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ON THE STRUCTURE OF OSCILLATORY SOLUTIONS OF A THIRD ORDER DIFFERENTIAL EQUATION

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Abstract. The aim of the paper is to study the structure of oscillatory solutions of a nonlinear third order differential equation $y''' + py'' + qy' + rf(y, y', y'') = 0$.

1. Introduction

The aim of the paper is to study the structure of oscillatory solutions of the nonlinear differential equation

$$y''' + p(t)y'' + q(t)y' + r(t)f(y, y', y'') = 0 \tag{1}$$

where $p, q \in C^0(R_+)$, $r \in L_{\text{loc}}(R_+)$, $f \in C^0(R^3)$, $R_+ = [0, \infty)$, $R = (-\infty, \infty)$,

$$f(x_1, x_2, x_3)x_1 > 0 \quad \text{for} \quad x_1 \neq 0 \quad \text{on} \quad R^3 \tag{2}$$

and

$$r \quad \text{does not change the sign on} \quad R_+ . \tag{3}$$

A function $y \in C^2(I)$ is said to be a solution of (1) if y'' is absolutely continuous and (1) holds almost everywhere on I. It is called proper if $I = R_+$ and $\sup\limits_{\tau \leqslant t < \infty} |y(t)| > 0$ holds for an arbitrary $\tau \in R_+$. A proper solution is said to be oscillatory if it has arbitrarily large zeros.

Motivation for the study of properties of oscillatory solutions of (1) comes from the papers [1] and [8]. In [1] the structure of solutions of (1) is studied for $p \equiv q \equiv 1$. It is shown that every nontrivial solution y may have at most one interval of (have no) double or triple zeros in case $r \geqslant 0$ ($r \leqslant 0$) and the zeros of y and y', with the possible exception of the multiplied ones, are separated.

Similar results are obtained for a special kind of (1)

$$y''' + p(t)y'' + q(t)y' + r(t)g(y) = 0 \tag{4}$$

where $p, q, r \in C^0(R_+)$, $g \in C^0(R)$, $g(x)x > 0$ for $x \neq 0$ and (3) holds. The following two theorems are due to Moravský [8]:

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Theorem A. *Let $q \in C^1(R_+)$, $p \geqslant 0$, $q \geqslant 0$ and $r \leqslant 0$ on R_+, and let a constant $k > 0$ exist such that $|g(x)| \leqslant k|x|$ on R, $q' + pq - 2kr \leqslant 0$ on R_+. Further, let the functions p and $q' + pq - 2kr$ be not equal to zero at any subinterval of R_+ at the same time. Let $t_0 \in R_+$ and $y : R_+ \to R$ be a solution of (4) for which*

$$2yy'' - y'^2 + qy^2\Big|_{t=t_0} < 0\,.$$

Then the zeros of y and y' are separated on (t_0, ∞).

Theorem B. *Let $r \geqslant 0$, $p \geqslant |q|$ on R_+ and let $k \in (0, \infty)$ exist such that $|g(x)| \geqslant k|x|$ on R. Let y be a solution of (4) for which $t_0 \in R_+$ exists such that*

$$y(t_0)y''(t_0) - \frac{1}{2}y'^2(t_0) < 0\,.$$

Further, let one of the following assumptions hold:

(i) *$q \leqslant 0$, $q + 2kr \geqslant 0$ on R_+ and the functions $q + 2kr$ and $p + q$ are not equal to zero on any subinterval of R_+ at the same time;*

(ii) *$q \geqslant 0$, $q \leqslant 2k^2 r$ on R_+ and the functions $p - q$ and $q - 2k^2 r$ are not, at the same time, equal to zero on any subinterval of R_+.*

Then the zeros of y and y' are separated on (t_0, ∞).

Our goal: To generalize and extend these results for Eq. (1), to study mutual position of zeros of an oscillatory solution y and its derivatives y' and y''. The paper does not deal with the existence of oscillatory solutions. As concern to this problem, see e.g. [3, 6, 8,9,10].

2. Structure of oscillatory solutions

The following equation plays an important role in investigations of (1):

$$h'' + ph' + qh = 0\,. \tag{5}$$

A solution $h : R_+ \to R$ of (5) is called nonoscillatory if it is different from zero in some neighbourhood of ∞. Eq. (5) is said to be nonoscillatory if every nontrivial solution is nonoscillatory. If Eq. (5) is nonoscillatory, then it is said to be disconjugate if each nontrivial solution has at most one zero on R_+. Note, that Eq. (5) is disconjugate if and only if it has a positive solution on $(0, \infty)$, see [7].

