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INTRODUCTION TO THE THEORY OF SEMI–HOLONOMIC JETS

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To Ivan Kolář on the occasion of his 60th birthday.

0. Preface

The usual jets were introduced by C. Ehresmann as a fundamental tool in Differential Geometry. They permit to globalize the theory of differential systems and to give a formulation of the "infinite groups" of E. Cartan; this leads to the theory of Lie pseudogroups; initiated by C. Ehresmann, this theory was studied by many mathematicians (the author, J. Pradines, S. Chern, D. Spencer, Guillemin-Sternberg, H. Goldschmidt, Kumpera, Qué, Molino and Albert etc). D. Spencer introduced cohomological methods.

When studying the prolongations of a differential system or higher order connections (for instance the iteration of a linear connection on the tangent bundle) C. Ehresmann was led to introduce what he called, using the terminology of Mechanics, *non holonomic* and *semi-holonomic* jets; the ordinary jets are called *holonomic* jets.

While the theory of holonomic jets is now classical, the theory of non holonomic and semi-holonomic jets seems "mysterious" to many mathematicians. The purpose of this paper is to explain how semi-holonomic jets occur naturally in Differential Geometry and to serve as an elementary introduction to the works devoted to semi-holonomic jets; we leave to the reader the task of studying these papers.

Among the mathematicians who have investigated semi-holonomic jets are C. Ehresmann pupils (J. Pradines and the author, P. Ver Eecke, P. C. Yuen). A very important contribution has been made by I. Kolář and his co-workers, especially concerning natural transformations and higher order connections, as well as G. Virsik and M. Modugno. For instance I. Kolář has introduced the notion of equivalence with respect to curves for semi-holonomic jets; in the case of holonomic jets, equivalent jets coincide. This is linked with the research of natural transformations

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existing in a bundle of semi-holonomic jets. On the subject of natural transformations, we refer to the book "Natural Operations in Differential Geometry" by I. Kolář, P. Michor, J. Slovák which contains a great list of references.

The non holonomic jets are obtained by iteration of 1-jets; among them semi-holonomic jets are obtained while "forgetting" the condition of Schwarz symmetry in higher order derivatives; they correspond to an iteration of linear maps in the following sense; the projection $\bar{J}_qE \to \bar{J}_{q-1}E$ (where $\bar{J}_q$ means the semi-holonomic prolongation of order q) is endowed with an affine bundle structure whose associated vector bundle is a bundle of multilinear maps from π^*TM to the vertical bundle $VE = \ker T\pi$. Here π denotes the projection $E \to M$. The "difference" between holonomic and semi-holonomic prolongations leads to the notion of *curvature*.

Utilizing these affine structures we show the existence of a contraction onto the holonomic prolongation and an involution $\mathcal{J}$ in the space $\bar{J}_{q,q-1}E$, inverse image of $J_{q-1}E$ in $\bar{J}_q$ E. So we generalize a result of J. Pradines in the case $q = 2$.

J. Pradines has attached with the notion of non holonomic jet and semi-holonomic jet the theory of double vector bundles. We give a very short abstract of this theory.

I. Some facts about holonomic prolongations

To simplify we shall assume that all manifolds and maps are C^∞, the manifolds being finite-dimensional and paracompact. Many of the results are valid under less restrictive assumptions.

For any fibered manifold (E, M, π) (i.e. for any triple (E, M, π) where E and M are manifolds and π a surjective submersion), the holonomic prolongation J_qE is the set of q-jets of local section of E. Given manifolds M and N, the set $J^q(M, N)$ of q-jets from M to N could be written J_qE, considering the fibered manifold $(E = M \times N, M, pr_1)$. It is known that the projection $J_qE \to J_{q-1}E$ defines an affine bundle structure. In particular the bundle $J_1E \to E$ admits as associated vector bundle the set of linear morphisms from TM to the vertical bundle $VE = \ker T\pi$. The prolongations of vector bundles are vector bundles.

For any manifold M, let $T^q_p(M)$ (resp. $T^{*q}_p(M)$) be the set of q-jets from $\mathbb{R}^p$ to M (resp. from M to $\mathbb{R}^p$), with source (resp. target) 0. For $q = p = 1$, we recover the tangent and cotangent bundles TM and T^*M. If G is a Lie group, so is $T^q_p(G)$ for any q and p.

The q-frame bundle $H^q(M)$ is the subset of $T^q_n(M)$ (where n = dimension of M) generated by the q-jets of local diffeomorphisms. It is a L^q_n-principal bundle, where L^q_n is the group of q-jets of local diffeomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^n$ with source and target 0. The q-coframe bundle $H^{*q}(M)$ is also a L^q_n-principal bundle.

For any fibered manifold (E, M, π), we define the submanifold $\mathfrak{C}^q_nE$ of T^q_nE, inverse image of $H^q(M)$ by the projection $T^q_n\pi : T^q_nE \to T^q_nM$. In particular if $E = M$ and $\pi = \mathrm{id}_M$, we obtain $\mathfrak{C}^q_nM = H^q(M)$. We have proved [L3]

Proposition I.1. *The manifold $\mathfrak{C}^q_nE$ is diffeomorphic to the fibered product $J_qE \times_M H^q(M)$. Moreover if (P, M, π) is a principal G-bundle, then $(\mathfrak{C}^q_nP, M, T^q\pi)$*

is a principal G_q-bundle with $G_q = T^q_n(G) \times L^q_n$.

Other authors, as for instance G. Virsik [V1] introduced such prolongations for principal bundles.

These prolongations are linked with the *prolongations of Lie groupoids* introduced by C. Ehresmann [E1] in the following way; let $\Phi \overset{a}{\underset{b}{\rightrightarrows}} \Phi_0$ be a Lie groupoid with Φ_0 as the set of units; an invertible a-section s is a section of Φ with respect of a, such that $b \circ s$ is a diffeomorphism. The set of all local invertible a-sections constitute a pseudogroup Γ; the q-prolongation of Φ is the groupoid $\Phi^q = J^q(\Gamma)$, set of q-jets of all elements of Γ.

We have proved [L3] that if Φ *is the Lie groupoid associated with the principal bundle* (P, M, π) i.e. the quotient of $P \times P$ by the diagonal action of G, *then the Lie groupoid associated with* $\mathfrak{C}^q_n P$ *is the prolongation* Φ^q of Φ. Here the set of units of Φ and Φ^q is the manifold M.

Utilizing the notion of "partial jet" which is subjacent in the paper [E1] by C. Ehresmann, we have proved

Proposition I.2 (Schwarz lemma for manifolds)**.** *There exists a natural diffeomorphism*

$$\psi^q_p : TT^q_p M \to T^q_p TM \tag{I.1}$$

which exchanges the projections of $TT^q_p M$ and $T^q_p TM$ on TM and $T^q_p M$; the diffeomorphism ψ^1_1 is the natural involution on TTM. Moreover for a fibered manifold (E, M, π), we get

$$\psi^q_n(T\mathfrak{C}^q_n E) = \mathfrak{C}^q_n TE\,; \tag{I.2}$$

in particular

$$\psi^q_n(TH^q(M)) = \mathfrak{C}^q_n(TM)\,. \tag{I.3}$$

For the proof, we considered the partial jets from $\mathbb{R} \times \mathbb{R}^p$ to M or E; in terms of local coordinates, we used the Schwarz lemma for partial derivatives.

