ARCHIVUM MATHEMATICUM (BRNO)
Tomus 33 (1997), 315 – 321

COMMUTATIVITY OF ASSOCIATIVE RINGS THROUGH A STREB'S CLASSIFICATION

Mohammad Ashraf

Abstract. Let $m \geq 0,\ r \geq 0,\ s \geq 0,\ q \geq 0$ be fixed integers. Suppose that R is an associative ring with unity 1 in which for each $x, y \in R$ there exist polynomials $f(X) \in X^2 Z[X]$, $g(X)$, $h(X) \in XZ[X]$ such that $\{1 - g(yx^m)\}[x,\ x^r y - x^s f(yx^m)x^q]\{1 - h(yx^m)\} = 0$. Then R is commutative. Further, result is extended to the case when the integral exponents in the above property depend on the choice of x and y. Finally, commutativity of one sided s-unital ring is also obtained when R satisfies some related ring properties.

1. Introduction

Throughout the present paper R will denote an associative ring. The symbol $[x, y]$ will denote the commutator $xy - yx$. As usual, $\mathbb{Z}[X]$ is the totality of polynomials in X with coefficients in $\mathbb{Z}$, the ring of integers. A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for all $x \in R$. A ring R is called s-unital if and only if $x \in Rx \cap xR$ for all $x \in R$. Consider the following ring properties:

(H) For each $x, y \in R$ there exists a polynomial $f(X) \in \mathbb{Z}[X]$ such that $[x - x^2 f(x), y] = 0$.

(CH) For each $x, y \in R$, there exist polynomials $f(X), g(X) \in \mathbb{Z}[X]$ such that $[x - x^2 f(x),\ y - y^2 g(y)] = 0$.

(P_1) For each $x, y \in R$ there exist polynomials $f(X) \in X^2\mathbb{Z}[X]$ and $g(X)$, $h(X) \in X\mathbb{Z}[X]$ such that $\{1 - g(yx^m)\}[x,\ x^r y - x^s f(yx^m)x^q]\{1 - h(yx^m)\} = 0$, where $m \geq 0,\ r \geq 0,\ s \geq 0,\ q \geq 0$ are fixed integers.

(P_1^*) For each $x, y \in R$ there exist integers $m = m(x, y) \geq 0$, $r = r(x, y) \geq 0$, $s = s(x, y) \geq 0$, $q = q(x, y) \geq 0$ and polynomials $f(X) \in X^2\mathbb{Z}[X]$,

---

1991 *Mathematics Subject Classification*: 16U80.

Key words and phrases: factorsubring, s-unital ring, commutativity, commutator, associative ring.

Received February 27, 1996

$g(X),\ h(X) \in X\mathbb{Z}[X]$ such that
$\{1-g(yx^m)\}[x,\ x^ry - x^sf(yx^m)x^q]\{1-h(yx^m)\} = 0$.

(P_2) For each $x, y \in R$ there exist polynomials $f(X) \in X^2\mathbb{Z}[X]$ and $g(X)$, $h(X) \in X\mathbb{Z}[X]$ such that $\{1-g(yx^m)\}[x, yx^r - x^sf(yx^m)x^q]\{1-h(yx^m)\} = 0$, where $m \geq 0,\ r \geq 0,\ s \geq 0,\ q \geq 0$ are fixed integers.

(P_2^*) For each $x, y \in R$ there exist integers $m = m(x,y) \geq 0$, $r = r(x,y) \geq 0$, $s = s(x,y) \geq 0$, $q = q(x,y) \geq 0$ and polynomials $f(X) \in X^2\mathbb{Z}[X]$, $g(X),\ h(X) \in X\mathbb{Z}[X]$ such that
$\{1-g(yx^m)\}[x, yx^r - x^sf(yx^m)x^q]\{1-h(yx^m)\} = 0$.