Let $T \in R_+$, $J = (R, \infty)$ and $h > 0$ be a solution of (5) on $J \subset R_+$. Together, with (1) let us consider the differential equation with quasiderivatives

$$y^{[3]} + rRhf_1(y^{[0]}, y^{[1]}, y^{[2]}) = 0 \tag{6}$$

on J where

$$R(t) = \exp\Big(\int_0^t p(s)\,ds\Big)\,, \tag{7}$$

$$y^{[0]} = y, \quad y^{[1]} = \frac{y'}{h}, \quad y^{[2]} = Rh^2(y^{[1]})' = R(y''h - y'h'), \quad y^{[3]} = (y^{[2]})'\,, \tag{8}$$

$$f_1(x_1, x_2, x_3) = f(x_1, hx_2, \frac{x_3}{Rh} + h'x_2) \quad \text{on} \quad R^3$$

and

$$f_1(x_1, x_2, x_3)x_1 > 0 \quad \text{for} \quad x_1 \neq 0 \quad \text{holds}.$$

A function $y \in C^2(J)$ is said to be a solution of (6) if y'' is absolutely continuous and (6) holds almost everywhere on J. It is called oscillatory if it has arbitrary large zeros and $\sup_{\tau \leqslant t < \infty} |y(t)| > 0$ holds for an arbitrary $\tau \in J$.

If (5) is nonoscillatory, then (1) can be transformed into the equation (6).

Lemma 1. *Let $h > 0$ be a solution of (5) on $J \subset R_+$. Then a function $y : J \to R$ is a solution of (1) on J if, and only if, y is a solution of the equation (6).*

Proof. The statement can be obtained by the direct computation similarly as in [3] for $p \equiv 0$. □

Remark 1. It follows from (8) that

$$y' = hy^{[1]}, \quad y'' = \frac{y^{[2]}}{Rh} + h'y^{[1]}. \tag{9}$$

If (5) is nonoscillatory, some results, obtained for Eq. (6), can be transformed into (1).

Lemma 2. *Let $\alpha \in \{0, 1\}$, $(-1)^\alpha r \geqslant 0$ on R_+ and $T \in R_+$. Let (5) be nonoscillatory and h be its solution such that*

$$h > 0 \quad \text{and} \quad (-1)^\alpha (ph + 3h') \geqslant 0 \quad \text{on} \quad [T, \infty). \tag{10}$$

Further, let y be a solution of (1) and

$$E(t) = (-1)^\alpha R[-2hy''y + 2h'yy' + hy'^2] \tag{11}$$

where R is defined by (7). Then E is nondecreasing on $[T, \infty)$ and

$$E'(t) = (-1)^\alpha R[2hryf(y, y', y'') + (ph + 3h')y'^2] \geqslant 0 \quad \text{on} \quad [T, \infty).$$

Proof follows by the direct computation using (1) and (5). □

First, let us sum up some results concerning Eq. (5).

Lemma 3. (i) *If $p \equiv 0$ on R_+ and $\limsup_{t \to \infty} t^2 q(t) \in [-\infty, \frac{1}{4})$, then (5) is nonoscillatory.*
(ii) *If $q > 0$ on R_+ and*

$$\int_0^\infty q(t)R(t) \int_0^t \frac{ds}{R(s)}\, dt < \infty, \int_0^\infty \frac{ds}{R(s)} = \infty, \tag{12}$$

where R is given by (7), then Eq. (5) is nonoscillatory and there exists its eventually positive and nondecreasing solution.

(iii) *Let $q \leqslant 0$ on R_+. Then Eq.* (5) *is disconjugate on R_+ and there exist solutions h_0 and h_1 such that*

$$h_0 > 0,\ h_0' \leqslant 0,\ h_1 > 0 \quad \text{and} \quad h_1' > 0 \quad \text{on} \quad R_+ .$$

Proof. Eq. (5) can be transformed into the equivalent equation

$$(R(t)y')' + q(t)R(t)y = 0 .$$

(i) See [7], Chap. XI, Th. 7.1.
(ii) See [5].
(iii) See [7], Chap. XI, Conseq. 6.4. □

Lemma 4. *Let Eq.* (5) *be nonoscillatory with a positive solution h on J and let y be an oscillatory solution of Eq.* (6).

(a) *Let $r \geqslant 0$ on J. Then there exists at most one number $\tau \in J$ such that either*

$$y(\tau) = y^{[1]}(\tau) = 0,\ y^{[2]}(\tau) \neq 0, \quad \tau_1 = \tau \tag{13}$$

or

$$\begin{gathered} y^{[1]}(t) = y^{[2]}(t) = 0\,,\ t \in [\tau_1, \tau] \cap J, \tau_1 \leqslant \tau \\ \text{and } \sup |y^{[1]}(t)| \neq 0 \text{ in any right (left) neighbourhood} \\ \text{of } t = \tau \ (\text{of } t = \tau_1 \text{ if } T < \tau_1) \end{gathered} \tag{14}$$

holds.