Let (P, M, π) be a principal G-bundle; the set TP/G of tangent vector to P mod the right translations by G is a vector bundle with base M. Then the tangent bundle TP may be identified with the fibered product $(TP/G) \times_M P$.

Using formula (I.2) and proposition I.1, we deduce the vector bundle isomorphism

$$T(\mathfrak{C}^q_n P)/G_q \Longleftrightarrow J_q(TP/G)\,; \tag{I.4}$$

in particular we recover the isomorphism

$$T(H^q(M))/L^q_n \Longleftrightarrow J_q TM\,, \tag{I.5}$$

which was proved previously in [L9] considering the one-parameter local groups of transformations generated by the vector fields tangent to M and their liftings to $H^q(M)$.

Let Φ be the groupoid associated with a principal bundle (P, M, π). An "infinitesimal displacement" in the sense of C. Ehresmann [E4] is a vector tangent to Φ which is a-vertical and whose origin lies in M (considered as the set of units). The set $\mathcal{A}(\Phi)$ of all infinitesimal displacements is a vector bundle; it was proved in [L3] that $\mathcal{A}(\Phi)$ is isomorphic to TP/G. So formula (I.4) may be written

$$J_q\mathcal{A}(\Phi) = \mathcal{A}(\Phi^q)\,. \tag{I.6}$$

Let (E, M, π) be a fibered manifold. A *regular differential system of order q* relative to E is a fibered submanifold $R_q \to M$ of $J_qE \to M$. A local solution of R_q is a local section $s : U \subset M \to E$ such that for any $x \in U$, $j^q_x s$ belongs to R_q. The differential system is said to be *completely integrable* if for any $X^q \in R_q$, there exists a local solution s of R_q such that $j^q_x s = X^q$ (where $x = \alpha(X^q)$). Then for any $k > 0$, the $(q+k)$-jet $j^{q+k}_x s$ (which can be identified with $j^k_x j^q f$) belongs to $J_kR_q \cap J_{q+k}E$ and the map $J_kR_q \cap J_{q+k}E \to R_q$ is surjective. For any differential system R_q the set $R_{q+k} = J_kR_q \cap J_{q+k}E$ is called the *holonomic prolongation* of order k of R_q; this prolongation is not necessarily a submanifold of J_kR_q and of $J_{q+k}E$.

Among the differential systems, those of *finite type* are characterized by the following property: there exists k such that R_{q+k} is a submanifold of $J_{q+k}E$ and R_{q+k} is a diffeomorphic to R_{q+k-1}. The systems which are not of finite type are said to be of *infinite type* (for instance the systems corresponding to the "infinite groups" of E. Cartan).

A *connection of order* 1 relative to a fibered manifold (E, M, π) (i.e. a lifting $C : E \to J_1E$) is of finite type because $R_1 = C(E)$ is diffeomorphic to E.

A *G-structure* on a manifold i.e. a principal G-subbundle H_G of the frame bundle $H(M)$, with G a subgroup of $L^1_n = GL(n, \mathbb{R})$, is a differential system which may be of finite type or of infinite type.

The obstructions to complete integrability of connections and G-structures lead to the notions of curvature and "structure tensor".

II. The notion of semi-holonomic jets

1. The introduction of semi-holonomic jets gives a good formulation of the notion of curvature and "structure tensor". We shall obtain prolongations of differential systems which are fibered manifolds.

Let (E, M, π) be a fibered manifold, J_1E its first prolongation. The *second non holonomic prolongation* is the set $\widetilde{J}_2E = J_1J_1E$. By iteration we define the *non holonomic prolongation of order q* by $\widetilde{J}_qE = J_1\widetilde{J}_{q-1}E$. These prolongations define fibered manifolds with base M.

Given two manifolds M and N, the set $\widetilde{J}^q(M, N)$ of non holonomic q-jets from M to N is defined considering the fibered manifold $(E = M \times N, M, pr_1)$.

The *semi-holonomic prolongation* $\bar{J}_2E \subset J_1J_1E$ is defined as follows; a local section $s : U \subset M \to J_1E$ is said to be *adapted* at $x \in U$ if $s(x) = j^1_x(\beta \circ s)$, where $\beta : J_1E \to E$ is the target map; then the jet $j^1_x s$ is called *semi-holonomic*. This definition was introduced by C. Ehresmann [E3]. As remarked by J. Pradines, the subset $\bar{J}_2E$ of $\widetilde{J}_2E$, set of all semi-holonomic jets can be defined in the following way. We have the commutative diagram

$$\begin{array}{ccc} J_1J_1E & \xrightarrow{\quad j^1\beta \quad} & J_1E \\ \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\beta} \\ J_1E & \xrightarrow[\quad \beta \quad]{} & E \end{array}$$

and $\bar{J}_2E = \{z_2 \in J_1J_1E; \beta(z_2) = j^1\beta(z_2)\}$. In other terms $\bar{J}_2E$ is the inverse image of the diagonal of $J_1E \times_E J_1E$ by the map $(\beta, j^1\beta)$. So $\bar{J}_2E$ is a submanifold of J_1J_1E.

If we consider a local section of J_1E which can be written $s = j^1f$ (where f is a local section of E), then s is adapted at each point of its source and $j^1_x s = j^1_x j^1 f = j^2_x f$. We get a holonomic 2-jet. So J_2E is contained in $\bar{J}_2E$.

By iteration we consider the commutative diagram

$$\begin{array}{ccc} J_1\bar{J}_{q-1}E & \xrightarrow{\quad j^1\beta \quad} & J_1\bar{J}_{q-2}E \\ \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\beta} \\ \bar{J}_{q-1}E & \xrightarrow[\quad \beta \quad]{} & \bar{J}_{q-2}E \end{array}$$

As $\bar{J}_{q-1}E \subset J_1\bar{J}_{q-2}E$, the projection $\bar{J}_{q-1}E \to \bar{J}_{q-2}E$ is the restriction to $\bar{J}_{q-1}E$ of the target map $\beta : J_1\bar{J}_{q-2}E \to \bar{J}_{q-2}E$; the projection $J_1\bar{J}_{q-1}E \to J_1\bar{J}_{q-2}E$ is the 1-jet prolongation of β. We define

$$\bar{J}_qE = \{z_q \in J_1\bar{J}_{q-1}E; \quad \beta(z_q) = j^1\beta(z_q)\}\,.$$

The semi-holonomic prolongation $\bar{J}_qE$ is a submanifold of $J_1\bar{J}_{q-1}E$ and of $\widetilde{J}_qE = J_1\widetilde{J}_{q-1}E$; J_qE is contained in $\bar{J}_qE$.

We also have to consider the *prolongation* $\bar{J}_{q,q-1}E$, inverse image of $J_{q-1}E$ by the projection $\bar{\pi}^q_{q-1} = \beta : \bar{J}_qE \to \bar{J}_{q-1}E$, as well as the *sesquiholonomic* prolongation $\check{J}_qE$ defined by

$$\check{J}_qE = \{z_q \in J_1J_{q-1}E; \quad \beta(z_q) = j^1\beta(z_q)\}\,.$$

This prolongation $\check{J}_qE$ is contained in $\bar{J}_{q,q-1}E$.

