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ON PERIODIC SOLUTIONS OF SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Ivan Kiguradze and Bedřich Půža

Abstract. This paper deals with the system of functional-differential equations

$$\frac{dx(t)}{dt} = p(x)(t) + q(t),$$

where $p : C_\omega(\mathbf{R}^n) \to L_\omega(\mathbf{R}^n)$ is a linear bounded operator, $q \in L_\omega(\mathbf{R}^n)$, $\omega > 0$ and $C_\omega(\mathbf{R}^n)$ and $L_\omega(\mathbf{R}^n)$ are spaces of n-dimensional ω-periodic vector functions with continuous and integrable on $[0, \omega]$ components, respectively. Conditions which guarantee the existence of a unique ω-periodic solution and continuous dependence of that solution on the right hand side of the system considered are established.

Introduction

Let us consider a system of functional-differential equations

$$\frac{dx(t)}{dt} = p(x)(t) + q(t)\,, \tag{0.1}$$

and its particular case

$$\frac{dx(t)}{dt} = P(t)x(\tau(t)) + q(t)\,, \tag{0.2}$$

where $p : C_\omega(\mathbf{R}^n) \to L_\omega(\mathbf{R}^n)$ is a linear operator for which there is $\eta \in L_\omega(\mathbf{R})$ such that

$$\|p(x)(t)\| \le \eta(t)\|x\|_C \quad \text{for } t \in \mathbf{R}, x \in C_\omega(\mathbf{R}^n)\,, \tag{0.3}$$

$$P = (p_{ik})_{i,k=1}^n \in L_\omega(\mathbf{R}^{n\times n})\,, \tag{0.4}$$

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$$q \in L_\omega(\mathbf{R}^n)\,. \tag{0.5}$$

As concerns the function $\tau : \mathbf{R} \to \mathbf{R}$, it is measurable and satisfies the condition

$$\tau(t+\omega) \equiv \mu(t)\omega + \tau(t)\,, \tag{0.6}$$

where μ is a function assuming only integer values.

A vector function $x : \mathbf{R} \to \mathbf{R}^n$ is called ω-periodic solution of the system (0.1) (of the system (0.2)) if it is absolutely continuous, periodic with the period ω, i.e.

$$x(t+\omega) = x(t)\,,$$

and satisfies the system (0.1) (the system (0.2)) almost everywhere on $\mathbf{R}$.

In the case $\tau(t) \equiv t$, the problem of ω-periodic solutions of the system (0.2) and an analogous problem for a system of nonlinear ordinary differential equations are treated in literature in sufficient details [2,5,8-10,13-15,21,23,25]. A general theory of linear boundary value problems for systems of functional-differential equations, including periodic problems, is presented in monographs [1,22], a periodic problem is studied in [3,4,6,7,12,18,20,25]. The present paper is based on results of [11] and it establishes new sufficient conditions for existence of a unique ω-periodic solution of the system (0.1) (of the system (0.2)). Theorems of J. Kurzweil - Z. Vorel type [16,17,24] and Z. Opial type [19] on continuous dependence of the solution mentioned on the right hand side of the system considered are proved.

Throughout the paper, the following notation is used:

$\mathbf{R}^n$ - space of n dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in \mathbf{R}$ $(i = 1,\dots,n)$ and the norm

$$\|x\| = \textstyle\sum_{i=1}^n |x_i|;$$

$\mathbf{R}^{n\times n}$ - space of $n \times n$ matrices $X = (x_{ik})_{i,k=1}^n$ with elements $x_{ik} \in \mathbf{R}$ $(i,k = 1,\dots,n)$ and the norm

$$\|X\| = \sum_{i,k=1}^n |x_{ik}|\,;$$

$$\mathbf{R}_+^n = \{(x_i)_{i=1}^n \in \mathbf{R}^n : x_i \ge 0 \qquad (i=1,\dots,n)\}\,;$$
$$\mathbf{R}_+^{n\times n} = \{(x_{ik})_{i,k=1}^n \in \mathbf{R}^{n\times n} : x_{ik} \ge 0 \qquad (i,k=1,\dots,n)\}\,;$$

if $x, y \in \mathbf{R}^n$ and $X, Y \in \mathbf{R}^{n\times n}$ then

$$x \le y \iff y - x \in \mathbf{R}_+^n,\ X \le Y \Leftrightarrow Y - X \in \mathbf{R}_+^{n\times n}\,;$$

if $x = (x_i)_{i=1}^n \in \mathbf{R}^n$ and $X = (x_{ik})_{i,k=1}^n \in \mathbf{R}^{n\times n}$ then

$$|x| = (|x_i|)_{i=1}^n,\ |X| = (|x_{ik}|)_{i,k=1}^n\,;$$

det (X) - determinant of the matrix X; X^{-1} - matrix inverse to X; $r(X)$ - spectral radius of the matrix X; E - unit matrix; Θ - zero matrix;