If τ exist, put $J_1 = (T, \tau_1)$, $J_2 = J_3 = (\tau, \infty)$. In the opposite case put

$$J_1 = (T, \bar\tau), \quad J_2 = (T, \infty), \quad J_3 = (\bar\tau, \infty)$$

in case that

$$y\, y^{[1]} < 0, \quad y\, y^{[2]} > 0 \quad \text{in a right neighbourhood of } T \tag{15}$$

holds where $\bar\tau$ is the smallest zero of $y^{[1]}$ on J and

$$J_1 = \emptyset, \quad J_2 = J_3 = (T, \infty) \quad \text{otherwise}.$$

Then $|y|$ is decreasing on J_1, $|y^{[1]}|$ and $|y^{[2]}|$ are nonincreasing on J_1,

$$y\, y^{[1]} < 0, \quad y\, y^{[2]} \geqslant 0 \quad \text{on} \quad J_1 \tag{16}$$

and y and $y^{[1]}$ have only simple zeros on J_2 which are separated on J_3.

(b) *Let $r \leqslant 0$ on J. Then the zeros of y and $y^{[1]}$ are simple and separated on J and for every zero σ of $y^{[1]}$ $y(\sigma)\, y^{[2]}(\sigma) < 0$ holds. Moreover, if $y \neq 0$ in a right* *neighbourhood of T and $\lim\limits_{t\to T_+} y^{[2]}(t)=0$, then $|y^{[1]}|\neq 0$ is nonincreasing in a right neighbourhood of T.*

Proof. The statement is a consequence of some results in [2]. Let us note, that in spite of Ths. 3(ii) and 6(ii) of [2] were proved on R, they are valid on J, too – the proof is identical. Further, using (8), the assumptions of Ths. 1, 3 (ii), 4 and 6 (ii) from [2] are fulfilled.

(a) According to (6) – (8) the following relations hold: Let $L\subset J$.

(i) Let $j\in\{1,2\}$ and $y^{[j]}\geqslant 0$ ($\leqslant 0$) on L. Then $y^{[j-1]}$ is nondecreasing (nonincreasing) on L.

(ii) If $y\geqslant 0$ ($y\leqslant 0$) on L, then $y^{[2]}$ is nonincreasing (nondecreasing) on L.

From this, from Th. 3 (ii) and Remark 5 (i) of [2] the structure of (oscillatory solution) y has three parts: $T\leqslant\sigma_1\leqslant\sigma_2<\infty$,

Part I.

$$y\neq 0,\quad y\,y^{[1]}\leqslant 0,\quad y\,y^{[2]}\geqslant 0\quad\text{on}\quad (T,\sigma_1)\,, \tag{17}$$

(Type V of [2]); this part may be missing ($\sigma_1=T$).

Part II.

$$y^{[i]}\equiv 0,\qquad i=0,1,2,\quad\text{on}\quad [\sigma_1,\sigma_2]$$

(Type VIII of [2]); this part may be missing ($\sigma_1=\sigma_2$), but if Parts I and II are present, then the inequalities (17) are sharp (see Th. 1(ii) of [2]).

Part III. All zeros of y and y' are simple and separated on (σ_2,∞): According to Th. 3(ii) and Remark 5(i) there exists $\sigma_3\in[\sigma_2,\infty)$ such that y is nonoscillatory on (σ_2,σ_3) with simple and separated zeros of y and y' on (σ_2,σ_3) and y is oscillatory on (σ_3,∞) with simple and separated zeros of y and y' on (σ_3,∞). The proof of Lemma 3 of [2] (or $\lim\limits_{t\to\sigma_3}$) shows that they are no problems with $t=\sigma_3$; if σ_3 is a zero of either y or y', it is simple and zeros of y and y' are separated on (σ_2,∞).

If Part II is present, we put $\tau=\sigma_2$, $\tau_1=\sigma_1$ and the conclusion holds. Let Part II be missing. Then all zeros of y and y' are isolated. If Part I is missing then the statement holds. Thus, suppose, that $\sigma_1>T$. According to (17), (i), (ii) there exists at most one τ such that either (13) or (14) with $y\neq 0$ on $(\tau_1,\tau]$ holds. If τ exists it is evident that $\tau=\sigma_1$ and the conclusion holds. If τ does not exist, then $y\,y^{[1]}<0$ on (T,σ_1). Moreover, if σ_1 and $\bar\sigma_1>\sigma_1$ are first two zeros of $y^{[1]}$ on J, then $y(\sigma_1)y^{[2]}(\sigma_1)>0$ and it is easy to prove that $y\neq 0$ on $[\sigma_1,\bar\sigma_1]$ (use (i), (ii) and $y\,y^{[3]}\leqslant 0$ a.e. on J_1) and zeros of y and $y^{[1]}$ are separated on (σ_1,∞) only.