For $q = 2$, $\bar{J}_{2,1}E$ and $\check{J}_2E$ coincide with $\bar{J}_2E$.

Other semi-holonomic prolongations can be obtained if we consider the non holonomic prolongations $\widetilde{T}^q_p(M) = T^1_p(\widetilde{T}^{q-1}_p M)$, $\widetilde{\mathfrak{C}}^q_n E = \mathfrak{C}^1_n(\widetilde{\mathfrak{C}}^{q-1}_n E)$; then the semi-holonomic prolongations $\bar{T}^q_p(M)$, $\bar{\mathfrak{C}}^q_n E$ are obtained by iteration, starting with adapted mappings from $\mathbb{R}^p$ to $\bar{T}_p^{q-1}(M)$ or from $\mathbb{R}^n$ to $\bar{\mathfrak{C}}_n^{q-1}(E)$. The functors $\widetilde{\mathfrak{C}}_n^q$ and $\bar{\mathfrak{C}}_n^q$ transform principal bundles into principal bundles. The relation between prolongations of principal bundles and prolongations of their associated groupoids as well as formulae I.4, I.5, I.6 are still valid in the semi-holonomic case.

For $p = 1$, $\widetilde{T}_1^2(M) = TT(M)$ (double tangent bundle), $\widetilde{T}_1^q(M) = T(\widetilde{T}_1^{q-1}(M))$. The semi-holonomic prolongations coincide with the holonomic prolongations.

We have to remark that $T^*T^*(M)$ is not a non holonomic prolongation because $T^*(M)$ is a set of 1-jets from M to $\mathbb{R}$, while $T^*T^*(M)$ is a set of 1-jets from $T^*(M)$ to $\mathbb{R}$. We shall introduce later $\bar{T}^{*2}(M)$, set of all semi-holonomic jets order 2, from M to $\mathbb{R}$ with target 0.

2. We shall give examples of semi-holonomic prolongations which occur naturally in differential geometry:

Example 1. Let (E, M, π) be a fibered manifold and $C : E \to J_1 E$ be a connection of order 1. We define the 1-jet extension $j^1 C : J_1 E \to J_1 J_1 E$. It is easy to check that the lifting $j^1 C \circ C : E \to J_1 J_1 E$ takes its values in $\bar{J}_2 E$. The connection C is integrable (as a differential system) if and only if $J^1 C \circ C$ takes its values into the holonomic prolongation $J_2 E$; indeed if we use local coordinates, we are in the situation of a "total differential" equation and we use Frobenius theorem.

If we start with a linear connection $TM \to J_1 TM$, by iteration we obtain a lifting $\bar{C}_q : \bar{J}_{q-1} TM \to \bar{J}_q TM$, called semi-holonomic connection of order q.

For a holonomic connection of order q i.e. a lifting $C_q : J_{q-1} E \to J_q E$, then $J^1 C_q \circ C_q$ takes its values in the sesquiholonomic prolongation $\check{J}_{q+1} E$. More generally for a regular differential system $R_q \to M$, fibered submanifold of $J_q E \to M$, we define the sesqui-holonomic prolongation of R_q by

$$\check{R}_{q+1} = J_1 R_q \cap \check{J}_{q+1} E \,;$$

when R_{q+1} is diffeomorphic to R_q (system of finite type), then according to Frobenius theorem, the system R_q is completely integrable if and only if $\check{R}_{q+1}$ coincides with R_{q+1}.

Example 2. Let $\omega : M \to T^*(M)$ be a Pfaffian form; then for any $x \in M$, there exists a function $f : U \subset M \to \mathbb{R}$ such that $f(x) = 0$ and $j_x^1 f = \omega(x)$; so the section ω is adapted at any point $x \in M$. The 1-jet $j_x^1 \omega$ belongs to $\bar{T}^{*2}(M)$ (space of all semi-holonomic 2-jets from M to $\mathbb{R}$ or semi-holonomic 2-covelocities). So we have proved that

$$J_1 T^*(M) \quad \text{is diffeomorphic to} \quad \bar{T}^{*2}(M)\,.$$

The jet $j_x^1 \omega$ belongs to $T^{*2}(M)$ at any x if and if $\omega = j^1 f$ i.e. $\omega = df$ in the neighbourhood of x. *So ω must be closed.*

In terms of local coordinates, we write in an open subset U of M

$$\omega = \sum a_i dx^i \,;$$

$j^1_x\omega$ is defined by

$$a_1(x), \ldots, a_n(x), \frac{\partial a_i}{\partial x^j}(x) .$$

The jet is a holonomic 2-jet if and only if

$$\frac{\partial a_i}{\partial x^j} = \frac{\partial a_j}{\partial x^i} \qquad (i, j = 1, \ldots, n) ,$$

with U assumed to be simply connected.

Example 3. For a principal bundle (P, M, π), the prolongation $J_1 P \to M$ is not necessarly a principal bundle (contrary to $\mathfrak{C}_n P \to M$). But in the case of the frame bundle $H(M)$, the prolongation $J_1 H(M) \to M$ is endowed with a principal bundle structure. Indeed we shall prove the following property (due to P. Ver Eecke [Ve]):

Proposition II.1. *There exists a natural diffeomorphism Φ from $J_1 H(M)$ onto the principal bundle $\bar{H}^2(M)$ where $\bar{H}^2(M)$ is the semi-holonomic frame bundle of order 2.*

Proof. The manifold $\bar{H}^2(M)$ is a submanifold of $\mathfrak{C}_n\mathfrak{C}_n H$, inverse image of the diagonal of $\mathfrak{C}_n M \times \mathfrak{C}_n M$ by the map $(\beta, j^1\beta)$; as β and $j^1\beta$ are principal bundle morphisms, we may deduce that $\bar{H}^2(M) \to M$ is a principal subbundle of $\mathfrak{C}_n\mathfrak{C}_n M \to M$. We remark that the projection $H(M) \to M$ is also a target map β (as $H(M) = \mathfrak{C}_n M$).

Let $s : U \subset M \to H(M)$ be a local section of β. For $x \in U$, the frame h_x is the 1-jet $j^1_0\varphi$ where φ is a local diffeomorphism from an open neighbourhood V of 0 in $\mathbb{R}^n$ onto a neighbourhood of x. The map $\psi = s \circ \varphi$ from $V' = V \cap \varphi^{-1}(U)$ to $H(M)$ is *adapted* at 0; indeed $\psi(0) = h_x$. Moreover $\beta \circ \psi = \beta \circ s \circ \varphi = \varphi | V'$ and $h_x = j^1_0(\beta \circ \psi)$; so $j^1_0\psi$ is an element of $\bar{H}^2(M)$.

Conversely given the adapted map ψ, the map $\varphi = \beta \circ \psi$ is a local diffeomorphism inducing a section $s = \psi \circ \varphi^{-1}$ and a jet $j^1_x s$.

