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ANTIDOMATIC NUMBER OF A GRAPH

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Abstract. A subset D of the vertex set $V(G)$ of a graph G is called dominating in G, if for each $x \in V(G) - D$ there exists $y \in D$ adjacent to x. An antidomatic partition of G is a partition of $V(G)$, none of whose classes is a dominating set in G. The minimum number of classes of an antidomatic partition of G is the number $\bar{d}(G)$ of G. Its properties are studied.

In this paper we introduce a new numerical invariant of a graph, called antidomatic number. We consider finite undirected graphs without loops and multiple edges.

In [1] the domatic number of a graph was introduced. Let D be a subset of the vertex set $V(G)$ of a graph G. The set D is called dominating in G, if for each $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. A partition of $V(F)$, all of whose classes are dominating sets in G, is called the domatic partition of G. The maximum number of classes of a domatic partition of G is called the domatic number of G and denoted by $d(G)$.

We introduce a similar concept; it was yet mentioned in [2]. If a subset $A \subseteq V(G)$ is not dominating in G, we call it non-dominating in G. A partition of $V(G)$, all of whose classes are non-dominating sets in G, is called an antidomatic partition of G. The minimum number of classes of an antidomatic partition of G is called the antidomatic number of G and denoted by $\bar{d}(G)$.

Proposition 1. *The antidomatic number of a graph G is well-defined if and only if G has no saturated vertex, i.e vertex adjacent to all others.*

Namely, a graph containing a saturated vertex v has no antidomatic partition, because each subset of its vertex set which contains v is dominating.

Proposition 2. *For every graph G without saturated vertices*

$$\bar{d}(G) \geq 2\,.$$

This follows from the fact that the set $V(G)$ is dominating in G (and thus not non-dominating).

The antidomatic number of a graph is closely related to its diameter.

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Theorem 1. *Let G be a graph without saturated vertices. The antidomatic number $\bar{d}(G) = 2$ if and only if diam $G \geq 3$.*

Proof. Suppose diam $G \geq 3$. Let x, y be two vertices whose distance in G is at least 3. (This includes the case when this distance is ∞, i.e. when no path connects x and y in G.) Let A_1 be the set consisting of x and of all vertices which are adjacent to x in G, let $A_2 = V(G) - A_1$. The set A_1 is not dominating in G, because $y \in V(G) - A_1$ and is adjacent to no vertex of A_1. The set A_2 is not dominating in G, because $x \in V(G) - A_2$ and is adjacent to no vertex of A_2. Therefore $\{A_1, A_2\}$ is an antidomatic partition of G and $\bar{d}(G) = 2$.

Now suppose $\bar{d}(G) = 2$ and let $\{B_1, B_2\}$ be an antidomatic partition of G. As B_1 (or B_2) is not dominating in G, there exists a vertex $u_2 \in B_2$ (or $u_1 \in B_1$) which is adjacent to no vertex of B_1 (or B_2 respectively). Obviously u_1, u_2 are not adjacent and their distance is not 1. If this distance were 2, there would exist a vertex v adjacent to both u_1, u_2. But v might be neither in B_1, nor in B_2, which would be a contradiction. Hence the distance between u_1 and u_2 is at least 3 and diam $G \geq 3$. □

Therefore at the study of $d(G)$ we may restrict ourselves to graphs with diam $G = 2$.

Now we relate $d(G)$ to degrees of vertices of (G).

Theorem 2. *Let G be a graph without saturated vertices, let n be its number of vertices, let $\delta(G)$ be the minimum degree of a vertex in G. Then*

$$\lceil (n/(n-1) - \delta(G)) \rceil \leq \bar{d}(G) \leq \delta(G) + 2$$

Proof. Let $A \subseteq V(G)$ be a non-dominating set in G. Then there exists at least one vertex $x \in V(G) - A$ which is adjacent to no vertex of A. The degree of x in G is at most $n-1-|A|$. We have $n-1-|A| \geq \delta(G)$, which implies $|A| \leq n-1-\delta(G)$. Every antidomatic partition of G has the property that each of its classes has at most $n-1-\delta(G)$ elements and this implies the first inequality.

Now let u be a vertex of G having the degree $\delta(G)$. Let $v_1, \dots, v_\delta$ be the vertices adjacent to u, let $A = V(G) - \{u, v_1, \dots, v_\delta\}$. As G has no saturated vertex, $A \neq \emptyset$. The set A is not dominating in G, because $u \in V(G) - A$ and is adjacent to no vertex of A. No vertex is saturated, therefore no one-element set is dominating in G. The partition $\big\{\{u\}, \{v_1\}, \dots, \{v_\delta\}, A\big\}$ is an antidomatic partition of G with $\delta(G) + 2$ vertices, which implies the second inequality. □

Corollary. *Let G be a graph without saturated vertices, let n be its number of vertices, let $\bar{d}(G) = n$. Then n is even and G is obtained from the complete graph K_n by deleting edges of a linear factor.*

The Zykov sum $G_1 \oplus G_2$ of two graphs G_1, G_2 is the graph obtained from disjoint graphs G_1, G_2 by joining each vertex of G_1 with each vertex of G_2 by an edge.