(b) The statement can be proved similarly. Note that the situation is much more simple and y is of Type II (from [2]) in a right neighbourhood of ∞. If $y\neq 0$ in a right neighbourhood of T and $\lim\limits_{t\to T_+} y^{[2]}(t)=0$ then y is either of Type II or of Type IV from [2]. □

Theorem 1. *Let Eq.* (5) *be nonoscillatory with a positive solution on* (T,∞), $T \geqslant 0$ *and let* y *be an oscillatory solution of* (1).

(a) *Let* $r \geqslant 0$ *on* R_+. *Then there exists at most one number* $\tau \in [T,\infty)$ *such that either*

$$y(\tau) = y'(\tau) = 0, \quad y''(\tau) \neq 0\,, \tau_1 = \tau \tag{18}$$

or

$$\begin{gathered} y'(t) = y''(t) = 0, \quad t \in [\tau_1, \tau],\ T \leqslant \tau_1 \leqslant \tau \\ \text{and } \sup |y'(t)| \neq 0 \text{ in any right (left) neighbourhood} \\ \text{of } t = \tau \text{ (of } t = \tau_1 \text{ if } T < \tau_1) \end{gathered} \tag{19}$$

holds.

If τ *exist, then put* $I = [T, \tau_1)$ *and* $I_1 = I_2 = (\tau, \infty)$. *In the opposite case denote by* $\bar\tau$ *the smallest zero of* y' *on* $[T,\infty)$ *and put* $I = [T,\bar\tau)$, $I_1 = (T,\infty)$, $I_2 = (\bar\tau,\infty)$ *in case that* $y(\bar\tau)\,y''(\bar\tau) > 0$ *and* $I = \emptyset$, $I_1 = [T,\infty)$, $I_2 = (T,\infty)$ *otherwise.*
Then $|y|$ *is decreasing and* $y\,y' < 0$ *on* I, *and* y *and* y' *have only simple zeros on* I_1 *which are separated on* I_2.

(b) *Let* $r \leqslant 0$ *on* R_+. *Then* y *and* y' *have only simple zeros on* $[T,\infty)$ *which are separated on* (T,∞).

Proof. Let h be a solution of (5), $h > 0$ on (T,∞) and $J = (T,\infty)$. Then the assumptions of Lemma 1 are fulfilled and Eq. (1) and Eq. (6) are equivalent on J. Moreover, according to (8) and (9) the relations

$$\begin{cases} y'(\tau) = 0 \Leftrightarrow y^{[1]}(\tau) = 0\,, \\ y(\tau) = y'(\tau) = 0,\ y''(\tau) \neq 0 \Leftrightarrow y(\tau) = y^{[1]}(\tau) = 0,\ y^{[2]}(\tau) \neq 0\,, \\ y(\tau) \neq 0,\ y'(\tau) = y''(\tau) = 0 \Leftrightarrow y(\tau) \neq 0\,,\ y^{[1]}(\tau) = y^{[2]}(\tau) = 0\,, \\ y^{(i)}(\tau) = 0\,,\ i = 0,1,2 \Leftrightarrow y^{[i]}(\tau) = 0\,,\ i = 0,1,2 \end{cases} \tag{20}$$

hold on J.

(a) The statement of the theorem on J follows from Lemma 4(a) and from (20). It is necessary to extend it to the interval $[T,\infty)$.

Let there exist $\tau \in (T,\infty)$ such that either (18) or (19) holds. Then, according to Lemma 4, (20) and (8) all zeros of y and y' are simple on I_1 and separated on I_2,

$$\begin{gathered} |y| \quad \text{is decreasing}, \quad y\,y^{[1]} < 0, \\ |y^{[j]}|, \quad j = 1,2 \quad \text{are nonincreasing on} \quad (T,\tau_1)\,. \end{gathered} \tag{21}$$

From this and from (8) $y(T) \neq 0$, $y\,y' < 0$ on (T,τ_1). We prove indirectly that $y'(T) \neq 0$. Thus, suppose that $y'(T) = 0$. If $h(T) > 0$ this result follows from (8) and (21). Let $h(t) = 0$. Then

$$\lim_{t\to T_+} y^{[2]}(t) = \lim_{t\to T_+} [R(t)\,(y''(t)h(t) - y'(t)h'(t))] = 0$$

and (21) yields $y^{[2]} \equiv 0$, $y^{[1]} \equiv \text{const}$ on $(T, \tau_1]$. As $y^{[1]}(\tau_1) = 0$ we have $y^{[1]} \equiv 0$ on $(T, \tau_1]$ that contradicts to (21). Thus $y'(T) \neq 0$, neither (18) nor (19) holds at $\tau = T$ and $y\,y' > 0$ on $[T, \tau_1)$.

Let neither (18) nor (19) hold for $\tau \in (T, \infty)$. Then all zeros of y and y' on (T, ∞) are simple.

First, suppose, that

$$\text{either (18) or (19) holds at} \quad \tau = T\,. \tag{22}$$

It is necessary to prove that the zeros of y and y' are separated on (T, ∞).

