ARCHIVUM MATHEMATICUM (BRNO)
Tomus 33 (1997), 157 – 165

SOME CLASSES OF LINEAR N–TH ORDER DIFFERENTIAL EQUATIONS

Valter Šeda

Dedicated to the memory of Professor Otakar Borůvka

Abstract. Sufficient conditions for the n-th order linear differential equation are derived which guarantee that its Cauchy function K, together with its derivatives $\frac{\partial^i K}{\partial t^i}$, $i = 1, \dots, n-1$, is of constant sign. These conditions determine four classes of the linear differential equations. Further properties of these classes are investigated.

1. Introduction

In the last fourty years the theory of ordinary linear differential equations has been intensively developed. O. Borůvka began the systematic study of global properties of the second order linear differential equations. He summarized his results in the monograph [1]. The results on higher order equations have been brought in monographs written by several authors, among them by M. Greguš [5], I. T. Kiguradze–T. A. Čanturija [7], F. Neuman [9]. The results from that theory are often used to solve the problems in nonlinear differential equations, see [12].

Consider the n-th order ($n \geq 1$) linear differential equation

$$(1) \qquad (L(y) \equiv)\ y^{(n)} + \sum_{k=1}^{n} p_k(t) y^{(n-k)} = 0,$$

where the coefficients $p_k \in C(I, R)$, $k = 1, \dots, n$, and $I = [a, \infty)$, $-\infty < a < \infty$. The sign of the Cauchy function for (1) $K = K(t,s)$, $t, s \in I$, plays an important role in the Čaplygin comparison theorem [10], p. 99. If (1) is disconjugate in I, then $K(t,s)$ has a constant sign for $t > s$ as well as for $t < s$, $t, s \in I$. In the paper sufficient conditions will be given in order that not only K, but also $\frac{\partial^i K}{\partial t^i}$, $i = 1, \dots, n-1$, be of constant sign in both mentioned cases. The considerations from [13] will be extended here to the full equation (1). Throughout the paper, only real functions will be dealt with.

1991 *Mathematics Subject Classification*: 34A30, 34A40, 34D05.

Key words and phrases: Cauchy function, Čaplygin comparison theorem, monotonic solutions, regularity of bands.

Sometimes the following assumptions will be required:

$$(H1)\quad p_1(s)+\sum_{k=2}^{n}\int_s^t p_k(u)\frac{(u-s)^{k-2}}{(k-2)!}du\le 0\quad \text{for all } s,t\in I,\ s\le t;$$

$$(H2)\quad p_1(s)+\sum_{k=2}^{n}\int_s^t p_k(u)\frac{(u-s)^{k-2}}{(k-2)!}du\ge 0\quad \text{for all } s,t\in I,\ s\le t,$$

$$\int_a^t[p_1(s)+\sum_{k=2}^{n}\int_s^t p_k(u)\frac{(u-s)^{k-2}}{(k-2)!}du]\,ds<1\quad \text{for all } t\in I;$$

$$(H3)\quad -p_1(s)+\sum_{k=2}^{n}\int_t^s p_k(u)\frac{(u-s)^{k-2}}{(k-2)!}du\le 0\quad \text{for all } s,t\in I,\ t\le s;$$

$$(H4)\quad -p_1(s)+\sum_{k=2}^{n}\int_t^s p_k(u)\frac{(u-s)^{k-2}}{(k-2)!}du\ge 0\quad \text{for all } s,t\in I,\ t\le s,$$

$$\int_t^T[-p_1(s)+\sum_{k=2}^{n}\int_t^s p_k(u)\frac{(u-s)^{k-2}}{(k-2)!}du]\,ds<1\quad \text{for all } t,T\in I,\ t<T.$$

Remark 1. If $p_1(t)\equiv 0$ in I, then hypothesis (A) in [13], p. 350, implies $(H1)$ and $(H3)$.

