ARCHIVUM MATHEMATICUM (BRNO)
Tomus 33 (1997), 245 – 251

A COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPPINGS ON A NORMED VECTOR SPACE

H. K. PATHAK AND BRIAN FISHER

Abstract. A common fixed theorem is proved for two pairs of compatible mappings on a normed vector space.

1. Results on common fixed points

The following definition was given by Jungck [1].

Definition. Let S and I be mappings of a metric space (X, d) into itself. Then S and I are said to be *compatible* if

$$\lim_{n\to\infty} d(SIx_n, ISx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ix_n = z$ for some z in X.

Jungck also proved the following proposition.

Proposition. *Let S and I be mappings of a metric space (X, d) into itself. If S and I are compatible mappings and $Sz = Iz$ for some z in X, then $SIz = ISz$.*

We now prove a theorem for two pairs of compatible mappings on a normed vector space.

Theorem 1. *Let S, I and T, J be two pairs of compatible mappings of a normed vector space X into itself, let C be a closed, convex subset of X such that*

$$(1-k)I(C) + kS(C) \subseteq I(C)\,, \tag{1}$$
$$(1-k')J(C) + k'T(C) \subseteq J(C)\,, \tag{2}$$

1991 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. normed vector space, compatible mappings and common fixed points.
Received July 4, 1995

where $0 < k, k' < 1$ *and suppose that*

(3) $\quad ||Sx - Ty||^p \leq$

$$\leq \Phi\left(\frac{a||Ix - Jy||^{2p} + (1-a)\max\{||Sx - Ix||^{2p}, ||Ty - Jy||^{2p}\}}{\max\{||Sx - Jy||^p, ||Ty - Ix||^p\}}\right),$$

for all $x, y \in C$ *for which* $\max\{||Sx - Jy||, ||Ty - Ix||\} \neq 0$, *where* $0 < a < 1$, $p > 0$ *and* Φ *is a function which is upper semi–continuous from the right of* R^+ *into itself such that* $\Phi(t) < t$ *for each* $t > 0$. *If for some* $x_0 \in C$, *the sequence* $\{x_n\}$ *in* X *defined inductively for* $n = 0, 1, 2, \ldots$ *by*

$$Ix_{2n+1} = (1-k)Ix_{2n} + kSx_{2n}\,, \tag{4}$$

$$Jx_{2n+2} = (1-k')Jx_{2n+1} + k'Tx_{2n+1} \tag{5}$$

converges to a point $z \in C$, *and if* I *and* J *are continuous at* z, *then* S, T, I *and* J *have the unique common fixed point* Tz *in* C. *Further, if* I *and* J *are continuous at* Tz, *then* S *and* T *are continuous at* Tz.

Proof. We will first of all prove that

$$Sz = Tz = Iz = Jz\,. \tag{6}$$

It follows from (4) that

$$kSx_{2n} = Ix_{2n+1} - (1-k)Ix_{2n}$$

and since I is continuous at z,

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Sx_{2n} = Iz\,.$$

Similarly,

$$\lim_{n\to\infty} Jx_n = \lim_{n\to\infty} Tx_{2n+1} = Jz\,.$$

Now suppose that $Iz \neq Jz$ so that for large enough n, $Sx_{2n} \neq Jx_{2n+1}$. Then using (3) we have

$$||Sx_{2n} - Tx_{2n+1}||^p \leq$$

$$\Phi\left(\frac{a||Ix_{2n} - Jx_{2n+1}||^{2p} + (1-a)\max\{||Sx_{2n} - Ix_{2n}||^{2p}, ||Tx_{2n+1} - Jx_{2n+1}||^{2p}\}}{\max\{||Sx_{2n} - Jx_{2n+1}||^p, ||Tx_{2n+1} - Ix_{2n}||^p\}}\right).$$

Letting n tend to infinity, it follows that

$$||Iz - Jz||^p \leq \Phi(a||Iz - Jz||^p) < a||Iz - Jz||^p,$$

a contradiction since $a < 1$. Thus $Iz = Jz$.

Now suppose that $Tz \neq Iz$ so that for large enough n, $Tz \neq Ix_{2n}$. Then using (3) again we have

$$||Sx_{2n} - Tz||^p \leq$$

$$\leq \Phi\left(\frac{a||Ix_{2n} - Jz||^{2p} + (1-a)\max\{||Sx_{2n} - Ix_{2n}||^{2p}, ||Tz - Jz||^{2p}\}}{\max\{||Sx_{2n} - Jz||^p, ||Tz - Ix_{2n}||^p\}}\right).$$

Letting n tend to infinity, it follows that

$$\begin{aligned} \|Iz - Tz\|^p &\le \Phi\left(\frac{a\|Iz - Jz\|^{2p} + (1-a)\|Tz - Jz\|^{2p}}{\max\{\|Iz - Jz\|^p, \|Tz - Iz\|^p\}}\right) \\ &= \Phi\left(\frac{(1-a)\|Tz - Jz\|^{2p}}{\|Tz - Iz\|^p}\right) < \frac{(1-a)\|Tz - Jz\|^{2p}}{\|Tz - Iz\|^p}, \end{aligned}$$

a contradiction. Thus $Tz = Iz$.