Let (18) be valid at $\tau = T$. Then $y\,y' > 0$ and, according to (8) $y\,y^{[1]} > 0$ is valid in some right neigbourhood of T. Thus, (15) does not hold and it follows from Lemma 4(a) that the zeros of y and $y^{[1]}$, and thus also the zeros of y and y' are separated on J.

Let (19) be valid at $\tau = T$. Then it follows from (8) and (22) that $\lim\limits_{t\to T_+} y^{[1]}(t) = \lim\limits_{t\to T_+} y^{[2]}(t) = 0$ (use L'Hospital rule in the first limit if $h(T) = 0$). From this (15) does not hold and according to Lemma 4 the zeros of y and $y^{[1]}$, and thus the zeros of y and y', are separated on J.

Finally, suppose, that there exists no $\tau \in [T, \infty)$ for which either (18) or (19) holds, i.e. all zeros of y and y' are simple on $[T, \infty)$.

If $y(T) \neq 0$ and $y'(T) \neq 0$ then the statement of the theorem follows from Lemma 4 and (8). In all other possible cases, i.e. if either $y(T) = 0$, $y'(T) \neq 0$ or $y(T) \neq 0$, $y'(T) = 0$, it is easy to see, using (8), that (15) is not valid and according to Lemma 4 the zeros of y and y' are separated on (T, ∞).

(b) Using Lemma 4 (b) and (20) the zeros of y and y' are simple and separated on J and we must only prove that $t = T$ is not multiplied zero of either y or y'.

Thus, suppose

$$y'(T) = 0\,. \tag{23}$$

Consider two cases:

1° There exists a sequence of zeros of y on J tending to T;

2° $y \neq 0$ in a right neighbourhood J_1 of T.

Ad 1°. We prove that this case is impossible. As $y(T)$ exists we have

$$y(T) = 0\,. \tag{24}$$

Let h_1 be a solution of Eq. (5) with the initial condition $h_1(T) = 1$, $h_1'(T) < -\frac{p(T)}{3}$. Let $J_2 = [T, \alpha]$ be such interval that $h_1(t) > 0$ on J_2 and

$$-ph_1 - 3h_1' \geqslant 0 \quad \text{on} \quad J_2\,.$$

It is evident, that $\alpha > T$ exists. Then the function

$$E_1 = R(2h_1 y'' y - 2h' y\,y' - hy'^2)$$

(see Lemma 2, $E = E_1$, $h = h_1$, $\alpha = -1$) is nondecreasing on J_2.

Let $\bar{\tau} \in (T, \alpha]$ be a zero of y'. Then, according to Lemma 4 (b), (8) and (9) $y(\bar{\tau})y''(\bar{\tau}) < 0$ and thus $E_1(\bar{\tau}) < 0$. The contradiction to $E_1(T) = 0$ (see (23), (24)) and E being nondecreasing proves that this case is impossible.

Ad 2°. Let $y(T) = y'(T) = 0$, $y''(T) \neq 0$. Then (8) yields $y\, y^{[j]} > 0$, $j = 1, 2$ in some right neighbourhood of T and as $y\, y^{[3]} \geqslant 0$ a.e. on J we can conclude that $y\, y^{[1]} > 0$, $j = 1, 2$ on J that contradicts to y being oscillatory.

Thus, let $y'(T) = y''(T) = 0$. Then, According to (8) (use L'Hospital rule if $h(T) = 0$) we have

$$\lim_{t \to T_+} y^{[1]}(t) = 0, \quad \lim_{t \to T_+} y^{[2]}(t) = 0\,.$$

But Lemma 4 (b) yields $y^{[1]} \neq 0$, $|y^{[1]}|$ is nonincreasing. A contradiction. □

According to Th. 1 an oscillatory solution y may have one interval on which y is trivial in case $r \geqslant 0$. All other zeros are isolated. But y' (y'') may have one interval of zeros on which $y \neq 0$ $(y \neq 0,\ y' \neq 0)$. The following lemma describes conditions, under which such intervals exist.

Lemma 5. *Let $I = [\tau_1, \tau_2]$, $\tau_1 < \tau_2$ and let y be a solution of (1) defined on I. Then*

(a) *$y' \equiv 0$ on I if, and only if $y \equiv C$ and $r(t)f(C, 0, 0) = 0$ on I.*

(b) *$y'' = 0$ on I if, and only if constants C and C_1 exist such that $y = Ct + C_1$ and*

$$Cq(t) + r(t)f(Ct + C_1, C, 0) = 0 \quad \text{on} \quad I\,. \tag{25}$$

Especially, $y \neq 0$, $y' = y'' = 0$ on I if, and only if $y \equiv C \neq 0$ and $r(t) \equiv 0$ on I.