We have proved in [L3], [L5] the existence of a diffeomorphism

$$\Phi^q : \bar{J}_q H(M) \to \bar{H}^{q+1}(M) ,$$

the image by Φ^q of $J_q H(M)$ being the principal bundle $T^q_n(H) \cap \bar{H}^{q+1}$.

3. A connection $C : H(M) \to J_1 H(M)$ is said to be *principal* if it is a principal bundle morphism; then the distribution of horizontal spaces on H is invariant for the translation of the structure group. We recover the Ehresmann connections. The connection C is said to be *symmetric* if C takes its values in $H^2(M)$, the holonomic frame bundle of order 2.

III. Affine bundle structures on the semi-holonomic prolongations

1. Let (E, M, π) be a fibered manifold. We shall denote by $\bar{\pi}^q$ the projection $\bar{J}_q E \to M$, by $\bar{\pi}^q_k$ the projection $\bar{J}_q E \to \bar{J}_k E$ (for $0 \leq k < q$ with the convention $J_0 E = E$). The main property of semi-holonomic prolongations is the following theorem (proved in [L3] for Banach manifolds).

Theorem III.1. *The projection $\bar{\pi}^q_{q-1} : \bar{J}_q E \to \bar{J}_{q-1} E$ defines an affine bundle structure whose associated vector bundle is $(\bar{\pi}^{q-1}_0)^* L^q_E(\pi^* TM, VE)$ i.e. the pull back to $\bar{J}_{q-1}E$ of the vector bundle, with base E, of the q-linear morphisms from π^*TM to $VE = \ker T\pi$.*

Proof. This theorem is known for $q = 1$. Let $z_q \in \bar{J}_q E$, $z_{q-1} = \bar{\pi}^q_{q-1}(z_q)$, $y = \pi^q_0(z_q)$, $x = \pi(y)$. As $\bar{J}_q E \subset J_1 \bar{J}_{q-1} E$, the projection $\bar{\pi}^q_{q-1}$ is a target map β. For any $z'_q \in J_1 \bar{J}_{q-1} E$ such that $\beta(z'_q) = \beta(z_q) = z_{q-1}$, the jets z_q and z'_q, which can be considered as linear maps from $T_x M$ to $T_{z_{q-1}} \bar{J}_{q-1} E$, satisfy the relation $z_q - z'_q \in L(T_x M, \ker T_{z_{q-1}} \bar{\pi}^{q-1})$. Now z'_q belongs to $\bar{J}_q E$ if and only if $\beta(z'_q) = j^1\beta(z'_q) = z_{q-1}$; $j^1\beta$ can be considered as a linear morphism $T\beta : T\bar{J}_{q-1}E \to T\bar{J}_{q-2}E$; so the condition $j^1\beta(z'_q) = j^1\beta(z_q)$ is equivalent to $T_{z_{q-1}}\beta \cdot (z_q - z'_q) = 0$ i.e. $z_q - z'_q \in L(T_x M, \ker T_{z_{q-1}} \bar{\pi}^{q-1}_{q-2})$. If the theorem is true for the projection $\bar{J}_{q-1}E \to \bar{J}_{q-2}E$, the condition is equivalent to: $z_q - z'_q \in L(T_x M, L^{q-1}(T_x M, V_y E))$ i.e.

$$z_q - z'_q \in L^q(T_x M, V_y E)\,.$$

□

As we deal with finite dimensional manifolds, we may write:

$$L^q(T_x M, V_y E) = V_y E \otimes \otimes^q T^*_x M\,.$$

In the case of the prolongations $\bar{J}_{q,q-1}E$ and $\check{J}_q E$ (see section II.1), the inverse image of $z_{q-1} \in J_{q-1}E$ by the projection $\bar{J}_{q,q-1}E \to J_{q-1}E$ (resp. $\check{J}_q E \to J_{q-1}E$) is an affine space whose associated vector space is $V_y E \otimes \otimes^q T^*_x M$ (resp. $V_y E \otimes \bigodot^{q-1} T^*_x M \otimes T^*_x M$); here $\bigodot$ is the multiplication in the symmetric algebra of the tensor algebra of $T^*_x M$. It is known that the inverse image of $z_{q-1} \in J_{q-1}E$ in $J_q E$ is an affine space whose associated vector space is $V_y E \otimes \bigodot^q T^*_x M$.

As was done by C. Ehresmann [E3] for $\bar{T}^q_n(\mathbb{R}^k)$, we define by iteration local coordinates on $\bar{J}_q E$. Let $(x^i, y^\alpha, y^\alpha_{j_1}, \ldots, y^\alpha_{j_1,\ldots,j_{q-1}})$ be local coordinates in the neighbourhood of $z_{q-1} \in \bar{J}_{q-1}E$; then a section $s : U \subset M \to \bar{J}_{q-1}E$ is adapted at $x = \bar{\pi}^{q-1}(z_{q-1})$ if it is defined by functions $y^\alpha(x^1, \ldots, x^n), y^\alpha_j(x^1, \ldots, x^n), \ldots, y^\alpha_{j_1,\ldots,j_{q-1}}(x^1, \ldots, x^n)$ such that for $x^i = a^i$ (where a^i are the values at x of $x^1, \ldots, x^n$), $dy^\alpha = \sum y^\alpha_j(a^i) dx^j, dy^\alpha_{j_1} = \sum y^\alpha_{j_1 j_2}(a^i) dx^{j_2}, \ldots, dy^\alpha_{j_1,\ldots,j_{q-1}}(a^i) = \sum y^\alpha_{j_1\ldots j_q}(a^i) dx^{j_q}$. Then $(x^i, y^\alpha_{j_1}, \ldots, y^\alpha_{j_1,\ldots,j_q})$ constitute a system of local coordinates on $\bar{J}_q E$ around $j^1_x s$.

For $J_q E$ we have local coordinates $(x^i, y^\alpha, y^\alpha_{j_1}, \ldots, y_{j_1\ldots j_{q-1}}, y^\alpha_{j_1\ldots j_q})$ with the conditions: for $k = 2, \ldots, j_q$, $y^\alpha_{\sigma(j_1)\ldots\sigma(j_k)} = y^\alpha_{j_1,\ldots,j_k}$ for any permutation on $(1, \ldots, k)$.

For $\bar{J}_{q,q-1}E$ we have this condition only for $k = 2, \ldots, j_{q-1}$. For $\check{J}_q E \subset \bar{J}_{q,q-1}E$ we must add the condition: the $y^\alpha_{j_1,\ldots,j_q}$ are symmetric with respect of the $(q-1)$ first indices.

2. In this section we fix an element $z_{q-1} \in J_{q-1}E$; let $y = \pi^{q-1}_0(z_{q-1})$, $x = \pi(y)$ its projections on E and M. We shall denote by z_q and z'_q elements of $\bar{J}_{q,q-1}E$ such that $\bar{\pi}^q_{q-1}(z_q) = \bar{\pi}^q_{q-1}(z'_q) = z_{q-1}$.