We can prove similarly that $Sz = Jz$, completing the proof of equations (6).

Now suppose that $S^2z \neq Tz$. Then using (3) again, the Proposition and equations (6), we have

$$\begin{aligned} &\|S^2z - Tz\|^p \le \\ &\le \Phi\left(\frac{a\|ISz - Jz\|^{2p} + (1-a)\max\{\|S^2z - ISz\|^{2p}, \|Tz - Jz\|^{2p}\}}{\max\{\|S^2z - Jz\|^p, \|Tz - ISz\|^p\}}\right) = \\ &= \Phi(a\|S^2z - Tz\|^p) < a\|S^2z - Tz\|^p, \end{aligned}$$

a contradiction since $a < 1$. Thus $S^2z = Tz$.

Using the Proposition and equations (6) we now have

$$S^2z = S(Tz) = SIz = ISz = I(Tz) = Tz$$

and so Tz is a fixed point of S and I. We can prove similarly that

$$T^2z = T(Sz) = TJz = JTz = J(Sz) = Sz$$

and so $Sz = Tz = w$ is also a fixed point of T and J.

Now let $\{y_n\}$ be an arbitrary sequence in C with the limit w and suppose that the sequence $\{Sy_n\}$ does not converge to Sw. Then for large enough n and using (3) we have

$$\begin{aligned} &\|Sy_n - Sw\|^p = \|Sy_n - Tw\|^p \le \\ &\le \Phi\left(\frac{a\|Iy_n - Tw\|^{2p} + (1-a)\max\{\|Sy_n - Iy_n\|^{2p}, \|Tw - Jw\|^{2p}\}}{\max\{\|Sy_n - Jw\|^p, \|Tw - Iy_n\|^p\}}\right). \end{aligned}$$

Since I and J are continuous at w, it follows that for arbitrary $\epsilon > 0$ and sufficiently large n

$$\|Sy_n - Sw\|^p \le \Phi\big((1-a)\|Sy_n - Sw\|^p + \epsilon\big) < (1-a)\|Sy_n - Sw\|^p + \epsilon,$$

a contradiction since $a < 1$. Thus the sequence $\{Sy_n\}$ must converge to Sw, proving the continuity of S at w. We can prove similarly that T is also continuous at w.

The uniqueness of the common fixed point follows easily on using inequality (3). This completes the proof of the theorem.

When $S = T$ and $I = J$ we have the following corollary:

Corollary 1. *Let T and I be two compatible mappings of a normed vector space X into itself, let C be a closed, convex subset of X such that*

$$(1-k)I(C) + kT(C) \subseteq I(C)$$

where $0 < k, < 1$ and suppose that

$$\|Tx - Ty\|^p \leq \Phi\left(\frac{a\|Ix - Iy\|^{2p} + (1-a)\max\{\|Tx - Ix\|^{2p}, \|Ty - Iy\|^{2p}\}}{\max\{\|Tx - Iy\|^p, \|Ty - Ix\|^p\}}\right),$$

for all $x, y \in C$ for which $\max\{\|Tx - Iy\|, \|Ty - Ix\|\} \neq 0$, where $0 < a < 1$, $p > 0$ and Φ is a function which is upper semi-continuous from the right of R^+ into itself such that $\Phi(t) < t$ for each $t > 0$. If for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined inductively for $n = 0, 1, 2, \ldots$ by

$$Ix_{n+1} = (1-k)Ix_n + kTx_n$$

converges to a point $z \in C$, and if I is continuous at z, then T and I have the unique common fixed point Tz in C. Further, if I is continuous at Tz, then T is continuous at Tz.