Proof follows directly from (1) and (2). □

Remark 2. If the Cauchy problem of (1) is unique and if $\sup\limits_{t \in I} |r(t)| > 0$, then $\tau_1 = 0$, $\tau_2 = \infty$ in case (a).

Theorem 2. *Let Eq. (5) be nonoscillatory with a solution $h > 0$ on (T, ∞), $T \in R_+$ and let y be an oscillatory solution of (1). Let I_2 be defined as in Th. 1(a) (let $I_2 = (T, \infty)$) if $r \geqslant 0$ $(r \leqslant 0)$ on R_+. Let τ_1 and τ_2 be two consecutive zeros of y on I_2, $\tau_1 < \tau_2$, i.e.*

$$y(\tau_1) = y(\tau_2) = 0, \quad y(t) \neq 0 \quad \text{on} \quad (\tau_1, \tau_2)\,.$$

Let $y'(\tau_1) \neq 0$. Denote by $\tau_3 \in (\tau_1, \tau_2)$ the only zero of y' on $[\tau_1, \tau_2]$.

(i) *Let $r \geqslant 0$ and $h' \geqslant 0$ on $[T, \infty)$. Then y'' has a zero on $[\tau_1, \tau_2]$ and all zeros of y'' from $[\tau_1, \tau_2]$ are lying on (τ_1, τ_3). Moreover, if $q \geqslant 0$, then y'' has the only interval of zeros on (τ_1, τ_3).*

(ii) *Let $r \leqslant 0$ and $h' \leqslant 0$ on $[T, \infty)$. Then y'' has a zero on $[\tau_1, \tau_2]$ and all zeros of y'' are lying on (τ_3, τ_2). Moreover, if $q \geqslant 0$, then y'' has the only interval of zeros on (τ_3, τ_2).*

Proof. The number τ_3 exists according to Th. 1.

(i) Let for the simplicity $y > 0$ on (τ_1, τ_2). According to Lemma 1 Eq. (1) is equivalent to Eq. (6) and thus it follows from [2, Th. 3] that there exists the only interval $[\tau_4, \bar\tau_4]$, $\tau_4 \leqslant \bar\tau_4$ of zeros of $y^{[2]}$ on $[\tau_1, \tau_2]$ and

$$(26)\qquad \begin{cases} \quad \tau_1 < \tau_4 \leqslant \bar\tau_4 < \tau_3 < \tau_2\,, \\ y^{[1]} > 0 \quad \text{on} \quad [\tau_1, \tau_3)\,, \qquad y^{[2]} > 0 \quad \text{on} \quad [\tau_1, \tau_4)\,, \\ \qquad < 0 \quad \text{on} \quad (\tau_3, \tau_2]\,, \qquad\quad < 0 \quad \text{on} \quad (\bar\tau_4, \tau_2]\,, \\ y^{[3]} \leqslant 0 \quad \text{on} \quad (\tau_1, \tau_2)\,. \end{cases}$$

Note, that according to (8) y' and $y^{[1]}$ have the same zeros and the same signs.

Using (26) and (9) we have $y''(\tau_1) > 0$, $y''(\tau_2) < 0$. Thus, y'' has a zero on $[\tau_1, \tau_2]$. Moreover, (26) and (9) yield

$$\operatorname{sign} y''(t) = \operatorname{sign} y^{[2]}(t) \quad \text{on} \quad [\tau_1, \tau_4] \cup [\tau_3, \tau_2]$$

and we can conclude that y'' has all zeros on $[\tau_4, \tau_3) \subset (\tau_1, \tau_3)$.

Further, suppose $q \geqslant 0$. Then according to (9), (8) and (26)

$$(Ry'')' = \frac{y^{[3]}}{h} + Ry^{[1]}(h'' + ph') = \frac{y^{[3]}}{h} - qRhy^{[1]} \leqslant 0 \quad \text{a.e. on} \quad [\tau_4, \tau_3)\,.$$

Thus Ry'' is nonincreasing and y'' has the only interval of zeros.

(ii) The proof is similar. We must use Th. 6 from [2] instead of Th. 3. □

Remark 3. According to Th. 3 the structure of zeros of a solution y and its derivatives y' and y'' of (1) is the same as in case $p \equiv q \equiv 0$, see [1].

Remark 4. If (5) is disconjugate, then it has a positive solution on $(0, \infty)$ and the conclusions of Ths. 1 and 2 are valid on R_+.

Corollary 1. *Let y be an oscillatory solution of* (1).