The following lemma is easy to prove.

Lemma. *Let G_1, G_2 be graphs without saturated vertices. Then*

$$\bar{d}(G_1 \oplus G_2) = \bar{d}(G_1) + \bar{d}(G_2)\,.$$

If G is a graph, then $\bar{G}$ denotes the complement of G.

Theorem 3. *Let k, n be integers such that $2 \le k \le n-2$. Then there exists a graph G with n vertices such that $d(G) = k$.*

Proof. If k is even, then G is the Zykov sum of $\frac{1}{2}k - 1$ copies of $\bar{K}_2$ and one copy of $\bar{K}_{n-k+2}$. If k is odd, then G is the Zykov sum of $\frac{1}{2}(k-1) - 3$ copies of $\bar{K}_2$, one copy of $C_5$and one copy of $\bar{K}_{n-k}$. Here C_5 denotes the circuit of length 5; its is easy to see that $\bar{d}(C_5) = 3$. □

Theorem 4. *Let n be an integer, $n \ge 2$. The graph G with n vertices and $\bar{d}(G) = n$ exists if and only if n is even. The graph H with n vertices and $\bar{d}(H) = n-1$ exists if and only if n is odd.*

Proof. The graph G is described in Corollary; it is the Zykov sum of $\frac{1}{2}n$ copies of $\bar{K}_2$. The graph H is the Zykov sum of $\frac{1}{2}(n-3)$ copies of $\bar{K}_2$ and one copy of $\bar{K}_3$ (or one copy of $\bar{P}_2$, where P_2 is a path of length 2).

Now we shall prove that the graph G for n odd does not exist. Consider a graph G with required properties. The unique antidomatic partition of G is the partition of $V(G)$ into one-element sets. Every subset of $V(G)$ having more than one vertex is dominating. This implies that the degree of each vertex must be $n-2$. It is possible only if n is even.

We shall prove that the graph H for n even does not exist. Consider a graph H with required properties. Evidently H has no non-dominating set with more than two vertices (and thus $\delta(H) = n-3$) and no two disjoint non-dominating sets with two vertices. It has at least one non-dominating set with two vertices; otherwise its antidomatic number would be n. If H contains exactly one non-dominating set with two vertices, then it has exactly one vertex of degree $n-3$ and all others of degree $n-2$; this is possible only if n is odd. Now suppose that H has two non-dominating sets which are not disjoint. Let them be $\{x, y\}$, $\{y, z\}$, where x, y, z are pairwise distinct. Let u be a vertex not contained in $\{x, y\}$ and not adjacent to any vertex of $\{x, y\}$; let v be an analogous vertex for $\{y, z\}$. If $u = v$, then $\{x, y, z\}$is a non- dominating set with three vertices, which is impossible. If $u \ne v$, then $\{u, v\}$ is another non-dominating set with two vertices, because y is adjacent neither to u, nor to v. It may not be disjoint with $\{x, y\}$ or with $\{y, z\}$ and this is possible only if $u = z$, $v = x$. Then the subgraph of G induced by $\{x, y, z\}$ is $\bar{K}_3$ and $H = H_0 \oplus \bar{K}_3$, where H_0 is the graph obtained from H deleting x, y, z. As $\bar{d}(H) = n-1$, $\bar{d}(\bar{K}_3) = 2$, we have $\bar{d}(H_0) = n-3$ and this is also its number of vertices. Such an equality is possible only if $n-3$ is even, i.e. if n is odd. □

Theorem 5. *Let G be a graph without saturated vertices, let $\chi(G)$ be its chromatic number. Then*

$$d(G) \le 2\,\chi(G)\,.$$

This bound is sharp.

Proof. Let $\chi(G) = h$. Suppose that G is coloured by n colours $1, \dots, h$ and let B_i be the set of vertices coloured by the colour i for $i = 1, \dots, h$. We shall construct an antidomatic partition of G. If a set B_i is non-dominating in G, then it will be a class of this partition. If B_i is dominating in G, then it has at least two elements, because no saturated vertices exist. We choose an arbitrary partition $\{B_i', B_i''\}$ of B_i into two classes; the sets B_i', B_i'' (evidently non-dominating in G) will be classes of the constructed partition. The resulting antidomatic partition has at most $2\,\chi(G)$ classes, which implies the inequality. If G is a complete h-partitie graph, then non-dominating sets are only proper subsets of its partities, therefore the equality occurs. $\square$

Corollary 2. *Let G be a bipartite graph different from a star. If G is a complete bipartite graph, then $\bar{d}(G) = 4$; otherwise $\bar{d}(G) = 2$.*

