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PERIODIC BOUNDARY VALUE PROBLEM OF A FOURTH ORDER DIFFERENTIAL INCLUSION

Marko Švec

Dedicated to the memory of Professor Otakar Borůvka

Abstract. The paper deals with the periodic boundary value problem (1) $L_4x(t)+a(t)x(t) \in F(t,x(t))$, $t \in J = [a,b]$, (2) $L_ix(a) = L_ix(b)$, $i = 0,1,2,3$, where $L_0x(t) = a_0x(t)$, $L_ix(t) = a_i(t)L_{i-1}x(t)$, $i = 1,2,3,4$, $a_0(t) = a_4(t) = 1$, $a_i(t)$, $i = 1,2,3$ and $a(t)$ are continuous on J, $a(t) \geq 0$, $a_i(t) > 0$, $i = 1,2$, $a_1(t) = a_3(t) \cdot F(t,x) : J \times R \to$ {nonempty convex compact subsets of R}, $R = (-\infty,\infty)$. The existence of such periodic solution is proven via Ky Fan's fixed point theorem.

In this paper we will discuss the periodic boundary value problem

$$L_4x(t) + a(t)x(t) \in F\left(t, x(t)\right), \ t \in [a,b] \tag{1}$$

$$L_i(x(a) = L_i(x(b), \ i = 0,1,2,3 \tag{2}$$

where

$$L_0x(t) = a_o(t)x(t), \ L_ix(t) = a_i(t)\left(L_{i-1}x(t)\right)', \ i = 1,2,3$$

$$L_4x(t) = \left(a_1(t)\left(a_2(t)\left(a_1(t)\left(a_0(t)x(t)\right)'\right)'\right)'\right)' \tag{3}$$

$a_0(t) = 1, \ a(t) \geq 0, \ a_i(t) > 0, \ i = 1,2, \ a_1(t) = a_3(t)$, *continuous on* $[a,b] = J$.
$F(t,x) : J \times R \to \{$*nonempty convex compact subsets of* $R\}$, $R = (-\infty,\infty)$.

(4) If $B \subset R$ then $\mid B \mid = sup\{\mid x \mid : \ x \in B\}$ and if D is a set, then cf (D) is the set of all convex closed subsets of D.

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The basic assumptions concerning $F(t,x)$ are as follows:
1^o $F(t,x)$ is upper semicontinuous on $J \times R$;
2^o To each measurable function $z(t) : J \to R$ there exists a measurable selector $v(t) : J \to R$ such that $v(t) \in F(t, z(t))$ a. e. on J. Denote $Mz(t) = \{$the set of all measurable selectors belonging to $z(t)\}$.

We start with two theorems. The proofs of these theorems are very easy, therefore, we shall not present them.

Theorem 1. *Let $y(t)$ be a nontrivial solution of the equation*

$$L_4 y(t) + a(t)y(t) = 0, \ t \in [a,b] \tag{5}$$

Then the function

$$F(y(t)) = L_0 y(t) L_3 y(t) - L_1 y(t) L_2 y(t) \tag{6}$$

is strictly decreasing on $[a,b]$. If t_0 is at least a double zero of $y(t)$, then $F(y(t)) > 0$ for $t \in [a,t_0)$ and $F(y(t)) < 0$ for $t \in (t_0, b]$. Thus every nontrivial solution $y(t)$ of (5) has at most one double zero on $[a,b]$.

From this follows

Theorem 2. *The boundary value problem*

$$L_4 y(t) + a(t)y(t) = 0 \tag{7}$$

$$L_i y(a) = L_i y(b), \ i = 0,1,2,3 \tag{8}$$

has only trivial solution $y(t) \equiv 0, \ t \in [a,b]$.

Theorem 3. *Let the assumptions 1^o and 2^o be satisfied. Moreover, let exist a continuous function $H(t) > 0, \ t \in [a,b]$ such that*

$$| F(t,z) | \leq H(t) \ \text{for each} \ (t,z) \in [a,b] \times R \tag{9}$$

Then the problem (1), (2) has a solution.