The famous Jacobson's "$x^{n(x)} = x$ theorem" was generalized by Herstein [6] (signified as Theorem H in sequel), who proved that a ring satisfying the property (H) must be commutative. It is natural to consider the related properties $[xy - p(xy), x] = 0$ and $[xy - q(yx), x] = 0$ for some $p(X),\ q(X) \in X^2\mathbb{Z}[X]$ depending on ring's elements x, y. Putcha and Yaqub [13] established that if for each $x, y \in R$ there exists a polynomial $f(X) \in X^2\mathbb{Z}[X]$ such that $xy - f(xy)$ is central, then R^2 must be central. Recently, Bell et al.(cf. [3] and [4]) obtained the commutativity of rings with unity 1 satisfying identities of the form $[xy - p(xy), x] = 0$ or $[xy - q(yx), x] = 0$, where the underlying polynomials $p(X),\ q(X) \in X^2\mathbb{Z}[X]$, are considered to be fixed. Inspired by these works, the author [2] obtained commutativity of rings with unity 1 satisfying the property $[x^my - x^pf(x^my)x^q, x] = 0$, where the polynomial $f(X) \in X^2\mathbb{Z}[X]$ depends on the pairs $x, y \in R$ and $m \geq 0,\ p \geq 0,\ q \geq 0$ are fixed integers. Thus, a natural question arises: what can we say about the commutativity of ring R, if the underlying condition is replaced by $[x^my - x^pf(yx^m)x^q, x] = 0$? In the present paper, we not only answer this question, but also we establish rather a more general result by proving that a ring with unity 1 satisfying either of the properties (P_1) or (P_2) is commutative. Further, results are obtained for one sided s-unital rings. Thus, we generalize considerably many well-known commutativity theorems to mention a few [3, Theorem 2], [11, Theorem 1], [12, Theorem 2], [14, Theorem], [15, Theorem A], [16, Theorem], [17, Theorem] and [19, Theorem] etc.

2. Some Preliminary Results

We begin by considering the following types of rings.

(i) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

$(\mathrm{i})_l$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime.

$(\mathrm{i})_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(ii) $M_\sigma(K) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} / \alpha, \beta \in K \right\}$, where K is a finite field with a non-trivial automorphism σ.

(iii) A non-commutative division ring.
(iv) $S = < 1 > + T$, T a non-commutative radial subring of S.
(v) $S = < 1 > + T$, T a non-commutative subring of S such that $T[T,T] = [T,T]T = 0$.

Recently Streb [18] classified non-commutative rings, which has been used effectively as a tool, to obtain a number of commutativity theorems (cf. [2], [9], [10], [11] and [12] etc.). From the proof of [18, Corollary 1] it can be easily, observed that if R is a non-commutative ring with unity 1, then there exists a factorsubring of R, which is of type (i), (ii), (iii), (iv) or (v). This observation gives the following result, which plays the key role in our subsequent study (cf. [11, Lemma 1]).

Lemma 1. *Let* P *be a ring property which is inherited by factorsubring. If no rings of type* (i), (ii), (iii), (iv) *or* (v) *satisfy* P, *then every ring with unity* 1 *satisfying* P *is commutative.*

For easy reference, we state the following known results which are essentially proved in [9, Corollary 1] and [12, Lemma 1] respectively.

Lemma 2. *Suppose that a ring R with unity* 1 *satisfies* (CH). *If R is non-commutative, then there exists a factrosubring of R which is of type* (i) *or* (ii).

Lemma 3. *Let R be a left (resp. right) s-unital not a right (resp. left) s-unital, then R has a factorsubring of type* $(\mathrm{i})_l$ (resp. $(\mathrm{i})_r$).

3. Main Results

We being with the following theorem.

Theorem 1. *Let R be a ring with unity* 1 *satisfying either of the properties* (P_1) *or* (P_2), *then R is commutative (and conversely).*

In order to develop the proof of the above theorem, first we prove the following lemma.

