'(\tau(t))\tau'(t)}{y(\tau(t))}\,.$$

Then in view of (8) and (1)

$$z'(t) = \frac{r(\tau(t))\tau'(t)}{R(\tau(t))}z(t) - R(\tau(t))p(t) - z(t)\left(\frac{y'(\tau(t))}{r(\tau(t))}\right)\frac{r(\tau(t))\tau'(t)}{y(\tau(t))}.$$

Since $\left(\frac{1}{r(t)}y'(t)\right)$ is decreasing, one gets $\frac{y'(\tau(t))}{r(\tau(t))} \geqslant \frac{y'(t)}{r(t)}$ and therefore

$$z'(t) \leqslant \frac{r(\tau(t))\tau'(t)}{R(\tau(t))}\Big(z(t) - z^2(t)\Big) - R(\tau(t))p(t).$$

It is easy to see that the polynomial $P(z) = z - z^2 \leqslant 1/4$. Thus

$$z'(t) \leqslant \frac{r(\tau(t))\tau'(t)}{4R(\tau(t))} - R(\tau(t))p(t).$$

Then integrating the last inequality from t_1 to t, we obtain

$$z(t) \leqslant z(t_1) - \int_{t_1}^{t}\left(R(\tau(s))p(s) - \frac{r(\tau(s))\tau'(s)}{4R(\tau(s))}\right)\mathrm{d}s.$$

Letting $t \to \infty$, we have in view of (7) that $z(t) \to -\infty$ as $t \to \infty$, which contradicts $z(t) > 0$ and the proof is complete.

As a consequence we have the following result, which is due to Ohriska [10].

Corollary 1. *Assume that (2) and (6) are satisfied and*

$$\liminf_{t\to\infty} \frac{R^2(t)p(t)}{r(t)} > \frac{1}{4}. \tag{9}$$

Then the equation

$$\left(\frac{1}{r(t)}y'(t)\right)' + p(t)y(t) = 0$$

is oscillatory.

Proof. Note that (9) implies (7) with $\tau(t) \equiv t$.

Remark. Theorem 1 improves the Corollary 4.5.2 in [7] and Theorem 1 in [13]. As we mentioned above Theorem 1 also covers the Euler's equation (5).

It is known ([1], [7]) that the condition

$$\int^{\infty} tp(t)\,\mathrm{d}t = \infty \tag{10}$$

is necessary (not sufficient) for oscillation of all solutions of the equation

$$y''(t) + p(t)y(t) = 0. \tag{11}$$

Note that criterion for oscillation of Eq.(11) presented in Theorem 1, namely the condition

$$\int^{\infty} \left(tp(t) - \frac{1}{4t} \right) \mathrm{d}t = \infty \tag{12}$$

is stronger than necessary condition (10) and weaker than known sufficient condition

$$\int^{\infty} t^{1-\epsilon} p(t) \, \mathrm{d}t = \infty \,, \qquad 0 < \epsilon < 1 \,. \tag{13}$$

The following result improves Chanturija and Kiguradze's [1] one known for (1) with $\tau(t) \equiv t$.

Corollary 2. *Assume that (2) and (6) hold. Further suppose that*

$$\liminf_{t \to \infty} R(\tau(t)) \int_t^{\infty} p(s) \, \mathrm{d}s > \frac{1}{4} \,. \tag{14}$$

Then Eq.(1) is oscillatory.