Theorem 6. *. Let G be a graph without saturated vertices, let $\gamma(G)$ be its dominating number. Then*

$$\bar{d}(G) \le n/(\gamma(G) - 1)\,.$$

Proof. The domination number $\gamma(G)$ is the minimum number of vertices of a dominating set in G. There exists a partition of $V(G)$ into $n/(\gamma(G) - 1)$ classes, each of which has at most $\gamma(G) - 1$ vertices and hence is non-dominating. This implies the assertion. $\square$

At the end we shall study regular graphs of degree $n-3$. where n is the number of vertices. Such a graph is a complement of a regular graph of degree 2 and this is a graph whose connected components are circuits. Hence a regular graph of degree $n - 3$ is either a complement of a circuit, or a Zykov sum of complements of circuits. Therefore we determine $\bar{d}(\bar{C}_n)$, where C_n is a circuit of length n.

Theorem 7. *Let C_n be a circuit of length n, let $\bar{C}_n$ be its complement. Then*

$$\begin{aligned} \bar{d}(\bar{C}_n) &= n/2 && \text{for} \quad n \equiv 0(\mod 4)\,,\\ \bar{d}(\bar{C}_n) &= n/2 + 1 && \text{for} \quad n \equiv 2(\mod 4)\,,\\ \bar{d}(\bar{C}_n) &= (n+1)/2 && \text{for} \quad n \ \text{odd}. \end{aligned}$$

Proof. The graph $\bar{C}_n$ is a regular graph of degree $n - 3$; by Theorem 3 we have $\bar{d}(\bar{C}_n) \ge \lceil n/2 \rceil$ and this is $\bar{d}(\bar{C}_n) \ge n/2$ for n even and $\bar{d}(\bar{C}_n) \ge (n+1)/2$ for n odd. Let $u_1, \dots, u_n$ be the vertices of C_n and let $u_i u_{i+1}$ for $i = 1, \dots, n-1$ and $u_n u_1$ be the edges of C_n. If $n \equiv 0 \pmod 4$, we take $n/2$ sets $\{u_{4i+1}, u_{4i+3}, u_{4i+2}, u_{4i+4}\}$ for $i = 0, \dots, n/4 - 1$; this is an antidomatic partition of $\bar{C}_n$ into $n/2$ classes and $\bar{d}(\bar{C}_n) = n/2$. If $n \equiv 1 \pmod 4$ we take the sets $\{u_{4i+1}, u_{4i+3}\}$, $\{u_{4i+2}, u_{4i+4}\}$ for $i = 0, \dots, (n-1)/4 - 1$ and the sets $\{u_n\}$. For $n \equiv 3 \pmod 4$ we take the sets $\{u_{4i+1}, u_{4i+3}\}$, $\{u_{4i+2}, u_{4i+4}\}$ for $i = 0, \dots, (n-3)/4 - 1$ and the sets $\{u_{n-2}, u_n\}$, $\{u_{n-1}\}$. In both these cases we have an antidomatic partition with $(n+1)/2$ classes and $d(C_n) = [(n+1)/2]$. Now the case $n \equiv 2 \pmod 4$ remains. Suppose that there exists an antidomatic partition A consisting only of two-element sets. Each of them has the form $\{u_i, u_{i+2}\}$, where $i+2$ is taken modulo 2. Let U_1 (or U_2) be the set of all vertices u_i (or u_{i+2} respectively) such that $\{u_i, u_{i+2}\} \in A$. Evidently $\{U_1, U_2\}$ is a partition of $V(C_n)$ and $|U_1| = |U_2| = \frac{1}{2}n$. No two vertices of U_1 can have the distance 2 in C_n and thus at least two of them are adjacent; without loss of generality let $u_1 \in U_1$, $u_2 \in U_1$. For $j \in \{0, 1, 2, 3\}$ let N_j be the set of numbers from $\{1, \dots, n\}$ which are congruent to j modulo 4. Let f be a permutation of $\{1, \dots, n\}$ such that for $i = 1, \dots, n$ the value $f(i)$ is such a number j that $j \equiv i$ and $\{u_i, u_j\} \in A$. Then f maps bijectively N_0 onto N_2, N_1 onto N_3, N_2 onto N_0, N_3 onto N_1. But $|N_0| = (n-2)/4$, $|N_2| = (n-2)/4 + 1$, which is a contradiction. Therefore $\bar{d}(\bar{C}_n) \geq n+1$. We take the sets $\{u_{i+1}, u_{4i+3}\}$, $\{u_{4i+2}, u_{4i+4}\}$ for $i = 0, \dots, n/2 - 2$ and the sets $\{u_{n-1}\}$, $\{u_n\}$; this is an antidomatic partition of C_n and $\bar{d}(\bar{C}_n) = n/2 + 1$.

For circuits themselves the values are clear: $\bar{d}(C_4) = 4$, $\bar{d}(C_5) = 3$, $\bar{d}(C_n) = 2$ for $n \geq 6$.

References

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