An element z_q of $\bar{J}_{q,q-1}E$ belongs to $\check{J}_qE$ (resp. J_qE) if and only if

$$\begin{aligned} & z_q - z'_q \in V_yE \otimes (\bigcirc^{q-1} T_xM \otimes T^*_xM) \quad \text{for} \quad z'_q \in \check{J}_qE \\ (\text{resp.} \quad & z - z'_q \in V_yE \otimes (\bigcirc^{q} T^*_xM) \quad \text{for} \quad z'_q \in J_qE)\,. \end{aligned}$$

These conditions are independent of the choice of z'_q in $\check{J}_qE$ (resp. J_qE). Let S_y be the projection:

$$V_yE \otimes (\otimes^q T^*_xM) \to V_yE \otimes (\bigcirc^q T^*_xM)$$

defined by

$$S_y(v \otimes u) = \frac{1}{q!} \sum_{\sigma \in \mathcal{P}_q} v \otimes \sigma u, \qquad \text{for} \quad v \in V_yE, \quad u \in \otimes^q T^*_xM\,,$$

$\mathcal{P}_q$ being the group of permutations of $[1, q]$.

Let A_y be the projection

$$V_yE \otimes (\otimes^q T^*_xM) \to \text{Ker } S_y$$

defined by

$$A_y(v \otimes u) = v \otimes u - S_y(v \otimes u)\,.$$

As Ker A_y =image S_y, if $z'_q \in J_qE$, then $A_y(z - z'_q)$ in independent of the choice of z'_q in J_qE. So we have defined a mapping

$$\mathcal{A}_{z_{q-1}} : \bar{J}_{q,q-1}E \cap (\bar{\pi}^q_{q-1})^{-1}(z_{q-1}) \to \text{Ker } S_y$$

such that $\qquad \mathcal{A}_{z_{q-1}}(z_q) = A_y(z_q - z'_q)\,.$

We remark that Ker $\mathcal{A}_{z_{q-1}} = J_qE \cap (\pi^q_{q-1})^{-1}(z_{q-1})$.

We shall show that the mapping

$$\mathcal{S}_{z_{q-1}} : z_q \to Z_q = z_q - \mathcal{A}_{z_{q-1}}(z_q)$$

takes its values in J_qE. Indeed for $z'_q \in J_qE$, we have $Z_q - z'_q = S_y(z_q - z'_q)$; from a previous remark, Z_q belongs to J_qE.

Let $W_q = \mathcal{J}_{z_{q-1}}(z_q) = z_q - 2\mathcal{A}_{z_{q-1}}(z_q)$; this element W_q satisfies the relation $Z_q = \frac{z_q + W_q}{2}$. So we have an involution $\mathcal{J}_{z_{q-1}} : z_q \to W_q$ acting on $\bar{J}_{q,q-1}E \cap (\bar{\pi}^q_{q-1})^{-1}(z_{q-1})$.

With the same process for all elements of $\bar{J}_{q,q-1}E$, we obtain a contradiction $\mathcal{S}$: $\bar{J}_{q,q-1}E \to J_qE$ and an involution $\mathcal{J} : \bar{J}_{q,q-1}E \to \bar{J}_{q,q-1}E$. Using local coordinates we see that $\mathcal{S}$ and $\mathcal{J}$ are differentiable.

We remark that the maps $\mathcal{S}$ and $\mathcal{S}\mathcal{J}$ coincide as $\mathcal{S}|J_qE$ is the identity map.

If we consider the restriction of $\mathcal{J}$ to $\check{J}_qE$, its image is $\check{J}_qE$. In this case the image of the mapping $\mathcal{A}_{z_{q-1}}$ is $V_yE \otimes (\bigcirc^{q-2} T^*_xM) \otimes (\Lambda^2 T^*_xM)$, as was proved in [L2] and [L3]. Then this operator coincides with the restriction of the cohomology operator δ introduced by D. Spencer. Using local coordinates $(x^i, y^\alpha, y^\alpha_{j_1}, \dots, y^\alpha_{j_1,\dots,j_{q-1}j_q})$ satisfying the conditions of symmetry explained in section III.1 we obtain for the expression of $\mathcal{A}_{z_{q-1}}$ the quantities $y^\alpha_{j_1,\dots,j_{q-1}j_q} - y^\alpha_{j_1,\dots,j_q j_{q-1}}$.

From these developments, we deduce

Theorem III.2. *Given a fibered manifold (E, M, π) there exists for the prolongation $\bar{J}_{q,q-1}E$ a natural contraction $\mathcal{S} : \bar{J}_{q,q-1}E \to J_qE$ and a natural involution $\mathcal{J} : \bar{J}_{q,q-1}E \to \bar{J}_{q,q-1}E$ such that $\mathcal{J}(\check{J}_qE) = \check{J}_qE$ and $\mathcal{J}|J_qE = \mathrm{id}$. The maps $\mathcal{S}$ and $\mathcal{J}$ may be expressed as follows:*

$$\mathcal{S}(z_q) = z_q - \mathcal{A}(z_q) \qquad \mathcal{J}(z_q) = z_q - 2\mathcal{A}(z_q)$$

where $\mathcal{A}$ is a mapping from $\bar{J}_{q,q-1}E$ onto the kernel of the projection: $VE \otimes (\otimes^q \pi^* TM) \to VE \otimes (\bigcirc^q \pi^* TM)$; this map $\mathcal{A}$ vanishes on J_qE.

Remarks.

1°) The projectors A_y and S_y were introduced in [L2] and [L3].

2°) If $q = 2$, then $\bar{J}_{2,1}E = \check{J}_2E = \bar{J}_2E$. The mapping $\mathcal{A}$ takes its values in $VE \otimes (\Lambda^2 T_x^* M)$. We recover the involution introduced by J. Pradines [P] utilizing other methods (see section V).

Previously I. Kolář has introduced the symmetrization $\mathcal{S}$ on $J_{q,q-1}E$ for any q [K5] as well as the notion of "equivalence with respect to curves".

3°) When (E, M, π) is a vector bundle, we have the following exact sequences of vector bundles with base M.

$$0 \to E \otimes (\bigcirc^q T^* M) \to J_qE \to J_{q-1}E \to 0$$
$$0 \to E \otimes (\otimes^q T^* M) \to \bar{J}_qE \to \bar{J}_{q-1}E \to 0$$
$$0 \to E \otimes (\otimes^q T^* M) \to \bar{J}_{q,q-1}E \to J_{q-1}E \to 0$$
$$0 \to E \otimes (\bigcirc^{q-1} T^* M \otimes T^* M) \to \check{J}_qE \to J_{q-1}E \to 0$$

4°) The theorem can be extended to the spaces $\bar{J}^{q,q-1}(M, N)$ and $\check{J}^q(M, N)$ in particular to $\bar{T}_p^{q,q-1}(M)$, $\check{T}_p^q(M)$, $\bar{\mathfrak{C}}_n^{q,q-1}(M)$, $\check{\mathfrak{C}}_n^q(M)$.

IV. Curvature, torsion, "structure" tensor

1. As was seen in example 1 of section II, given a connection $C : E \to J_1E$, where (E, M, π) is a fibered manifold, the map $j^1C \circ C$ takes its values in $\bar{J}_2E$. From theorem III. 2, we deduce a morphism $\mathcal{A} : \bar{J}_2E \to VE \otimes (\Lambda^2 \pi^* TM)$. Combining with the lifting $j^1C \circ C$ we get a morphism

$$\varrho : E \to VE \otimes (\Lambda^2 \pi^* TM)$$

which vanishes if and only if $j^1C \circ C$ takes its values in J_2E i.e. if the connection is *integrable*. The map ϱ is called the <u>curvature</u> of the connection.