Proof. Let $C[a,b]$ be the space of all continuous functions defined on $[a,b]$ with the supremum norm, i. e. for $u(t) \in C[a,b]$ it is $\|u(t)\| = sup\{| u(t) |, t \in [a,b]\}$. Let be $Y = \{u(t) \in C[a,b], \ L_i u(a) = L_i u(b), \ i = 0,1,2,3\}$. Then to $u(t) \in Y$ belongs the set of all measurable selectors $Mu(t)$. Let be $v(t) \in Mu(t)$. Evidently, $v(t) \in F(t, u(t))$. Then we seek a solution $x(t)$ of the problem

$$L_4 x(t) + a(t)x(t) = v(t) \tag{10}$$

$$L_i x(a) = L_i x(b), \ i = 0,1,2,3 \tag{11}$$

The problem (7), (8) has only trivial solution (see Theorem 2); therefore, there exists the Green function $G(t,s)$ for the problem (7, 8) and

$$x(t) = \int_a^b G(t,s)v(s)\,ds \tag{12}$$

is a solution of the problem (10), (11). Thus we have the multivalued operator A defined on the set Y as follows: for $u(t) \in Y$ it is

$$Au(t) = \{x(t) = \int_a^b G(t,s)v(s)\,ds,\ v(t) \in Mu(t)\} \tag{13}$$

Evidently $Au(t) \subset Y$ and $Au(t)$ is nonempty and it is easy to see that $Au(t)$ is convex.

We will prove that: $A : Y \to cf(Y)$; A is upper semicontinuous on Y; AY is compact.

Let be $u(t) \in Y,\ \zeta(t) \in Au(t)$. Then

$$\zeta(t) = \int_a^b G(t,s)v(s)\,ds,\ v(t) \in Mu(t),\ v(t) \in F(t,u(t))$$

and

$$\mid v(t) \mid \leq \mid F(t,u(t)) \mid \leq H(t)\ respecting\ (9)$$

and

$$\mid \zeta(t) \mid \leq \int_a^b \mid G(t,s) \mid H(s)\,ds \ \leq\ \|G(t,s)\| \int_a^b H(s)\,ds \ =\ K$$

where $\|G(t,s)\| = max_{[a,b]\times[a,b]} \mid G(t,s) \mid$ on $[a,b]\ \times\ [a,b]$. Furthermore,

$$\mid \zeta'(t) \mid \leq \int_a^b \mid G'_t(t,s) \mid \mid v(s) \mid\ ds \ \leq\ \int_a^b \|G'_t(t,s)\| H(s)\,ds \ =\ K_1$$

where $\|G'(t,s)\| = \max_{[a,b]\times[a,b]} \mid G'_t(t,s)$. We note that $G(t,s)$ and $G'_t(t,s)$ are continuous on $[a,b]\ \times\ [a,b]$. Thus we have that the elements $\zeta(t) \in Au(t)$ as well as the elements $\zeta(t) \in AY$ are uniformly bounded and equicontinuous on $[a,b]$. Therefore, the sets $Au(t), u(t) \in Y$ as well as the set AY are compact in the topology of $C[a,b]$. Thus it is easy to see that $Au(t) \in cf(Y)$.

Let $u_i(t) \in Y,\ i = 1,2,\ldots$ and let the sequence $\{u_i(t)\}$ converge to $u(t)$ in $C[a,b]$. Furthermore, let $z_i(t) \in Au_i(t) \subset AY$. The set AY being compact, there exists a subsequence $\{z_{i_j}(t)\}$ of $\{z_i(t)\}$ which converges to a function $z(t) \in AY$ in the topology of $C[a,b]$. We have

$$z_i(t) = \int_a^b G(t,s)v_i(s)\,ds, \;\; v_i(s) \in Mu_i(t), \;\; t \in [a,b]$$