Remark 2. If all $p_k(t)\le 0$, $t\in I$, $k=1,\dots,n$ (all $p_k(t)\ge 0$, $t\in I$, $k=1,\dots,n$, $\sum_{k=1}^n\int_a^t p_k(u)\frac{(u-a)^{k-1}}{(k-1)!}du<1$ for all $t\ge a$), then assumption $(H1)$ ($(H2)$) is satisfied. Similarly, if $(-1)^k p_k(t)\le 0$ for all $t\in I$, $k=1,\dots,n$ ($(-1)^k p_k(t)\ge 0$ for all $t\in I$, $k=1,\dots,n$, $\sum_{k=1}^n\int_t^T(-1)^k p_k(u)\frac{(T-u)^{k-1}}{(k-1)!}du<1$ for all $t,T\in I$, $t<T$), then assumption $(H3)$ ($(H4)$) is satisfied.

We shall also use the assumptions:

$(H1')$ All $p_k(t)\le 0$, $t\in I$, $k=1,\dots,n$;

$(H2')$ All $p_k(t)\ge 0$, $t\in I$, $k=1,\dots,n$,

$$\sum_{k=1}^{n}\int_a^t p_k(u)\frac{(u-a)^{k-1}}{(k-1)!}du<1 \text{ for all } t\ge a;$$

$(H3')$ The functions $(-1)^k p_k(t)\le 0$, $t\in I$, $k=1,\dots,n$;

$(H4')$ The functions $(-1)^k p_k(t)\ge 0$, $t\in I$, $k=1,\dots,n$,

$$\sum_{k=1}^{n}\int_t^T(-1)^k p_k(u)\frac{(T-u)^{k-1}}{(k-1)!}du<1 \text{ for all } t,T\in I,\ t<T.$$

2. Preliminaries

The following lemma is a slight modification of Lemma 1 in [13], p. 351.

Lemma 1. *Let $t_0 \in I$, y_0^i, $i = 0, \ldots, n-1$, be arbitrary numbers. Then the initial value problem*

$$L(y) = 0,\ y^{(i)}(t_0) = y_0^i,\ i = 0, \ldots, n-1, \tag{2}$$

is equivalent to the following Volterra's integral equation

$$y^{(n-1)}(t) = g(t) + \int_{t_0}^{t} A(t,s) y^{(n-1)}(s) ds,\ t \in I, \tag{3}$$

where

$$g(t) = y_0^{n-1} - \sum_{j=0}^{n-2} y_0^j \sum_{k=n-j}^{n} \int_{t_0}^{t} p_k(s) \frac{(s-t_0)^{j-n+k}}{(j-n+k)!} ds, \tag{4}$$

$$A(t,s) = -p_1(s) - \sum_{k=2}^{n} \int_{s}^{t} p_k(u) \frac{(u-s)^{k-2}}{(k-2)!} du, \quad t, s \in I. \tag{5}$$

Proof. Integrating equation (1) from t_0 to t and taking initial conditions (2) into consideration, we get

$$\begin{aligned} y^{(n-1)}(t) =& y_0^{n-1} - \int_{t_0}^{t} p_1(s) y^{(n-1)}(s) ds \\ & - \sum_{k=2}^{n} \int_{t_0}^{t} p_k(s) \left(\sum_{l=0}^{k-2} \frac{y_0^{n-k+l}}{l!} (s-t_0)^l + \int_{t_0}^{s} \frac{(s-u)^{k-2}}{(k-2)!} y^{(n-1)}(u) du \right) ds. \end{aligned} \tag{6}$$

Comparing (6) with (5) in [13], p. 351, we obtain that (6) can be put into (3) where g is determined by (4) and A by (5), respectively.

We remind that it suffices to consider continuous solutions of (3).

Let $t_0 \in I$, $g_1 \in C(I)$ and $A_1 \in C(I \times I)$. We shall study the equation

$$x(t) = g_1(t) + \int_{t_0}^{t} A_1(t,s) x(s) ds \tag{7}$$

either on the interval $[t_0, \infty)$ or on the interval $[a, t_0]$ if $a < t_0$. In both cases equation (7) has a unique continuous solution and the method of successive approximations can be applied on each compact subinterval $[b, c]$ of these two intervals in the space $C([b,c])$ provided by the sup-norm ([8], pp. 15, 26). Both spaces $C([t_0, \infty))$ and $C([a, t_0])$ can be partially ordered by natural ordering. Then the linear operator

$$Tx(t) = \int_{t_0}^{t} A_1(t,s) x(s) ds$$

is nondecreasing (nonincreasing) in $(C([t_0,\infty)),\leq)$ if $A_1(t,s)\geq 0$ $(A_1(t,s)\leq 0)$ for $t_0\leq s\leq t<\infty$, while T is nondecreasing (nonincreasing) in $(C([a,t_0]),\leq)$ when $A_1(t,s)\leq 0$ $(A_1(t,s)\geq 0)$ for $a\leq t\leq s\leq t_0$. Choosing the zero approximation $x_0=0$ we get the following lemma which extends Lemma 3 in [11], p. 331 (compare with Lemma 2 in [13], p. 352). Another proof of that lemma is given in [2], [3] and [4].