Similarly for a connection of order q i.e. a lifting $C_q : J_{q-1}E \to J_qE$, the map $j^1C_q \circ C_q$ takes its values in the sesquiholonomic prolongation $\check{J}_{q+1}E$. We have seen that the connection, considered as a differential system, is integrable if $j^1C_q \circ C_q$ takes its values in $J_{q+1}E$. The <u>curvature</u>, obstruction to integrability, is the map $\varrho_q = \mathcal{A} \circ j^1C_q \circ C_q$. This map ϱ_q is a morphism of fibered manifolds with base E;

$$\varrho_q : J_{q-1}E \to VE \otimes (\bigcirc^{q-1} \pi^* T^* M) \otimes (\Lambda^2 T^* M)\,.$$

2. Let us consider example 2 of II. 1. A Pfaffian form $\omega : M \to T^*M$ induces a connection $C : E_0 \to J_1 E_0$, where E_0 is the trivial bundle $M \times \mathbb{R}$ and $J_1 E_0 = T^*M \oplus E_0$; this connection is defined by $C(x,t) = (\omega(x), t)$; the curvature is the differential $d\omega$.

3. Let us consider example 3 of II. 1. Let $C : H(M) \to J_1 H(M) = \bar{H}^2(M)$ be a principal connection on $H(M)$. As $\bar{H}^2(M) = \bar{\mathfrak{C}}_n^2(M)$, according to remark 4°) concerning theorem III.2, we define a contraction $\mathcal{S} : \bar{H}^2(M) \to H^2(M) = \mathfrak{C}_n^2(M)$ and an involution $\mathcal{J} : \bar{H}^2(M) \to \bar{H}^2(M)$. The mappings $H(M) = \mathfrak{C}_n(M) \to \mathfrak{C}_n^2(M)$ and $\mathfrak{C}_n(M) \to \bar{\mathfrak{C}}_n^2(M)$ defined by $\mathcal{S} \circ C$ and $\mathcal{J} \circ C$ are also principal connections. The connection $\mathcal{S} \circ C$ is symmetric. The obstruction to symmetry for C is the *torsion*.

It is known that C induces linear connections $TM \to J_1 TM$ and $T^*M \to J_1 T^*M$, defined locally by Christoffel symbols Γ^i_{jk}. The connection defined locally by $\widetilde{\Gamma}^i_{jk} = \frac{\Gamma^i_{jk}+\Gamma^i_{kj}}{2}$ is symmetric. The torsion is defined by $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$; it depends on the connection itself while the curvature depends on $j^1C \circ C$; locally the curvature is function of the Christoffel symbols and their first order derivatives. It is an obstruction to the integrability of the connection i.e. to the property that $j^1C \circ C$ takes its values in $J_2 H$.

4. Let us consider a G-structure on M i.e. a principal G-subbundle $H_G(M)$ of the frame bundle $H(M)$. To simplify we shall write H_G and H instead of $H_G(M)$ and $H(M)$. The *semi-holonomic prolongation* of order q of H_G is the intersection

$$\bar{H}_G^{q+1} = \bar{\mathfrak{C}}_n^q(H_G) \cap \bar{H}^{q+1} \,;$$

so it is defined by iteration as the kernel of the double arrow

$$\begin{tikzcd}
\mathfrak{C}_n \bar{H}_G^q \arrow[r] \arrow[dr] & \mathfrak{C}_n \bar{H}_G^{q-1} \\
 & \bar{H}_G^q
\end{tikzcd}$$

and the mapping $\bar{H}_G^{q+1} \to M$ defines a principal bundle structure. Moreover the mapping $\bar{H}_G^{q+1} \to H_G$ is surjective. The prolongation $\bar{H}_G^{q+1}$ is also the image of $\bar{J}_q H_G$ by the diffeomorphism $\Phi_q : \bar{J}_q H \to \bar{H}^{q+1}$ (see section II.1).

The *holonomic prolongation* H_G^{q+1} of H_G is defined by

$$H_G^{q+1} = H^{q+1} \cap \bar{H}_G^{q+1} \,.$$

It is not necessarily a principal bundle.

A necessary condition for the G-structure H_G to be integrable in the sense of differential systems (see I) is the surjectivity of the mapping $H_G^{q+1} \to H_G$; if this condition is satisfied, the G-structure is said to be q-integrable.

It was proved in [L5] and [L7] that the following conditions are equivalent

a) the G-structure is q-integrable

b) H_G^{q+1} is a principal subbundle of H^{q+1} and of $\bar{H}_G^{q+1}$

c) there exists principal bundle morphisms $H_G \to \bar{H}_G^{q+1}$ which take their values in H_G^{q+1}.

The "structure tensor" was introduced to express the q-integrability of a G-structure. First C. Ehresmann [E2] and D. Bernard [B] introduced this "tensor" at the order 2. Then many mathematicians worked on this higher order; among them D. Lehmann [Leh], P. Molino [M] without utilizing semi-holonomic jets, P. Yuen [Y] as well as I. Kolář.

In [L2] and [L5] we proceeded as follows. Let L_n^{q+1} (resp. $\bar{L}_n^{q+1}$) be the structural group of the principal bundle H^{q+1} (resp. $\bar{H}^{q+1}$). To the projections $H^{q+1} \to H$ and $\bar{H}^{q+1} \to H$, there corresponds the projections $L_n^{q+1} \to L_n$ and $\bar{L}_n^{q+1} \to L_n$, where $L_n = GL(n, \mathbb{R})$. The group L_n^{q+1} is the semi-direct product $L_n \times N_n^{q+1}$, where N_n^{q+1} is the kernel of the projection $L_n^{q+1} \to L_n$. Similarly the group $\bar{L}_n^{q+1}$ is the semi-direct product $L_n \times \bar{N}_n^{q+1}$ and $\bar{G}^{q+1}$ (structural group of $\bar{H}_G^{q+1}$) is the semi-direct product $G \times \bar{G}_1^{q+1}$, where $\bar{G}_1^{q+1}$ is the kernel of the projection $\bar{G}^{q+1} \to G$.

For every $h \in H_G$, let $(H^{q+1})_h$ and $(\bar{H}_G^{q+1})_h$ the inverse images of h by the projections $H^{q+1} \to H$, $\bar{H}_G^{q+1} \to H_G$. If we fix $h_{q+1} \in (H^{q+1})_h$, then for $\bar{h}_{q+1}$ and $\bar{h}'_{q+1}$ elements of $(\bar{H}_G^{q+1})_h$, we have $\bar{h}_{q+1} = h_{q+1}s$, $\bar{h}_{q+1} = h_{q+1}sg$ where $s \in \bar{N}_n^{q+1}$, $g \in \bar{G}_1^{q+1}$. So with $(\bar{H}_G^{q+1})_h$ is associated an element $\varrho_{h_{q+1}}$ of $\bar{N}_n^{q+1}/\bar{G}_1^{q+1}$. When h_{q+1} describes $(H^{q+1})_h$, then $\varrho_{h_{q+1}}(h)$ described an orbit of the group N_n^{q+1} acting on $\bar{N}_n^{q+1}/\bar{G}_1^{q+1}$. We deduce the "structure tensor"

$$\alpha_q : H_G \to \left(\bar{N}_n^{q+1}/\bar{G}_1^{q+1}\right)/N_n^{q+1} .$$

We say that α_q "vanishes" if $\alpha_q(H_G)$ is the orbit of the equivalence class of the unity of $\bar{N}_n^{q+1}$, condition equivalent to the following: for every $h \in H_G$, the intersection $(\bar{H}_G^{q+1})_h \cap (H^{q+1})_h$ is non empty.