When $I = J = I_X$, the identity mapping on X, we have the following corollary:

Corollary 2. *Let S and T be two mappings of a normed vector space X into itself, let C be a closed, convex subset of X such that*

$$(1-k)C + kS(C) \subseteq C, \tag{7}$$

$$(1-k')C + k'T(C) \subseteq C, \tag{8}$$

where $0 < k, k' < 1$ and suppose that

$$\|Sx - Ty\|^p \leq \Phi\left(\frac{a\|x - y\|^{2p} + (1-a)\max\{\|Sx - x\|^{2p}, \|Ty - y\|^{2p}\}}{\max\{\|Sx - y\|^p, \|Ty - x\|^p\}}\right), \tag{9}$$

for all $x, y \in C$ for which $\max\{\|Sx - y\|, \|Ty - x\|\} \neq 0$, where $0 < a < 1$, $p > 0$ and Φ is a function which is upper semi-continuous from the right of R^+ into itself such that $\Phi(t) < t$ for each $t > 0$. If for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined inductively for $n = 0, 1, 2, \ldots$ by

$$x_{2n+1} = (1-k)x_{2n} + kSx_{2n}, \tag{10}$$

$$x_{2n+2} = (1-k')x_{2n+1} + k'Tx_{2n+1} \tag{11}$$

converges to a point $z \in C$, then S and T have the unique common fixed point Tz in C. Further, S and T are continuous at Tz.

When $I = J = I_X$ the identity mapping on X and $\phi(t) = \alpha t$, for all $t > 0$ and $0 < \alpha < 1$, we have the following corollary:

Corollary 3. *Let S and T be two mappings of a normed vector space X into itself, let C be a closed, convex subset of X satisfying the inclusions (7) and (8) and suppose that*

$$(12)\qquad \|Sx - Ty\|^p \le \alpha \frac{a\|x-y\|^{2p} + (1-a)\max\{\|Sx-x\|^{2p}, \|Ty-y\|^{2p}\}}{\max\{\|Sx-y\|^p, \|Ty-x\|^p\}},$$

for all $x, y \in C$ for which $\max\{\|Sx-y\|, \|Ty-x\|\} \ne 0$, where $0 < a, \alpha < 1$ and $p > 0$. If for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined by (10) and (11) converges to a point $z \in C$, then S and T have the unique common fixed point Tz in C. Further, S and T are continuous at Tz.

The following example shows the validity of Theorem 1.

Example 1. Let $X = [0, \infty)$ with the Euclidean norm and let $C = [0, 1]$. Define the mappings I, J, S and T of X into itself by

$$Ix = \begin{cases} 1 & \text{if} \quad x \in [0, \frac{1}{2}), \\ x & \text{if} \quad x \in [\frac{1}{2}, \infty), \end{cases} \qquad Sx = \begin{cases} 1 & \text{if} \quad x \in [0, 1], \\ 1 + x^2 & \text{if} \quad x \in (1, \infty), \end{cases}$$

$$Jx = \begin{cases} 1 & \text{if} \quad 1 \in [0, \frac{1}{2}), \\ x^2 & \text{if} \quad x \in [\frac{1}{2}, \infty), \end{cases} \qquad Tx = \begin{cases} 1 & \text{if} \quad x \in [0, 1], \\ 1 + x^3 & \text{if} \quad x \in (1, \infty). \end{cases}$$

Then I and J are not continuous at $\frac{1}{2}$ and S and T are not continuous at 1. Consider a sequence $\{x_n\}$ converging to 0. Then

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Sx_n \quad \text{and} \quad \lim_{n\to\infty} \|ISx_n - SIx_n\| = 0,$$

since

$$\lim_{n\to\infty} ISx_n = \lim_{n\to\infty} SIx_n = 1.$$

Thus I and S are compatible mappings. Similarly, J and T are compatible mappings. Moreover, J is not linear in C and

$$\|Jx - Jy\| = \|x^2 - y^2\| = (x + y)\|x - y\| > \|x - y\|$$

for all $x, y \in (\frac{1}{2}, 1]$. Therefore, J is not non–expansive in C. For fixed $k, k' \in (0, 1)$, we have

$$\begin{aligned} (1-k)I(C) + kS(C) &= [\tfrac{1}{2} + \tfrac{1}{2}k, 1] \subseteq I(C) = [\tfrac{1}{2}, 1], \\ (1-k')J(C) + k'T(C) &= [\tfrac{1}{4} + \tfrac{3}{4}k', 1] \subseteq J(C) = [\tfrac{1}{4}, 1] \end{aligned}$$

and

$$\|Sx - Ty\|^p = 0$$

for all $x, y \in C$ and $p > 0$. Also, for any $x_0 \in C$, we can show that the sequence $\{x_n\}$ in C such that

$$\begin{aligned} Ix_{2n+1} &= (1-k)Ix_{2n} + kSx_{2n}, \\ Jx_{2n+2} &= (1-k')Jx_{2n+1} + k'Tx_{2n+1}, \end{aligned}$$

for $n = 0, 1, 2, \ldots$ converges to the point 1. Clearly, $T1$ is a common fixed point of I, J, S and T.

The condition that I and T be compatible mappings is necessary in Corollary 1 is shown by the following example.