Lemma 4. *Let R be a division ring satisfying either of the properties* (P_1) *or* (P_2). *Then R is commutative.*

Proof. Suppose that R satisfies the property (P_1). Let u be a unit in R. Then for each $y \in R$, there exist polynomials $f(X) \in X^2\mathbb{Z}[X]$ and $g(X),\ h(X) \in X\mathbb{Z}[X]$ such that

$$\begin{aligned} 0 &= \{1 - g(yu^{-m}u^m)\}[u, u^r yu^{-m} - u^s f(yu^{-m}u^m)u^q]\{1 - h(yu^{-m}u^m)\} \\ &= \{1 - g(y)\}[u, u^r yu^{-m} - u^s f(y)u^q]\{1 - h(y)\}\,. \end{aligned}$$

This implies that either $y-yg(y)=0, y-yh(y)=0$ or $[u, u^r yu^{-m}-u^s f(y)u^q]=0$. In the first two cases R is commutative by Theorem H. Hence, we assume that for a unit $u\in R$ and arbitrary $y\in R$, $[u, u^r yu^{-m}-u^s f(y)u^q]=0$, which implies that

$$u^r[u,y] = u^s[u,f(y)]u^{q+m}. \tag{1}$$

Further, choose polynomial $f_1(X)\in X^2\mathbb{Z}[X]$ such that $u^{-r}[u^{-1},y] = u^{-s}[u^{-1},f_1(y)]u^{-(q+m)}$. This yields that

$$u^s[u,y]u^{q+m} = u^r[u,f_1(y)]. \tag{2}$$

In view of (1), there exists a polynomial $f_2(X)\in X^2\mathbb{Z}[X]$ such that $u^r[u,f_1(y)] = u^s[u,f_2(f_1(y))]u^{q+m}$. Thus, if $f_3(X)=f_2(f_1(X))\in X^2\mathbb{Z}[X]$, then we have

$$u^r[u,f_1(y)] = u^s[u,f_3(y)]u^{q+m}. \tag{3}$$

Comparing equations (2) and (3), we arrive at $u^s[u,y]u^{q+m} = u^s[u,f_3(y)]u^{q+m}$. But, since u is a unit in R and hence $[u, y-f_3(y)]=0$ for some $f_3(X)\in X^2\mathbb{Z}[X]$. Again using Theorem H, we see that R is commutative.

Using similar techniques with necessary variations, we get the required result, if R satisfies the property (P_2). We are now well-equipped to prove our main theorem.

Proof of Theorem 1. Suppose that R satisfies the property (P_1). In view of Lemma 1, it suffices to show that R can not be of type (i), (ii), (iii), (iv) or (v).

First consider the ring of type (i). Then in $(GF(p))_2$, p a prime, we see that for each $f(X)\in X^2\mathbb{Z}[X]$, $g(X),\ h(X)\in X\mathbb{Z}[X]$,

$$\{1-g(e_{12}e_{11}^m)\}[e_{11}, e_{11}^r e_{12}-e_{11}^s f(e_{12}e_{11}^m)e_{11}^q]\{1-h(e_{12}e_{11}^m)\} = e_{12} \neq 0,$$

a contradiction. Hence no rings of type (i) satisfy (P_1).

Now, consider the ring $M_\sigma(K)$, a ring of type (ii), and choose $x = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma(\alpha) \end{pmatrix}$, $(\alpha\neq\sigma(\alpha))$, $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then for each $f(X)\in X^2\mathbb{Z}[X]$, $g(X),\ h(X)\in X\mathbb{Z}[X]$, we see that

$$\{1-g(yx^m)\}[x, x^r y-x^s f(yx^m)x^q]\{1-h(yx^m)\} = \alpha^r(\alpha-\sigma(\alpha))e_{12} \neq 0.$$

Hence, R can not be of type (ii).

Further, let R be a ring of type (iii). Then by Lemma 4, R is commutative, a contradiction.

Assume that R is a ring of type (iv). Then a careful scrutiny of the proof of Lemma 4 shows that, for a unit $u \in R$ and arbitrary $y \in R$, there exist polynomials $f_3(X) \in X^2\mathbb{Z}[X]$, $g(X)$, $h(X) \in X\mathbb{Z}[X]$ such that either $y - yg(y) = 0$, $y - yh(y) = 0$ or $[u, y - f_3(y)] = 0$. Let $a, b \in T$. Then $1 + a$ is a unit and there exist $f_3(X) \in X^2\mathbb{Z}[X]$, $g(X)$, $h(X) \in X\mathbb{Z}[X]$ such that either $b - bg(b) = 0$, $b - bh(b) = 0$ or $[1 + a, b - f_3(b)] = 0$. Hence, by Theorem H T is commutative, a contradiction.