For $q = 1$, we recover the structure tensor of C. Ehresmann. Similarly we could define another "structure tensor" as a mapping

$$\beta_q : H_G \to \left(\bar{N}_n^{q+1}/N_n^{q+1}\right)\Big/\bar{G}_1^{q+1} ,$$

fixing first $\bar{h}_{q+1} \in (\bar{H}_G^{q+1})_h$ and considering h_{q+1}, h'_{q+1} in $(H^{q+1})_h$.

For $q = 1$, the groups $\bar{N}_n^2$, N_n^2, $\bar{G}_1^2$ are abelian. Then β_2 takes its values in

$$\mathbb{R}^n \otimes \Lambda^2\mathbb{R}^{n*}/A(\mathbb{R}^{n*} \otimes \underline{g}),$$

where $A = 1 - S$ and $\underline{g}$ is the Lie algebra of G. We recover the "tensor" introduced by D. Bernard [B].

V. Double vector bundles (J. Pradines [P])

1. Let us consider the triplet $(\mathcal{E}, M, \pi)$ where $\mathcal{E}$ is a set, M a differentiable manifold and $\pi : \mathcal{E} \to M$ a surjective map.

Definition V.1. The set $\mathcal{E}$ is endowed with a *double vector bundle structure* (with double base M) if there exists an atlas $\mathcal{A}$ on $\mathcal{E}$ satisfying the following conditions:

1°) Each local chart C_α is a bijection θ_α from $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_0}$ compatible with the projections π and pr_1 on U_α. Here U_α is an open subset of M and the family $(U_\alpha)_{\alpha \in I}$ is an open covering of M.

2°) The change of local chart $\theta_\beta \circ \theta_\alpha^{-1}$ over $U_\alpha \cap U_\beta$ may be expressed as

$$(x, X_1, X_2, X_0) \to (x, u_1(x) \cdot X_1, u_2(x) \cdot X_2, u_0(x) \cdot X_0 + \omega(x) \cdot (X_1, X_2)) \tag{V.1}$$

with $x \in U_\alpha \cap U_\beta$; $u_j (j = 1, 2, 0)$ is a differentiable map from $U_\alpha \cap U_\beta$ to $GL(n_j, \mathbb{R})$, ω is a differentiable map from $U_\alpha \cap U_\beta$ to the set $\mathcal{L}^2(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}; \mathbb{R}^{n_0})$ of bilinear maps: $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_0}$. This atlas defines on $\mathcal{E}$ a differentiable manifold structure for which π is a submersion.

2. From formula (V.1), it can be deduced the following properties:

1°) The local conditions $X_1 = 0$, $X_0 = 0$ are independent of the chart and define a submanifold $\mathcal{E}_1$ of $\mathcal{E}$. Likewise the conditions $X_2 = 0$, $X_0 = 0$ (resp. $X_1 = 0, X_2 = 0$) define a submanifold $\mathcal{E}_2$ (resp. $\mathcal{E}_0$).

Moreover the restrictions π_1, π_2, π_0 of π to $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_0$ define on these submanifolds vector bundles structures, with base M.

2°) There exists a projection ϖ_1 (resp. ϖ_2) from $\mathcal{E}$ onto $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) whose local expression is

$$(x, X_1, X_2, X_0) \to (x, 0, X_2, 0) \quad (\text{resp. } (x, X_1, X_2, X_0) \to (x, X_1, 0, 0)) .$$

These projections ϖ_1 and ϖ_2 define on $\mathcal{E}$ vector bundle structures with bases $\mathcal{E}_1$ and $\mathcal{E}_2$. For the first one the operations are

$$\lambda(x, X_1, X_2, X_0) = (x, \lambda X_1, X_2, \lambda X_0)$$
$$(x, X_1, X_2, X_0) + (x, X_1', X_2, X_0') = (x, X_1 + X_1', X_2, X_0 + X_0') .$$

For the second one the operations are

$$\lambda(x, X_1, X_2, X_0) = (x, X_1, \lambda X_2, \lambda X_0)$$
$$(x, X_1, X_2, X_0) + (x, X_1, X_2', X_0') = (x, X_1, X_2 + X_2', X_0 + X_0') .$$

We remark that

$$\pi = \pi_1 \circ \varpi_1 = \pi_2 \circ \varpi_2 \quad \text{and} \quad \varpi_1 | \mathcal{E}_2 = \pi_2, \quad \varpi_2 | \mathcal{E}_1 = \pi_1 . \tag{V.2}$$

If we consider the vector bundle $\mathcal{E}_1 \oplus \mathcal{E}_2$ with base M and projection (π_1, π_2), we obtain a surjective map $\varpi = (\varpi_1, \varpi_2)$ from $\mathcal{E}$ to $\mathcal{E}_1 \oplus \mathcal{E}_2$ such that

$$\mathcal{E}_0 = \{y \in \mathcal{E};\ \varpi(y) = \pi(y)\} .$$

In this section we have identified the zero section 0_B of any vector bundle with its base B for $B = M, \mathcal{E}_1, \mathcal{E}_2$. According to J. Pradines [P], the vector bundle $(\mathcal{E}_0, \pi_0, M)$ will be called the "heart" of the double vector bundle.

Examples. 1) Let $P : E \to M$ be a vector bundle; then $\mathcal{E} = TE$ is endowed with a double vector bundle for which

$$\pi = P \circ p,\ \mathcal{E}_1 = TM,\ \mathcal{E}_2 = E,\ \varpi_1 = TP,\ \varpi_2 = p$$

(where p is the projection $TE \to E$).

As was proved in [L8], the heart $\mathcal{E}_0$ is the restriction to the zero section 0_M of the vertical bundle $VE = \ker TP$. This vector bundle $\mathcal{E}_0$ is isomorphic to E. If we consider E as a groupoid for which $\alpha = \beta = P$, the heart $\mathcal{E}_0$ is the Lie algebroid of E.

2) In the particular case of the tangent bundle $p : TM \to M$, then $\mathcal{E} = TTM$ and $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_0$ are isomorphic to TM as vector bundles; but we have remarked in [L8] that $\mathcal{E}_0$ is a subspace of T^2M (set of holonomic 2-jets from $\mathbb{R}$ to M with source 0) and the action of $\mathbb{R}$ on $\mathcal{E}_0$ (considered as a subspace of $\mathcal{E}_0$) is different from the action of $\mathbb{R}$ on TM.