Proof. It is sufficient to prove that (14) implies (7). Suppose that (7) fails, that is for all $\epsilon > 0$ there exists a t_1 such that for all $t \geqslant t_1$

$$\int_t^{\infty} \left(R(\tau(s)) p(s) - \frac{r(\tau(s))\tau'(s)}{4R(\tau(s))} \right) \mathrm{d}s < \epsilon \,.$$

Hence

$$\int_t^{\infty} R(\tau(s)) \left(p(s) - \frac{r(\tau(s))\tau'(s)}{4R^2(\tau(s))} \right) \mathrm{d}s < \epsilon \,.$$

Since $R(\tau(t))$ is nondecreasing the last inequality implies

$$R(\tau(t)) \int_t^{\infty} \left(p(s) - \frac{r(\tau(s))\tau'(s)}{4R^2(\tau(s))} \right) \mathrm{d}s < \epsilon \,.$$

Therefore

$$R(\tau(t)) \int_t^{\infty} p(s) \, \mathrm{d}s < \epsilon + \frac{1}{4}, \qquad \text{for all } \epsilon > 0 \,,$$

which contradicts to (14).

Example 1. Consider the second order delay differential equation

$$y''(t) + \frac{a}{t\sqrt{t}} y(\sqrt{t}) = 0, \quad a > 0 \,, \quad t \geqslant 1 \,. \tag{15}$$

Theorem 1 guarantees oscillation of (15) if $a > 1/8$. On the other hand Erbe [4] and Ohriska [11] showed that the equation

$$y''(t) + p(t) y(\tau(t)) = 0 \tag{16}$$

is oscillatory provided that

$$\liminf_{t\to\infty} t \int_t^\infty p(s)\frac{\tau(s)}{s}\,\mathrm{d}s > \frac{1}{4} \tag{17}$$

or

$$\limsup_{t\to\infty} t \int_t^\infty p(s)\frac{\tau(s)}{s}\,\mathrm{d}s > 1\,, \tag{18}$$

respectively. Applying (17) and (18) to Eq.(15), one gets that (15) is oscillatory if $a > 1/4$ and $a > 1$, respectively. Theorem 1 provides better result. Really, if $\tau(t)/t$ is decreasing then (17) implies (14) with $R(t) \equiv t$ and (14) as we mentioned above implies (7).

According to general theory of Trench [15] condition (2) is no restriction and Theorem 1 can be easily rewritten also for noncanonical equation (1), namely when

$$\int^\infty r(s)\,\mathrm{d}s < \infty\,. \tag{19}$$

When this situation occurs, we define

$$\rho(t) = \int_t^\infty r(s)\,\mathrm{d}s\,.$$

By Lemma 1 in [15] every noncanonical equation of the form (1) can be represented in the following canonical form

$$\left(\frac{\rho^2(t)}{r(t)}\left(\frac{y(t)}{\rho(t)}\right)'\right)' + \rho(t)p(t)y(\tau(t)) = 0\,. \tag{20}$$

The transformation $y(t) = \rho(t)z(t)$ preserves oscillation and reduces (20) to the equation

$$\left(\frac{\rho^2(t)}{r(t)}z'(t)\right)' + \rho(t)\rho(\tau(t))p(t)z(\tau(t)) = 0\,. \tag{21}$$

If (19) is satisfied then Eq.(1) is oscillatory if and only if Eq.(21) is oscillatory. Applying Theorem 1 to (21) one gets sufficient condition for oscillation of non-canonical Eq.(1).

Oscillation of the second order equations of the form (1) is very useful when studying asymptotic properties of the *n-th* order delay differential equations of the form

$$L_n u(t) + q(t)u(\tau(t)) = 0\,, \tag{22}$$

where L_n denotes the general disconjugate operator:

$$L_n u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r_{n-1}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\cdots\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r_1(t)}\frac{\mathrm{d}}{\mathrm{d}t}u(t)\,.$$

By property (A) of (22) is meant the situation when for n even Eq.(22) is oscillatory and for n odd $|u(t)|$ is decreasing function for any nonoscillatory solution $u(t)$ of (22) and $\lim_{t\to\infty} u(t) = 0$.

Kusano, Naito and Tanaka ([8], [14]) discussed property (A) of (22) by comparing (22) with a set of second order differential equations of the form (1) in the sense that if all second order equations are oscillatory then (22) enjoys property (A).

Combining Theorem 1 with the results from [8] and [14] we get sufficient conditions for property (A) of (22).

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