$C([0,\omega];\mathbf{R}^n)$-space of continuous vector functions $x:[0,\omega]\to\mathbf{R}^n$ with the norm

$$\|x\|_C=\max\{\|x(t)\|:0\le t\le\omega\}\,;$$

$C_\omega(\mathbf{R}^n)$, where $\omega>0$ - space of continuous ω-periodic vector functions $x:\mathbf{R}\to\mathbf{R}^n$ with the norm

$$\|x\|_{C_\omega}=\max\{\|x(t)\|:0\le t\le\omega\}\,,$$

if $x=(x_i)_{i=1}^n\in C_\omega(\mathbf{R}^n)$ then

$$|x|_{C_\omega}=(\|x_i\|_{C_\omega})_{i=1}^n\,;$$

$L([0,\omega];\mathbf{R}^n)$-space of vector functions $x:\mathbf{R}\to\mathbf{R}^n$ with elements summable on $[0,\omega]$ with the norm

$$\|x\|_L=\int_0^\omega\|x(t)\|\,dt\,;$$

$L_\omega(\mathbf{R}^n)$ - space of ω-periodic vector functions $x:\mathbf{R}\to\mathbf{R}^n$ with elements summable on $[0,\omega]$ with the norm

$$\|x\|_{L_\omega}=\int_0^\omega\|x(t)\|\,dt\,;$$

$L_\omega(\mathbf{R}^{n\times n})$ - space of matrix functions $X:\mathbf{R}\to\mathbf{R}^{n\times n}$ with elements from $L_\omega(\mathbf{R})$.

If $Z:\mathbf{R}\to\mathbf{R}^{n\times n}$ is an ω-periodic continuous matrix function with columns $z_1,\dots,z_n$ and $g:C_\omega(\mathbf{R}^n)\to L_\omega(\mathbf{R}^n)$ is a linear operator then by $g(Z)$ we shall understand the matrix function with columns $g(z_1),\dots,g(z_n)$.

§1. Existence and uniqueness

In the whole subsequent text, we will assume that $p:C_\omega(\mathbf{R}^n)\to L_\omega(\mathbf{R}^n)$ is a linear operator satisfying the condition (0.3) and that P,q and τ satisfy the conditions (0.4)-(0.6).

For almost all $t\in\mathbf{R}$, let us denote by $\nu(t)$ the integer part of the number $\frac{\tau(t)}{\omega}$ and set

$$\zeta(t)=\tau(t)-\nu(t)\omega\,. \tag{1.1}$$

Then in view of (0.6),

$$\zeta\in L_\omega(\mathbf{R}),\ 0\le\zeta(t)<\omega\quad\text{ for } t\in\mathbf{R}\,. \tag{1.2}$$

For an arbitrary continuous vector function $x:[0,\omega]\to\mathbf{R}^n$, we denote by $v_\omega(x)$ the vector function defined by the equality

$$v_\omega(x)(t)=x(t-j\omega)+\frac{t-j\omega}{\omega}[x(0)-x(\omega)]\quad\text{ for } j\omega\le t<(j+1)\omega \\ (j=0,1,-1,2,-2,\dots) \tag{1.3}$$

and set

$$p_0(x)(t) = p(v_\omega(x))(t) \quad \text{for } t \in [0,\omega]\,. \tag{1.4}$$

Obviously, v_ω is a bounded linear operator acting from $C([0,\omega];\mathbf{R}^n)$ into $C_\omega(\mathbf{R}^n)$. Therefore by (0.3), $p_0 : C([0,\omega];\mathbf{R}^n) \to L([0,\omega];\mathbf{R}^n)$ is a linear operator satisfying the inequality

$$\|p_0(x)(t)\| \le \eta_0(t)\|x\|_C \quad \text{for } t \in [0,\omega],\ x \in C([0,\omega],\mathbf{R}^n)\,,$$

where $\eta_0(t) = 3\eta(t)$.