Finally, suppose that R is a ring of type (v). Then for each $a, b \in T$, there exist polynomials $f(X) \in X^2\mathbb{Z}[X]$, $g(X)$, $h(X) \in X\mathbb{Z}[X]$ such that

$$\begin{aligned} 0 &= \{1 - g(b(1+a)^m)\}[1+a, (1+a)^r b - (1+a)^s f(b(1+a)^m)(1+a)^q] \\ &\quad \{1 - h(b(1+a)^m)\} \\ &= \{1 - h(b(1+a)^m)\}[1+a, (1+a)^r b]\{1 - h(b(1+a)^m)\} \\ &= \{1 - h(b(1+a)^m)\}[1+a, b]\{1 - h(b(1+a)^m)\} \\ &= [a, b] \end{aligned}$$

This is a contradiction, and R can not be of type (v).

Now, let R satisfy the property (P_2). If R is of type (i), then there exist polynomials $f(X) \in X^2\mathbb{Z}[X]$, $g(X)$, $h(X) \in X\mathbb{Z}[X]$ such that

$$\{1 - g(e_{12}e_{22}^m)\}[e_{22}, e_{12}e_{22}^r - e_{22}^s f(e_{12}e_{22}^m)e_{22}^q]\{1 - h(e_{12}e_{22}^m)\} = -e_{12} \neq 0 .$$

Accordingly, no rings of type (i) satisfy the property (P_2). Using similar arguments as above, one can show that no rings of type (ii), (iii), (iv) or (v) satisfy the property (P_2) and by Lemma 1, we get the required result.

Corollary 1. *Let $r \geq 0$, $s \geq 0$, $q \geq 0$ be fixed integers and let R be a ring with unity 1. If for each $x, y \in R$ there exists a polynomial $f(X) \in X^2\mathbb{Z}[X]$ such that $[x^r y - x^s f(y) x^q, x] = 0$, then R is commutative.*

Remark 1. If the integral exponents m, r, s, q in the properties (P_1) and (P_2) are allowed to vary with the pair of elements $x, y \in R$, i.e. R satisfies either of the properties (P_1^*) or (P_2^*), then a careful scrutiny of the proof of Theorem 1 shows that R has no factorsubring of type (i) or (ii). Thus, in addition, if R satisfies the property (CH), then in view of Lemma 2, we get the following.

Theorem 2. *Let R be a ring with unity 1 satisfying either of the properties (P_1^*) or (P_2^*). Moreover, if R satisfies the property (CH), then R is commutative (and conversely).*

Remark 2. The non-commutative ring of 3×3 strictly upper triangular matrices over a ring satisfies the property $[x^r y - x^s f(yx^m)x^q, x] = 0$, and hence rules out the possible generalization of the above theorems for arbitrary rings. However, we can prove commutativity results for one sided s-unital rings, if R satisfies some related ring properties.

Theorem 3. *Let R be a left (resp. right) s-unital ring in which for each $x, y \in R$ there exists a polynomial $f(X) \in X^2\mathbb{Z}[X]$ such that $[x^r y - x^s f(yx^m)x^q, x] = 0$ (resp. $[yx^r - x^s f(yx^m)x^q, x] = 0$), where $m \geq 0$, $r \geq 0$, $s \geq 0$, $q \geq 0$ are fixed integers. Then R is commutative (and conversely).*

Proof. If R is a left (resp. right) s-unital ring satisfying $[x^r y - x^s f(yx^m)x^q, x] = 0$ (resp. $[yx^r - x^s f(yx^m)x^q, x] = 0$), then a careful scrutiny of the proof of Theorem 1 shows that no rings of type $(i)_l$ (resp. $(i)_r$) satisfy the property $[x^r y - x^s f(yx^m)x^q, x] = 0$ (resp. $[yx^r - x^s f(yx^m)x^q, x] = 0$). Hence, by Lemma 3, R is right (resp. left) s-unital. Thus, in both the cases R is s-unital and in view of Proposition 1 of [7], we can assume that R has unity 1, and commutativity of R follows by Theorem 1.