3) In the particular case of the cotangent bundle $q : T^*M \to M$, then $\mathcal{E} = TT^*M$ and $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_0$ are isomorphic to TM, T^*M, T^*M. The Liouville form μ on T^*M may be defined through the projection $\varpi : TT^*M \to TM \oplus T^*M$. As seen in [L8] the duality induced by the symplectic form $d\mu$ maps $\mathcal{E}_0$ onto T^*M.

These examples lead to the notion of *soldering* (strict soldering in the terminology of J. Pradines).

Definition V.2. Let $\mathcal{E}$ be a double vector bundle with base M; a 1-soldering (resp. 2-soldering) of $\mathcal{E}$ is a vector bundle isomorphism σ from $\mathcal{E}_2$ (resp. $\mathcal{E}_1$) onto $\mathcal{E}_0$.

In example 1, there exists a 1-soldering as $\mathcal{E}_2 = E$ and $\mathcal{E}_0$ is isomorphic to E. In example 3, the 1-soldering is defined by the symplectic duality. In example 2 ($\mathcal{E} = TTM$) there exists also a 2-soldering.

For a double vector bundle $\mathcal{E}$ endowed with a 1-soldering, there exists adapted local charts such that the local expression of the 1-soldering σ is $(x, X, 0, 0) \to (x, 0, 0, X)$.

Definition V.3. Let $(\mathcal{E}, M, \pi)$ and $(\mathcal{E}', M', \pi')$ be double vector bundles, f a differentiable mapping from M to M'. A mapping $g : \mathcal{E} \to \mathcal{E}'$ will be called a *f-double vector bundle morphism* if for any $x \in M$, there exists a local chart $c = (U, \theta, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_0})$ and a local chart $c' = (U', \theta', \mathbb{R}^{n'_1} \times \mathbb{R}^{n'_2} \times \mathbb{R}^{n'_0})$ of $\mathcal{E}'$ such that the local expression $\theta' \circ f \circ \theta^{-1}$ of g by means of these local charts is written:

$$\text{(V.3)}\quad (x, X_1, X_2, X_0) \to (f(x), a_1(x) \cdot X_1, a_2(x) \cdot X_2, a_0(x) \cdot X_0 + b(x)(X_1, X_2))\,,$$

where $x \in U, f(U) \subset U'$, a_j (j=1,2,0) is a differentiable map from U to $L(\mathbb{R}^{n_i}, \mathbb{R}^{n'_i})$, b is a differentiable map from U to the set $L^2(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}; \mathbb{R}^{n'_0})$ of bilinear maps: $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n'_0}$. The set of all these morphisms will be denoted by $\mathcal{L}(\mathcal{E}, \mathcal{E}')$.

The local components of g are $(f, a_1, a_2, (a_0, b))$. Double vector bundle morphisms can be composed. If $g \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ and $g' = \mathcal{L}(\mathcal{E}', \mathcal{E}'')$, then $g' \circ g \in \mathcal{L}(\mathcal{E}, \mathcal{E}'')$, with the composition law

$$\text{(V.4)} \qquad \begin{cases} (f', a_1', a_2', (a_0', b)) \circ (f, a_1, a_2, (a_0, b)) \\ =(f' \circ f, (a_1' \circ f) \cdot a_1, (a_2' \circ f) \cdot a_2, (a_0' \circ f) \cdot a_0, b'') \end{cases}$$

with $b'' = (a_0' \circ f) \cdot b + (b' \circ f) \cdot (a_1, a_2)$.

For any element of $\mathcal{L}(\mathcal{E}, \mathcal{E}')$ the restriction of g to each $\mathcal{E}_i$ $(i = 1, 2, 0)$ is a vector bundle map g_i into $\mathcal{E}_i'$; also g is a vector bundle map for the vector bundles $(\mathcal{E}, \mathcal{E}_1, \varpi^1)$ and $(\mathcal{E}, \mathcal{E}_2, \varpi^2)$.

If $\mathcal{E}$ and $\mathcal{E}'$ are endowed with 1-solderings the morphism g will be said to be 1-soldered if

$$g_0 \circ \sigma = \sigma' \circ g_2 ,$$

where $\sigma : \mathcal{E}_2 \to \mathcal{E}_0$ and $\sigma' : \mathcal{E}_2' \to \mathcal{E}_0'$ are the 1-solderings. In terms of adapted local charts, the local expression $(f, a_1, a_2, (a_0, b))$ of a 1-soldered morphism is characterized by $a_1 = a_0$. Double vector bundles with base reduced to a point (for instance the restriction $TTM|_x$ of the double tangent bundle to a fiber T_xM) and morphisms between them are called *elementary*. Utilizing formula (V.4) and the composition law between non holonomic jets introduced by C. Ehresmann [E3], J. Pradines has proved the following

Proposition V.1. *There exists a diffeomorphism between the manifold $\widetilde{J}^2_{x,y}(M, N)$ of non holonomic 2-jets from M to N with source $x \in M$, target $y \in N$ and the manifold $\mathcal{L}_\sigma(TTM|_x, TTN|_y)$ of elementary double vector bundle morphisms which are 1-soldered.*

Utilizing the notion of symmetry in double vector bundles, J. Pradines has introduced an *involution* $\mathcal{J}$ in the set $\mathcal{L}(TTM|_x, TTN|_y)$ of all elementary double vector bundle morphisms. But this involution does not keep invariant $\mathcal{L}_\sigma(TTM|_x, TTN|_y)$. On the other hand the subset of $\mathcal{L}_\sigma(TTM|_x, TTN|_y)$ which corresponds to the set $\bar{J}^2_{x,y}(M, N)$ is transformed into itself by $\mathcal{J}$. *So a natural involution $\mathcal{J} : \bar{J}^2_{x,y}(M, N) \to \bar{J}^2_{x,y}(M, N)$ is obtained and each holonomic jet is invariant by $\mathcal{J}$.*

These results have been interpreted by Janyška–Kolář [JK] (see also [J]) as follows: an element Y of $\widetilde{J}^2_{x,y}(M, N)$ is a 1-jet $j^1_x s$ where s is a local section $U \subset M \to J^1_{x,y}(M, N)$ such that $s(x) \in L(T_xM, T_yN)$. For any $u \in U$, $s(u)$ is a map $S_u : T_uM \to TN$ which defines a map $S : TM|U \to TN$; for $v \in TM|U$, $S(v) = s(p(v)) \cdot v$ where p is the projection $TM \to M$. The 2-jet Y is represented by an element μY of the restriction of TS to $TTM|_x$.

Utilizing local coordinates, the authors show that the composition law in non holonomic 2-jets can be expressed as

$$\mu(Z \circ Y) = (\mu Z) \circ (\mu Y) .$$

The natural involution $\mathcal{J} : TTM \to TTM$ transforms $TTM|_x$ into itself and induces an involution $\mathcal{J}_x : TTM|_x \to TTM|_x$.

Similarly we define $\mathcal{J}_y$ on $TTN|_y$. The involution

$$\mathcal{J} : \bar{J}^2(M,N) \to \bar{J}^2(M,N) \quad \text{is defined by}$$
$$\mu(\mathcal{J}X) = \mathcal{J}_y \circ \mu X \circ \mathcal{J}_x\,.$$

The natural transformations of $\bar{J}_{3,2}E$ have been studied by G. Vosmanská.

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