Let x be an arbitrary ω-periodic solution of the system (0.1). Then in view of (1.3) and (1.4), the restriction of x to $[0,\omega]$ is a solution of the periodic boundary value problem

$$\frac{dx(t)}{dt} = p_0(x)(t) + q(t)\,, \tag{1.5}$$

$$x(\omega) = x(0)\,. \tag{1.6}$$

The inverse statement is obvious: the ω-periodic continuation of an arbitrary solution of the boundary value problem (1.5), (1.6) represents an ω-periodic solution of the system (0.1). Therefore, Theorem 1.1 in the paper [11] implies

Theorem 1.1. *The system (0.1) has a unique ω-periodic solution if and only if the system of differential equations*

$$\frac{dx(t)}{dt} = p(v_\omega(x))(t) \tag{1.7}$$

with the boundary conditions (1.6) has only the trivial solution.

The system (0.1) coincides with the system (0.2) if

$$p(x)(t) = P(t)x(\tau(t))\,. \tag{1.8}$$

From (1.4) in view of (1.1) and (1.3), we obtain

$$p(v_\omega(x))(t) = P(t)v_\omega(x)(\tau(t)) = P(t)v_\omega(x)(\nu(t)\omega + \zeta(t)) =$$
$$= P(t)v_\omega(x)(\zeta(t)) = P(t)[x(\zeta(t)) + \frac{\zeta(t)}{\omega}(x(0) - x(\omega))] \quad \text{for } 0 \le t \le \omega\,.$$

Now, it is clear that the problem (1.7), (1.6) has only the trivial solution if and only if the system

$$\frac{dx(t)}{dt} = P(t)x(\zeta(t)) \tag{1.9}$$

with the boundary conditions (1.6) has only the trivial solution. That is why Theorem 1.1 implies

Corollary 1.1. *The system (0.2) has a unique ω-periodic solution if and only if the problem (1.9), (1.6) has only the trivial solution.*

Let us introduce sequences of operators $p^k : C_\omega(\mathbf{R}^n) \to C([0,\omega];\mathbf{R}^n)$ and matrices $\Lambda_k \in \mathbf{R}^{n\times n}$:

$$p^0(x)(t) = x(t), p^k(x)(t) = \int_0^t p(v_\omega(p^{k-1}(x)))(s)\,ds \qquad (k = 1, 2, \dots), \tag{1.10}$$

$$\Lambda_1 = \Theta,\ \Lambda_k = \sum_{i=1}^{k-1} p^i(E)(\omega) \qquad (k = 1, 2, \dots).$$

It is clear that

$$\Lambda_2 = \int_0^\omega p(E)(s)\,ds\,.$$

If the matrix Λ_k is non-singular for some $k \geq 2$ then we set

$$\begin{gathered} p^{k,0}(x)(t) = x(t), p^{k,m}(x)(t) = p^m(x)(t) - \\ -[p^0(E)(t) + \dots + p^{m-1}(E)(t)]\Lambda_k^{-1} p^k(x)(\omega)\,. \end{gathered} \tag{1.11}$$

Theorem 1.2 in [11] and Theorem 1.1 imply

Theorem 1.2. *The system (0.1) has a unique ω-periodic solution if there exist a matrix $A \in \mathbf{R}_+^{n\times n}$ and positive integers $k \geq 2$ and m such that the matrix Λ_k is non-singular,*

$$r(A) < 1 \tag{1.12}$$

and

$$|p^{k,m}(x)(t)| \leq A|x|_{C\omega} \quad \text{for } t \in [0,\omega], x \in C_\omega(\mathbf{R}^n)\,. \tag{1.13}$$

Corollary 1.2. *Let the matrix*

$$\Lambda_2 = \int_0^\omega p(E)(s)\,ds$$

be non-singular and let there exist a matrix $B \in \mathbf{R}_+^{n\times n}$ such that

$$\int_0^\omega |p(x)(s)|ds \leq B|x|_{C\omega} \quad \text{for } x \in C_\omega(\mathbf{R}^n) \tag{1.14}$$

and

$$r(B + |\Lambda_2^{-1}|B^2) < 1\,. \tag{1.15}$$

Then the system (0.1) has a unique ω-periodic solution.

Proof. In view of (1.10), (1.11) and (1.14)

$$p^1(x)(t) = \int_0^t p(x)(s)\,ds,$$

$$p^{2,1}(x)(t) = p^1(x)(t) - \Lambda_2^{-1}\int_0^\omega p(v_\omega(p^1(x)))(s)ds,$$

and

$$|p^{2,1}(x)(t)| \le B|x|_{C\omega} + |\Lambda_2^{-1}|B|v_\omega(p^1(x))|_{C\omega} \text{ for } t\in[0,\omega], x\in C_\omega(\mathbf{R}^n)\,.$$