(i) *Let either $p \equiv 0$ and $\limsup_{t\to\infty} t^2 q(t) \in [-\infty, \frac14)$ or $q > 0$ on R_+ and (12) be valid. Then the zeros of y and y' are separated in some neighbourhood of ∞.*

(ii) *Let $q \leqslant 0$ and $r \leqslant 0$ on R_+. Then the zeros of y and y' are separated on $(0, \infty)$.*

(iii) *Let $p \geqslant 0$, $q \leqslant 0$ and $r \geqslant 0$ on R_+. Let $t_0 \in R_+$ be such that*

$$(27)\qquad -2y''(t_0)\, y(t_0) + y'^2(t_0) > 0, \quad y(t_0)\, y'(t_0) \geqslant 0$$

holds. Then the zeros of y and y' are simple and separated on $[t_0, \infty) \cap (0, \infty)$.

(iv) *Let $q > 0$ and $r \geqslant 0$ on R_+ and let (12) be valid. Then there exists $T \geqslant 0$ such that for arbitrary consecutive zeros τ_1 and τ_2, $T \leqslant \tau_1 < \tau_2$ of y the functions y' and y'' have the only zero τ_3 and the only interval $[\bar\tau_4, \tau_4]$ of zeros on $[\tau_1, \tau_2]$, respectively, and $\tau_1 < \bar\tau_4 \leqslant \tau_4 < \tau_3 < \tau_2$ holds.*

Proof. (i) See Th. 1 and Lemma 3 (i), (ii).

(ii) See Th. 1 (b), Lemma 3 (iii) and Remark 4.

(iii) It follows from Lemma 3 (iii) that there exists a solution h of (5) such that the assumptions of Lemma 2 are fulfilled and $h' \geqslant 0$ on R_+. Let E be given by (11). Then $E(t_0) > 0$. Let I_2 and $\bar{\tau}$ be defined as in Th. 1(a). Let τ be a double zero of either y or y'. Then $E(\tau) = 0$. As E is nondecreasing, then $\tau < t_0$ and $t_0 \in I_2$. Further, let all zeros of y and y' are simple on R_+. If $y(\bar{\tau})y''(\bar{\tau}) > 0$ then $E(\bar{\tau}) < 0$ and thus $\bar{\tau} < t_0$, $t_0 \in I_2$; in the opposite case $I_2 = (0, \infty)$. The conclusion follows from Th. 1 (a) and Remark 4.

(iv) See Th. 2 (i) and Lemma 3 (ii). □

Remark 5. (i) Let the assumption of Cor. 1 (iii) be valid. Then there exists $t_0 \in R_+$ such that (27) holds; t_0 can be choosen as an arbitrary simple zero of y.

(ii) Cor. 1 extends the results of Ths. A and B. Moreover, Cor. 1 (iii), in fact, generalizes Th. B: Let there exist $t_0 \in R_+$ such that $y(t_0)y''(t_0) - \frac{1}{2}y'^2(t_0) < 0$ is valid and let $t_1 \geqslant t_0$ be the first simple zero of $y\,y'$. Then (27) is valid.

Ths. 1 and 2 solve our problem in case that Eq. (5) is nonoscillatory. In the opposite case the transformation, described in Lemma 1, can not be used. Thus, further, let us turn our attention to the case that Eq. (5) may be oscillatory.

Lemma 6. *Let $q \in C^1(R_+)$, $\alpha \in \{0,1\}$,*

$$(28) \qquad (-1)^\alpha p \leqslant 0, \quad (-1)^\alpha r \leqslant 0, \quad (-1)^\alpha (q' + pq) \geqslant 0 \quad \text{on} \quad R_+$$

and $y : R_+ \to R$ be a solution of (1). Then the function

$$(29) \qquad F(t) = (-1)^\alpha R(t) \left[2y''(t)\,y(t) - y'^2(t) + q(t)\,y^2(t)\right], \qquad t \in R_+$$

is nondecreasing on R_+ and

$$(30) \qquad \begin{aligned} F'(t) = (-1)^\alpha R(t) \big[&-p(t)\,y'^2 + (q'(t) + q(t)\,p(t))\,y^2 - \\ &-2r(t)\,y(t)\,f(y(t), y'(t), y''(t))\big] \geqslant 0 \end{aligned}$$

holds where R is defined by (7).

Proof follows by the direct computation from (28) and (2).

Theorem 3. *Let $q \in C^1(R_+)$, $\alpha \in \{0,1\}$, (28) be valid and let the functions p, r and $q' + pq$ are not equal to zero on any subinterval of R_+ at the same time. Let y be an oscillatory solution of (1).*

(i) Let $\alpha = 0$. Then all zeros of y are simple on R_+. Moreover, if $q \geqslant 0$ on R_+, then all zeros of y' are simple, too, on R_+, and the zeros of y and y' are separated on R_+.

(ii) Let $\alpha = 1$. Then at most one maximal interval $[\tau_1, \tau_2]$, $0 \leqslant \tau_1 \leqslant \tau_2 < \infty$ exists such that

$$(31) \qquad y(t) = y'(t) = 0, \qquad t \in [\tau_1, \tau_2]\,.$$

If this interval exists and $\tau_1 > 0$, then y has no zero on $[0, \tau_1)$.