Following is an immediate consequence of the above theorem:

Corollary 2. *Let $m \geq 0$, $r \geq 0$, $s \geq 0$, $q \geq 0$ be fixed integers. If R is a left (resp. right) s-unital ring in which for every $x, y \in R$ there exists an integer $t = t(x, y) > 1$ such that $[x^r y - x^s (yx^m)^t x^q, x] = 0$ (resp. $[yx^r - x^s (yx^m)^t x^q, x] = 0$). Then R is commutative (and conversely).*

Using similar arguments as used to get Theorem 2, we can prove the following:

Theorem 4. *Let R be a left (resp. right) s-unital ring in which for every $x, y \in R$ there exist integers $m = m(x, y) \geq 0$, $r = r(x, y) \geq 0$, $s = s(x, y) \geq 0$, $q = q(x, y) \geq 0$ and a polynomial $f(X) \in X^2\mathbb{Z}[X]$ such that $[x^r y - x^s f(yx^m)x^q, x] = 0$ (resp. $[yx^r - x^s f(yx^m)x^q, x] = 0$). Moreover, if R satisfies the property (CH). Then R is commutative (and conversely).*

References

[1] Abujabal, H. A. S., Ashraf, M., *Some commutativity theorems through a Streb's classification*, Note Mat. **14**, No.1 (1994) (to appear).

[2] Ashraf, M., *On commutativity of one sided s-unital rings with some polynomial constraints*, Indian J. Pure and Appl. Math. **25** (1994), 963-967.

[3] Bell, H. E., Quadri, M. A., Khan, M. A., *Two commutativity theorems for rings*, Rad. Mat. **3** (1994), 255-260.

[4] Bell, H. E., Quadri, M. A., Ashraf, M., *Commutativity of rings with some commutator constraints*, Rad. Mat. **5** (1989), 223-230.

[5] Chacron, M., *A commutativity theorem for rings*, Proc. Amer. Math. Soc., **59** (1976), 211-216.

[6] Herstein, I. N., *Two remakrs on commutativity of rings*, Canad. J. Math. **7** (1955), 411-412.

[7] Hirano, Y., Kobayashi, Y., Tominaga, H., *Some polynomial identities and commutativity of s-unital rings*, Math. J. Okayama Univ. **24** (1982), 7-13.

[8] Jacobson, N., *Structure theory of algebraic algebras of bounded degree*, Ann. Math. **46** (1945), 695-707.

[9] Komatsu, H., Tominaga, H., *Chacron's conditions and commutativity theorems*, Math. J. Okayama Univ. **31** (1989), 101-120.
[10] Komatsu, H., Tominaga, H., *Some commutativity theorems for left s-unital rings*, Resultate Math. **15** (1989), 335-342.
[11] Komatsu, H., Tominaga, H., *Some commutativity conditions for rings with unity*, Resultate Math. **19** (1991), 83-88.
[12] Komatsu, H., Nishinaka, T., Tominaga, H., *On commutativity of rings*, Rad. Math. **6** (1990), 303-311.
[13] Putcha, M. S., Yaqub, A., *Rings satisfying polynomial constraints*, J. Math. Soc., Japan **25** (1973), 115-124.
[14] Quadri, M. A., Ashraf, M., Khan, M. A., *A commutativity condition for semiprime ring-II*, Bull. Austral. Math. Soc. **33** (1986), 71-73.
[15] Quadri, M. A., Ashraf, M., *Commutativity of generalized Boolean rings*, Publ. Math. (Debrecen) **35** (1988), 73-75.
[16] Quadri, M. A., Khan, M. A., Asma Ali, *A commutativity theorem for rings with unity*, Soochow J. Math. **15** (1989), 217-227.
[17] Searcoid, M. O., MacHale, D., *Two elementary generalizations for Boolean rings*, Amer. Math. Monthly **93** (1986), 121-122.
[18] Streb, W., *Zur struktur nichtkommutativer Ringe*, Math. J. Okayama Univ. **31** (1989), 135-140.
[19] Tominaga, H., Yaqub, A., *Commutativity theorems for rings with constraints involving a commutative subset*, Resultate Math. **11** (1987), 186-192.

Department of Mathematics
Aligarh Muslim University
Aligarh-202 002, INDIA