On the other hand, by (1.3)

$$v_\omega(p^1(x))(t) = \int_0^t p(x)(s)\,ds - \frac{t}{\omega}\int_0^\omega p(x)(s)\,ds =$$
$$= (1-\frac{t}{\omega})\int_0^t p(x)(s)\,ds - \frac{t}{\omega}\int_t^\omega p(x)(s)\,ds\,.$$

Therefore

$$|v_\omega(p^1(x))(t)| \le \int_0^\omega |p(x)(s)|\,ds \le B|x|_{C_\omega} \quad \text{for } t\in[0,\omega], x\in C_\omega(\mathbf{R}^n)$$

and

$$|p^{2,1}(x)(t)| \le (B + |\Lambda_2^{-1}|B^2)|x|_{C\omega} \quad \text{for } t\in[0,\omega], x\in C_\omega(\mathbf{R}^n)\,.$$

Consequently, the condition (1.13) is satisfied for $k=2$ and $m=1$, where the matrix $A = B + |\Lambda_2^{-1}|B^2$ satisfies the inequality (1.12). □

For arbitrary matrix function $V\in L_\omega(\mathbf{R}^{n\times n})$, set

$$[V(t)]_{\zeta,0} = \Theta, [V(t)]_{\zeta,1} = V(t), [V(t)]_{\zeta,i+1} =$$
$$V(t)\int_0^{\zeta(t)} [V(s)]_{\zeta,i}\,ds \ (i=1,2,\dots)\,.$$

Then Theorem 2.2 in [11] and Theorem 1.1 imply the following

Corollary 1.3. *Let there exist positive integers $k\ge 2$ and m such that the matrix*

$$\Lambda_k = \sum_{i=1}^{k-1}\int_0^\omega [P(s)]_{\zeta,i}\,ds$$

is non-singular and

$$r(A_{k,m}) < 1\,,$$

where

$$A_{k,m} = \int_0^\omega [|P(s)|]_{\zeta,m}\,ds + (E + \sum_{i=0}^{m-1}\int_0^\omega [|P(s)|]_{\zeta,i}\,ds)|\Lambda_k^{-1}|\int_0^\omega [|P(s)|]_{\zeta,k}\,ds\,.$$

Then the system (0.2) has a unique ω-periodic solution.

For $k=2$ and $m=1$, Corollary 1.3 has the following form

Corollary 1.4. *Let the matrix*

$$\Lambda_2 = \int_0^\omega P(s)\,ds$$

be non-singular and

$$r(A_{2,1}) < 1,$$

where

$$A_{2,1} = \int_0^\omega |P(s)|ds + |\Lambda_2^{-1}| \int_0^\omega (|P(s)| \int_0^{\zeta(s)} |P(t)|\,dt)\,ds.$$

Then the system (0.2) has a unique ω-periodic solution.

Together with (0.1) and (0.2) under the conditions (0.3) - (0.6), let us consider differential systems

$$\frac{dx(t)}{dt} = \varepsilon p(x)(t) + q(t) \tag{1.16}$$

and

$$\frac{dx(t)}{dt} = \varepsilon P(t)x(\tau(t)) + q(t)\,, \tag{1.17}$$

where ε is a small positive parameter.

Corollary 1.5. *If the matrix*

$$\Lambda_2 = \int_0^\omega p(E)(s)\,ds$$

is non-singular then there is $\varepsilon_0 > 0$ such that the system (1.16) has a unique ω-periodic solution for each $\varepsilon \in]0, \varepsilon_0[$.

Proof. Since the operator $p : C_\omega(\mathbf{R}^n) \to L_\omega(\mathbf{R}^n)$ is bounded, there exists a matrix $B \in \mathbf{R}_+^{n\times n}$ satisfying the inequality (1.14). Let

$$A = B + |\Lambda_2^{-1}|B^2$$

and

$$\varepsilon_0 = \frac{1}{r(A)}\,. \tag{1.18}$$

Set

$$p_\varepsilon(x)(t) = \varepsilon p(x)(t), \Lambda_{2,\varepsilon} = \int_0^\omega p_\varepsilon(E)(s)\,ds,\ B_\varepsilon = \varepsilon B\,.$$

Then $\Lambda_{2,\varepsilon} = \varepsilon\Lambda_2$ is non-singular for each $\varepsilon > 0$. On the other hand, in view of (1.14) and (1.18),

$$\int_0^\omega |p_\varepsilon(x)(s)|\,ds \le B_\varepsilon |x|_{C_\omega} \quad \text{for } x \in C_\omega$$

and

$$r(B_\varepsilon + |\Lambda_{2,\varepsilon}^{-1}|B_\varepsilon^2) = \varepsilon r(A) < 1 \text{ for } \varepsilon \in]0, \varepsilon_0[\,.$$

In virtue of Corollary 1.2, the last two inequalities yield that (1.16) has a unique ω-periodic solution for each $\varepsilon \in]0, \varepsilon_0[$. □

For the system (1.17), Corollary 1.5 takes the following form:

Corollary 1.6. *If the matrix*

$$\int_0^\omega P(s)\,ds$$

is non-singular, then there is $\varepsilon_0 > 0$ such that the system (1.17) has a unique ω-periodic solution for each $\varepsilon \in]0, \varepsilon_0[$.