Example 2. Let $X = [0,\infty)$ with the Euclidean norm and let $C = [0,1]$. Define the mappings I and T of X into itself by

$$Ix = \begin{cases} 1 + \frac{1}{2}x & \text{if} \quad x \in [0,1]\,, \\ 1 & \text{if} \quad x \in (1,\infty)\,, \end{cases} \qquad Tx = 1\,.$$

Then we see that $\|Tx - Ty\|^p = 0$ for all $x, y \in C$ with $p > 0$. For some $k \in (0,1)$, we also have

$$(1-k)I(C) + kT(C) = [1, \tfrac{3}{2} - \tfrac{1}{2}k] \subset I(C) = [1, \tfrac{3}{2}]\,.$$

Further, if $\{x_n\}$ is a sequence in X converging to 0, then

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = 1$$

but

$$\lim_{n\to\infty} \|ITx_n - TIx_n\| = \tfrac{1}{2} \neq 0$$

and so I and T are not compatible mappings. On the other hand, I and T have no common fixed point in C.

2. An application to a product space

We now apply Corollary 3 to establish the following result.

Theorem 2. *Let C be a closed, convex subset of a normed vector space X, let P and Q be two mappings of $X \times X$ into X such that*

$$(1-k)C + kP(C \times C) \subseteq C\,, \tag{13}$$

$$(1-k')C + k'Q(C \times C) \subseteq C\,, \tag{14}$$

where $0 < k, k' < 1$ and suppose that

$$\|P(x,y) - Q(u,v)\|^p \leq \tag{15}$$

$$\leq \alpha\Big[\|y-v\|^p + \frac{a\|x-u\|^{2p} + (1-a)\max\{\|P(x,y)-x\|^{2p}, \|Q(u,v)-u\|^{2p}\}}{\max\{\|P(x,y)-u\|^p, \|Q(u,v)-x\|^p\}}\Big]$$

for all $x, y, u, v \in C$ for which $\max\{\|P(x,y)-u\|, \|Q(u,v)-x\|\} \neq 0$, where $0 < a < 1$, $0 < \alpha < (1+a)^{-1}$ and $p > 0$. If for each fixed $y \in C$ and some $x_0(y) \in C$, the sequence $\{x_n(y)\}$ in X defined inductively for $n = 0, 1, 2, \ldots$ by

$$x_{2n+1}(y) = (1-k)x_{2n}(y) + kP(x_{2n}(y), y)\,, \tag{16}$$

$$x_{2n+2}(y) = (1-k')x_{2n+1}(y) + k'Q(x_{2n+1}(y), y) \tag{17}$$

converges to a point $z \in C$, then there exists a unique point $w \in C$ such that

$$P(w,w) = w = Q(w,w)\,.$$

Proof. It follows from inequality (15) that

$$\|P(x,y)-Q(u,y)\|^p \leq$$
$$\leq \alpha \frac{a\|x-u\|^{2p}+(1-a)\max\{\|P(x,y)-x\|^{2p},\|Q(u,y)-u\|^{2p}\}}{\max\{\|P(x,y)-u\|^p,\|Q(u,y)-x\|^p\}},$$

for all $x, y, u \in C$. Therefore, by Corollary 3, for each $y \in C$, there exists a unique $z(y) \in C$ such that

$$P(z(y),y) = z(y) = Q(z(y),y)\,. \tag{18}$$

Now for any $y, y' \in C$, we obtain from (15)

$$\|P(z(y),y)-Q(z(y'),y')\|^p \leq \alpha\Big[\|y-y'\|^p +$$
$$\frac{a\|z(y)-z(y')\|^{2p}+(1-a)\max\{\|P(z(y),y)-z(y)\|^{2p},\|Q(z(y'),y')-z(y')\|^{2p}\}}{\max\{\|P(z(y),y)-z(y')\|^p,\|Q(z(y'),y')-z(y)\|^p\}}\Big]$$
$$=\alpha(\|y-y'\|^p + a\|z(y)-z(y')\|^p)$$

and so

$$\|z(y)-z(y')\| \leq [\alpha/(1-\alpha a)]^{1/p}\|y-y'\|\,.$$

Since $\alpha/(1-\alpha a) < 1$, it follows from the celebrated Banach contraction principle that the mapping $z(.)$ of C into itself has a unique fixed point $w \in C$, i.e. $z(w) = w$, which by (18) implies that

$$w = z(w) = P(w,w) = Q(w,w)\,.$$

It is not hard to prove that P and Q can only have one such point $w \in C$. This completes the proof of the theorem.

References

1. Jungck, G., *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci., **9**(1986), 771–779.

H. K. Pathak
Department of Mathematics, Kalyan
Mahavidyalaya
Bhilai Nagar, 490 006 (MP), INDIA

B. Fisher
Department of Mathematics and Computer Science
University of Leicester
Leicester, LE1 7RH ENGLAND