(iii) *Let $\alpha = 1$ and $q \geqslant 0$ on R_+, τ be a simple zero of y. Then all zeros of y' on (τ, ∞) are simple, i.e. the relation*

$$y'(\bar{\tau}) = 0, \quad \bar{\tau} > \tau \Rightarrow y''(\bar{\tau}) \neq 0 \tag{32}$$

holds. Moreover, the zeros of y and y' are separated on (τ, ∞).

Proof. First, solve the problem when

$$F \equiv 0 \qquad \text{on} \qquad [T_1, T_2], \quad T_1 < T_2$$

holds where F is defined by (29). In this case, $F' \equiv 0$ and with respect to the fact that all three terms in (30) are nonnegative, we can conclude that they are equal to zero on $[T_1, T_2]$. From this and from the assumptions of the theorem

$$y \equiv K = \text{ const.}, \quad y' \equiv y'' \equiv 0 \quad \text{on} \quad [T_1, T_2] \tag{33}$$

must be valid.

(i) Let, on the contrary, τ be a zero of y for which $y(\tau) = y'(\tau) = 0$ is valid. As y is oscillatory, there exists its zero $\bar{\tau}$ greater than τ, $\bar{\tau} > \tau$, $y(\bar{\tau}) = 0$, such that

$$\max_{\tau \leqslant t \leqslant \bar{\tau}} |y(t)| > 0\,. \tag{34}$$

Hence, Lemma 6 yields

$$F(\tau) = 0\,, \tag{35}$$

$F(\bar{\tau}) \leqslant 0$ and F is nondecreasing; thus $F \equiv 0$ on $[\tau, \bar{\tau}]$ and (33) holds on $[T_1, T_2] = [\tau, \bar{\tau}]$. As $y(\tau) = 0$, then $K = 0$ and $y \equiv 0$ on $[\tau, \bar{\tau}]$. The contradiction to (34) proves that all zeros of y are simple.

Let $q \geqslant 0$ on R_+. The conclusion, that all zeros of y' are simple, can be proved similarly to the same result for y. Only $F(\tau) \geqslant 0$ must be used instead of (35).

Further, let $t_1 < t_2$ be two consecutive zeros of y. Then y' has a zero according to the Role's theorem. Let $t_3 < t_4$ be two consecutive zeros of y'. Suppose, on the contrary, that

$$y(t) \neq 0 \quad \text{on} \quad [t_3, t_4]\,. \tag{36}$$

Let t_5 be an arbitrary zero of y greater then t_4. As t_5 is simple zero, then it follows from Lemma 6 that $F(t_5) < 0$ and F is nondecreasing; thus

$$F(t_3) < 0, \quad F(t_4) < 0$$

and using $q \geqslant 0$ we can conclude

$$y''(t_3)\, y(t_3) < 0, \quad y''(t_4)\, y(t_4) < 0\,.$$

Thus $y''(t_3)\,y''(t_4) > 0$ and t_3 and t_4 can not be the consecutive zeros of y'.

(ii) Let $t_1 < t_2$ be such that

$$y^{(i)}(t_1) = y^{(i)}(t_2) = 0, \quad i = 0, 1, \qquad \max_{t_1 \leqslant t \leqslant t_2} |y(t)| > 0\,. \tag{37}$$

Hence, Lemma 6 yields $F(t_1) = F(t_2) = 0$ and $F \equiv 0$ on $[t_1, t_2]$. From this (33) is valid and, using $y(t_1) = 0$, $y \equiv 0$ on $[t_1, t_2]$ is valid. The contradiction with (37) proves that there exists at most one maximal interval with the property (31). Suppose that such interval, say $[\tau_1, \tau_2]$, exists and $0 < \tau_1$. Let, conversely, τ_3 be a zero of y on $[0, \tau_1)$. According to the proved part τ_3 is simple, $y'(\tau_3) \neq 0$ and $F(\tau_3) > 0$. This, with $F(\tau_1) = 0$, contradicts to F being nondecreasing. Thus τ_3 does not exist and $y \neq 0$ on $[0, \tau_1)$.

(iii) On the contrary, suppose, that (32) is not valid. Thus there exist $\bar\tau$, $\bar\tau > \tau$ such that $y'(\bar\tau) = y''(\bar\tau) = 0$. Then (29) yields $F(\tau) > 0$, $F(\bar\tau) \leqslant 0$ that contradicts to F being nondecreasing on $[\tau, \bar\tau]$. The fact, that the zeros of y and y' are separated can be proved similarly as the same result in (i) (we use $t_5 = \tau$). ☐

Remark 6. If the Cauchy problem for (1) is unique, then $\tau_1 = \tau_2$ in (31).

Remark 7. Th. 3 extends the results of Ths. A and B. Note, that the assumptions of Ths. 1, 2 and 3 are posed on p, q, r only and not on the nonlinearity f.

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