As we noticed above, the ω-periodic continuation of an arbitrary solution of the problem (1.7), (1.6) represents an ω-periodic solution of the system

$$\frac{dx(t)}{dt} = p(x)(t)\,. \tag{1.19}$$

That is why Corollary 1.5 in [11] implies

Corollary 1.7. *Let there exist a matrix function $P_0 \in L_\omega(\mathbf{R}^n)$ such that the equality*

$$\left(\int_s^t P_0(\xi)\,d\xi\right) P_0(t) = P_0(t)\left(\int_s^t P_0(\xi)\,d\xi\right) \tag{1.20}$$

holds for almost all s and $t \in I$, let the matrix

$$A_0 = E - \exp\left(\int_0^\omega P_0(s)\,ds\right) \tag{1.21}$$

be non-singular and let the following inequality be satisfied for arbitrary ω-periodic solution of the system (1.19):

$$\int_{t-\omega}^t |A_0^{-1}\exp\left(\int_s^t P_0(\xi)\,d\xi\right)[p(x)(s) - P_0(s)x(s)]|\,ds \le A|x|_C \quad \text{for } t \in [0,\omega]\,,$$

where $A \in \mathbf{R}_+^{n\times n}$ is a matrix satisfying the condition (1.12). Then the system (0.1) has a unique ω-periodic solution.

If $p(x)(t) = P(t)x(\tau(t))$, then any ω-periodic solution of the system (1.19) represents also a solution of the system (1.9). Therefore for each such solution, we have

$$|p(x)(t) - P_0(t)x(t)| =$$
$$= |(P(t) - P_0(t))x(\zeta(t)) + P_0(t)\int_t^{\zeta(t)} P(s)x(\zeta(s))ds| \le Q(t)|x|_{C_\omega}\,,$$

where

$$Q(t) = |P(t) - P_0(t)| + |P_0(t)||\int_t^{\zeta(t)} |P(s)\,|ds|\,. \tag{1.22}$$

In virtue of the fact mentioned, Corollary 1.7 implies

Corollary 1.8. *Let there exist a matrix function $P_0 \in L_\omega(\mathbf{R}^n)$ such that the equality (1.20) holds for almost all s and $t \in I$, let the matrix (1.21) be non-singular and*

$$\int_{t-\omega}^{t} |A_0^{-1} \exp\left(\int_s^t P_0(\xi)\, d\xi\right) |Q(s)\, ds \le A \quad \text{for } t \in [0,\omega]\,, \tag{1.23}$$

where Q is the matrix function defined by the equality (1.22) and let $A \in \mathbf{R}_+^{n\times n}$ be the matrix satisfying the condition (1.12). Then the system (0.2) has a unique ω-periodic solution.

Corollary 1.9. *Let there be numbers $\sigma_i \in \{-1,1\}$, $b_{0i} > 0$ and $b_{ik} \in \mathbf{R}_+$ $(i,k = 1,\dots,n)$ such that the real parts of the eigenvalues of the matrix*

$$(b_{ik} - \delta_{ik} b_{0i})_{i,k=1}^n\,, \tag{1.24}$$

where δ_{ik} is the Kronecker's delta symbol, are negative, and the inequalities

$$\sigma_i p_{ii}(t) \ge b_{0i} \qquad (i = 1,\dots,n) \tag{1.25}$$

and

$$(1-\delta_{ik})|p_{ik}(t)| + |p_{ii}(t)| \left| \int_t^{\zeta(t)} |p_{ik}(s)| ds \right| \le b_{ik} \qquad (i,k = 1,\dots,n) \tag{1.26}$$

are satisfied almost everywhere on $[0,\omega]$. Then the system (0.2) has a unique ω-periodic solution.

Proof. It can be shown easily that the real parts of the eigenvalues of the matrix (1.24) are negative if and only if the matrix

$$A = \left(\frac{b_{ik}}{b_{0i}}\right)_{i,k=1}^n \tag{1.27}$$

satisfies the inequality (1.12).