Respecting (9) we get

$$|\, z_i(t) \,| \le \|G(t,s)\| \int_a^b H(s)\,ds = K$$

Denote by $L_1[a,b]$ the set of all measurable functions f defined on $[a,b]$ such that

$$\|f\|_1 = \int_a^b \|G(t,s)\| \;\; |\, f(s) \,| \;\; ds \;\; < \infty$$

We see that the sequence $\{v_i(t)\}$ is bounded in the space $L_1\,[a,b]$. Let $\{E_m\}$, $E_m \subset [a,b]$ be a decreasing sequence of the sets such that $\bigcap_{m=1}^{\infty} E_m = \emptyset$. Then we have

$$\lim_{m\to\infty} |\int_{E_m} \|G(t,s)\| v_i(s)\,ds \,| \le \lim_{m\to\infty} \|G(t,s)\| \int_{E_m} |\, v_i(s) \,| \; ds$$
$$\le \|G(t,s)\| lim_{m\to\infty} \int_{E_m} H(s)\,ds = 0$$

Therefore, (see [1], Th. IV.8.9) it is possible to choose a subsequence $\{v_{i_j}\}$ of $\{v_i(t)\}$ which weakly converges to some $v(t) \in L_1[a,b]$.

Evidently $\{u_{i_j}\}$ converges to $u(t)$ in $C[a,b]$. For $v_{i_j} \in F(t, u_{i_j}(t), j = 1,2,...$ using the assumption 1^o, to a given $\epsilon > 0$ and $t \in [a,b]$ there exists $N = N(t,\epsilon)$ such that for any $i_j \ge N$ we have $F(t, u_{i_j}(t)) \subset O_\epsilon(F(t,u(t))$, where $O_\epsilon(F(t,u(t))$ is the ϵ-neighbourhood of the set $F(t,u(t))$.

Consider now the sequence $\{v_{i_j}(t)\}$, $i_j \ge N$. Then (see [1], Corollary V.3.14) it is possible to construct such convex combinations from v_{i_j}, $i_j \ge N$, denoted by $g_m(t)$, $m = 1,2,...$ that the sequence $\{g_m(t)\}$ converges to $v(t)$ in $L_1[a,b]$. By Riesz theorem we get the existence of a subsequence $\{g_{m_i}(t)\}$ of $\{g_m(t)\}$ which converges to $v(t)$ a. e. on $[a,b]$. From the convexity of $O_\epsilon(F(t,u(t))$ and from the fact that $v_{i_j} \in O_\epsilon(F(t,u(t))$ it follows that $g_{m_i}(t) \in O_\epsilon(F(t,u(t))$, $i = 1,2,...$ and consequently $v(t) \in \bar{O}_\epsilon(F(t,u(t))$. For $\epsilon \to 0$ we get $v(t) \in F(t,u(t))$. We note that in our considerations t was a fixed point of $[a,b]$.

Thus we have that the function

$$z(t) = \int_a^b G(t,s)v(s)\,ds$$

is well defined and $z(t) \in Au(t)$, $t \in [a,b]$. Furthermore, it follows from the weak convergence of $v_{i_j}(t)$ to $v(t)$ in $L_1[a,b]$ that the subsequence $\{z_{i_j}(t)\}$ of $\{z_i(t)\}$ converges to $z(t)$ a. e. on $[a,b]$. The functions $z_{i_j}(t)$ belong to the compact set

AY. Therefore, there exists a subsequence of the sequence $\{z_{i_j}(t)\}$ which converges to a function $\bar{z}(t)$ in the topology of $C[a,b]$. This means that $\bar{z}(t) = z(t) \in Au(t)$ a. e. on $[a,b]$. This finishes the proof of the upper semicontinuity of the operator A on Y.

Using the similar considerations as in the proof of the upper semicontinuity of A on Y, made for the case that $z_i(t) \in Au(t)$ and $\{z_i(t)\}$ converges to $z(t)$ in $C[a,b]$ gives us that $z(t) \in Au(t)$. It means that $Au(t)$ is closed. Thus we have proven that $Au(t)$ is compact and A maps Y into $cf(Y)$.

Finally, the use of Ky Fan's theorem finishes the proof that A has a fixed point in Y, i. e. there exists $u(t) \in Y$ such that $u(t) \in Au(t)$.

References

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Department of Mathematical Analysis
Comenius University
Mlynská dolina
842 15 Bratislava, SLOVAKIA