Let us denote by $P_0(t)$ the diagonal matrix with the diagonal elements $p_{11}(t)$, $\dots, p_{nn}(t)$. Then in view of (1.25), the matrix A_0 defined by equality (1.21) is non-singular,

$$A_0^{-1} \exp\left(\int_s^t P_0(\xi)\, d\xi\right) = (\delta_{ik} g_i(t,s))_{i,k=1}^n\,, \tag{1.28}$$

where

$$g_i(t,s) = \exp\left(\int_s^t p_{ii}(\xi)\, d\xi\right)\left[1 - \exp\left(\int_0^\omega p_{ii}(\zeta) d\zeta\right)\right]^{-1}$$

and

$$\int_{t-\omega}^{t} |g_i(t,s)| \le \frac{\sigma_i}{b_{0i}} \int_{t-\omega}^{t} p_{ii}(s)|g_i(t,s)|\,ds =$$

$$= \frac{1}{b_{0i}} \left|1 - \exp\left(\int_{t-\omega}^{t} p_{ii}(\xi)\,d\xi\right)\right| \left|1 - \exp\left(\int_{0}^{\omega} p_{ii}(\xi)\,d\xi\right)\right|^{-1} \quad \text{for } t \in [0,\omega]\,.$$

But since p_{ii} is ω-periodic,

$$\int_{t-\omega}^{t} p_{ii}(\xi)\,d\xi = \int_{0}^{\omega} p_{ii}(\xi)\,d\xi\,.$$

Therefore

$$\int_{t-\omega}^{t} |g_i(t,s)|\,ds \le \frac{1}{b_{0i}} \quad \text{for } t \in [0,\omega]\,. \tag{1.29}$$

On the other hand in view of (1.22) and (1.26), the unequality

$$Q(t) \le (b_{ik})_{i,k=1}^{n} \tag{1.30}$$

is satisfied almost everywhere on $[0,\omega]$.

(1.27) - (1.30) yield the inequality (1.23). Consequently, all assumptions of Corollary 1.7 are satisfied. □

The requirement of negativity of the real parts of the eigenvalues of the matrix (1.24) is optimal and it can't be weakened. Indeed, let $p_{ii} = 0$ $(i = 1,\dots,n)$ and let the matrix (1.24) have at least one eigenvalue with nonnegative real part. Then the matrix (1.27) satisfies the inequality

$$r(A) \ge 1\,.$$

Therefore there are complex numbers λ and c_i $(i = 1,\dots,n)$ such that

$$|\lambda| \ge 1, \quad \sum_{i=1}^{n} |c_i| > 0$$

and

$$\sum_{k=1}^{n} b_{ik}c_k = \lambda b_{0i}c_i \qquad (i = 1,\dots,n)\,.$$

Therefore

$$\sum_{k=1}^{n} \eta_i b_{ik}|c_k| = b_{0i}|c_i| \qquad (i = 1,\dots,n)\,,$$

where $\eta_i \in [0,1]$ $(i = 1,\dots,n)$. Consequently, $(|c_i|)_{i=1}^{n}$ represents a non-trivial ω-periodic solution of the differential system

$$\frac{dx(t)}{dt} = P(t)x(t)\,,$$

where $P(t) \equiv (\eta_i b_{ik} - \delta_{ik}b_{0i})_{i,k=1}^{n}$. On the other hand, the considered system satisfies all assumptions of Corollary 1.9 except the negativity of the real parts of the eigenvalues of the matrix (1.24).

§2. Continuous dependence of solution on the right hand side of differential system

In this section, statements concerning continuous dependence of periodic solutions of the system (0.1), (0.2) on its right hand side are proved.

For each positive integer k, let us consider the systems

$$\frac{dx(t)}{dt} = p_k(x)(t) + q_k(t) \tag{2.1}$$

and

$$\frac{dx(t)}{dt} = P_k(t)x(\tau_k(t)) + q_k(t)\,, \tag{2.2}$$

where $p_k : C_\omega(\mathbf{R}^n) \to L_\omega(\mathbf{R}^n)$ is a linear operator for which there exists a function $\eta_k \in L_\omega(\mathbf{R})$ such that

$$\|p_k(x)(t)\| \le \eta_k(t)\|x\|_{C_\omega} \quad \text{for } t \in \mathbf{R}, x \in C_\omega(\mathbf{R}^n)$$

and

$$q_k \in L_\omega(\mathbf{R}^n),\ P_k \in L_\omega(\mathbf{R}^{n\times n})\,.$$

As concerns $\tau_k : \mathbf{R} \to \mathbf{R}$, it is measurable and it satisfies the unequality

$$\tau_k(t+\omega) = \mu_k(t)\omega + \tau_k(t)\,,$$

where μ_k is a function assuming integers values only. Let us denote by ν_k the integer part of the number $\frac{\tau_k(t)}{\omega}$ and set

$$\zeta_k(t) = \tau_k(t) - \nu_k(t)\omega\,.$$

Let $g : C_\omega(\mathbf{R}^n) \to L_\omega(\mathbf{R}^n)$ be an arbitrary linear bounded operator and let us denote by $|||\cdot|||$ its norm and by M_g^ω a set of all absolutely continuous ω-periodic vector functions $y : \mathbf{R} \to \mathbf{R}^n$ allowing the following representation:

$$y(t) = z(0) + \int_0^t g(z)(s)\,ds - \frac{t}{\omega}\int_0^\omega g(z)(s)\,ds \quad \text{for } t \in [0,\omega]\,,$$

where

$$z \in C_\omega(\mathbf{R}^n),\ \|z\|_{C_\omega} = 1\,. \tag{2.3}$$

Theorem 2.1. *Let the system (0.1) have a unique ω-periodic solution x,*
(2.4)
$$\sup\left\{\|\int_0^t [p_k(y)(s)-p(y)(s)]\,ds\| : t\in[0,\omega],\, y\in M^{\omega}_{p_k}\right\}\to 0 \quad \text{for } k\to+\infty$$

and let

$$\lim_{k\to+\infty}\left((1+|||p_k|||)\int_0^t [p_k(y)(s)-p(y)(s)]\,ds\right)=0 \quad \text{uniformly on } [0,\omega] \tag{2.5}$$

for any absolutely continuous ω-periodic function $y:\mathbf{R}\to\mathbf{R}^n$. Let further

$$\lim_{k\to+\infty}\left((1+|||p_k|||)\int_0^t [q_k(s)-q(s)]\,ds\right)=0 \quad \text{uniformly on } [0,\omega]\,. \tag{2.6}$$

Then there is a positive integer k_0 such that for each $k\geq k_0$ the system (2.1) also has a unique ω-periodic solution x_k and

$$\lim_{k\to+\infty}\|x-x_k\|_{C_\omega}=0\,. \tag{2.7}$$

Proof. Let $p_0: C([0,\omega];\mathbf{R}^n)\to L([0,\omega];\mathbf{R}^n)$ be the operator defined by (1.3), (1.4) and

$$p_{0k}(y)(t)=p_k(v_\omega(y))(t) \quad \text{for } y\in C(I,\mathbf{R}^n)\,. \tag{2.8}$$

Let us denote by $M_{p_{0k}}$ the set of all absolutely continuous vector functions $y:[0,\omega]\to\mathbf{R}^n$ allowing the representation

$$y(t)=z(0)+\int_0^t p_k(v_\omega(z))(s)\,ds\,, \tag{2.9}$$

where

$$z\in C([0,\omega];\mathbf{R}^n),\ \|z\|_C=1\,. \tag{2.10}$$

According to Theorem 1.4 in the paper [11], it is sufficient to verify the following conditions for completing the proof:
(2.11)
$$\sup\left\{\|\int_0^t [p_{0k}(y)(s)-p_0(y)(s)]\,ds\| : t\in[0,\omega],\, y\in M_{p_{0k}}\right\}\to 0 \quad \text{for } k\to+\infty\,,$$

$$\lim_{k\to+\infty}\left((1+|||p_{0k}|||)\int_0^t [q_k(s)-q(s)]\,ds\right)=0 \quad \text{uniformly on } [0,\omega] \tag{2.12}$$

and

$$\lim_{k\to+\infty}\left((1+|||p_{0k}|||)\int_0^t [p_{0k}(y)(s)-p_0(y)(s)]\,ds\right)=0 \tag{2.13}$$

for any absolutely continuous $y:[0,\omega]\to\mathbf{R}^n$.

In view of (1.3),

$$\|v_\omega(y)\|_{C_\omega}\le 3\|y\|_C\,.$$

Therefore (2.8) implies

$$|||p_{0k}|||\le 3|||p_k|||\qquad (k=1,2,\dots)\,. \tag{2.14}$$

Consequently, (2.6) yields the condition (2.12).

Let $y:[0,\omega]\to\mathbf{R}^n$ be arbitrary absolutely continuous function. Then $\tilde y=v_\omega(y)$ is an ω-periodic absolutely continuous function. On the other hand in view of (1.4) and (2.8),

$$\int_0^t [p_{0k}(y)(s)-p_0(y)(s)]\,ds=\int_0^t [p_k(\tilde y)(s)-p(\tilde y)(s)]\,ds. \tag{2.15}$$

From this, in view of (2.5) and (2.14), condition (2.13) follows.

Thus it remains to show that the condition (2.11) is satisfied. Let k be a positive integer and $y\in M_{p_{0k}}$. Then the representation (2.9) with z satisfying the condition (2.10) is valid.

If we set

$$\tilde y(t)=v_\omega(y)(t),\ \ \tilde z(t)=v_\omega(z)(t)\,,$$

then we have

$$\tilde y(t)=\tilde z(0)+\int_0^t p_k(\tilde z)(s)\,ds-\frac{t}{\omega}\int_0^\omega p_k(\tilde z)(s)\,ds$$

and

$$\|\tilde z\|_{C_\omega}\le 3\,. \tag{2.16}$$

If $\tilde z(t)\equiv 0$ then $y(t)\equiv\tilde y(t)\equiv 0$. If $\tilde z(t)\not\equiv 0$ then

$$y_0=\|\tilde z\|_{C_\omega}^{-1}\tilde y\in M_{p_k}^\omega\,.$$

Therefore (2.15) and (2.16) yield

$$\left|\int_0^t [p_{0k}(y)(s)-p_0(y)(s)]\,ds\right|\le 3\left|\int_0^t [p_k(y_0)(s)-p(y_0)(s)]\,ds\right|\quad \text{for } t\in[0,\omega]\,.$$

From this, in view of (2.4), condition (2.11) follows. □

The theorem just proved implies

Corollary 2.1. *Let the system (0.1) have a unique ω-periodic solution x and let the following condition be satisfied for any absolutely continuous ω-periodic vector function $y : \mathbf{R} \to \mathbf{R}^n$:*

$$\lim_{k\to+\infty} \int_0^t [p_k(y)(s) - p(y)(s)]\, ds = 0 \qquad \textit{uniformly on } [0,\omega]\,.$$

Let further

$$\lim_{k\to+\infty} \int_0^t [q_k(s) - q(s)] ds = 0 \textit{ uniformly on } [0,\omega]$$

and let there be a summable function $\eta : [0,\omega] \to \mathbf{R}_+$ such that

$$\|p_k(y)(t)\| \le \eta(t)\|y\|_{C_\omega}$$

almost everywhere on $[0,\omega]$ for any $y \in C_\omega(\mathbf{R}^n)$. Then the conclusion of Theorem 2.1 holds.

The restriction of an ω-periodic solution of the systems (0.2) and (2.2) to $[0,\omega]$ is a solution of differential systems

$$\frac{dx(t)}{dt} = P(t)x(\zeta(t)) + q(t) \tag{2.17}$$

and

$$\frac{dx(t)}{dt} = P_k(t)x(\zeta_k(t)) + q_k(t) \tag{2.18}$$

with the boundary conditions (1.6), respectively. On the other hand, ω-periodic continuations of solutions of the problems (2.17), (1.6) and (2.18), (1.6) represent solutions of the systems (0.2) and (2.2), respectively. That is why Corollary 2.1 implies

Corollary 2.2. *Let the system (0.2) have a unique ω-periodic solution x,*

$$\lim_{k\to+\infty} \int_0^t [P_k(s) - P(s)]\, ds = 0 \qquad \textit{uniformly on } [0,\omega]\,,$$

$$\lim_{k\to+\infty} \int_0^t [q_k(s) - q(s)]\, ds = 0 \qquad \textit{uniformly on } [0,\omega]$$

and

$$\operatorname{ess\,sup}\{|\zeta_k(t) - \zeta(t)| : t \in I\} \to 0 \qquad \textit{for } k \to +\infty\,.$$

Further let there be a summable function $\eta : [0,\omega] \to \mathbf{R}_+$ such that

$$\|P_k(t)\| \le \eta(t) \qquad (k = 1, 2, \dots)$$

almost everywhere on $[0,\omega]$. Then there is a positive integer k_0 such that for each $k \ge k_0$, the system (2.2) has a unique ω-periodic solution x_k and the equality (2.7) is satisfied.

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Ivan Kiguradze
Razmadze Mathem. Institute
Georgian Academy of Sciences
Rukhadze St. 1
380 093 Tbilisi, REPUBLIC OF GEORGIA

Bedřich Půža
Department of Mathem. Analysis
Faculty of Science Masaryk University
Janáčkovo nám. 2a
662 95 Brno, CZECH REPUBLIC
E-mail: puza@math.muni.cz