Lemma 2. *Let the functions $g_1\in C(I)$, $A_1\in C(I\times I)$. Then equation (7) has a unique continuous solution x in I. Moreover, if $g_1(t)\geq 0$ in $[t_0,\infty)$ and*

$$A_1(t,s)\geq 0 \text{ for } t_0\leq s\leq t<\infty \quad (A_1(t,s)\leq 0 \text{ for } t_0\leq s\leq t<\infty \text{ and}$$

$$g_1(t)+\int_{t_0}^{t}A_1(t,s)g_1(s)ds\geq 0 \text{ for } t_0\leq t<\infty), \tag{8}$$

then

$$x(t)\geq g_1(t)\quad (x(t)\geq g_1(t)+\int_{t_0}^{t}A_1(t,s)g_1(s)ds),\ t_0\leq t<\infty. \tag{9}$$

If $a<t_0$, $g_1(t)\leq 0$ in $[a,t_0]$ and

$$A_1(t,s)\leq 0 \text{ for } a\leq t\leq s\leq t_0 \quad (A_1(t,s)\geq 0 \text{ for } a\leq t\leq s\leq t_0 \text{ and}$$

$$g_1(t)+\int_{t_0}^{t}A_1(t,s)g_1(s)ds\leq 0 \text{ for } a\leq t\leq t_0), \tag{10}$$

then

$$x(t)\leq g_1(t)\quad (x(t)\leq g_1(t)+\int_{t_0}^{t}A_1(t,s)g_1(s)ds),\ a\leq t\leq t_0. \tag{11}$$

Now we apply Lemma 2 to integral equation (3) where we take the solution y of equation (1) satisfying the conditions

$$y^{(i)}(t_0)=0,\quad i=0,\dots,n-2 \text{ (if } n\geq 2) \quad y^{(n-1)}(t_0)=y_0^{(n-1)}\neq 0. \tag{12}$$

Lemma 3. *Let y be the solution of (1) satisfying initial conditions (12). Then the following statements are true:*

(i) If $y_0^{n-1}>0$ and (H1) ((H2)) holds, then

$$y^{(n-1)}(t)\geq y_0^{n-1}\quad (y^{(n-1)}(t)\geq y_0^{n-1}(1-\int_{t_0}^{t}[p_1(s)$$

$$+\sum_{k=2}^{n}\int_{s}^{t}p_k(u)\frac{(u-s)^{k-2}}{(k-2)!}du]\,ds),\ t_0\leq t<\infty \tag{13}$$

and

$$y^{(i)}(t) > 0, \quad i = 0, \ldots, n-2 \ (\text{if } n \geq 2),\ t_0 < t < \infty. \tag{14}$$

(ii) If $y_0^{n-1} < 0$ and (H3) ((H4)) holds, then

$$y^{(n-1)}(t) \leq y_0^{n-1} \quad (y^{(n-1)}(t) \leq y_0^{n-1}(1 - \int_t^{t_0} [-p_1(s) + \sum_{k=2}^{n} \int_t^s p_k(u) \frac{(u-s)^{k-2}}{(k-2)!} du]\, ds),\ a \leq t \leq t_0 \tag{15}$$

and

$$(-1)^i y^{(n-i)}(t) > 0,\ a \leq t < t_0,\ i = 2, \ldots, n \ (\text{if } n \geq 2). \tag{16}$$

Proof. In Lemma 2 we put $g_1(t) = y_0^{n-1}$, $t \in I$, $A_1(t,s) = A(t,s)$, $t, s \in I$ and $x(t) = y^{(n-1)}(t)$, $t \in I$. A is determined by (5). If $y_0^{n-1} > 0$, then $(H1)$ $((H2))$ implies (8) and, in case $y_0^{n-1} < 0$, from hypotheses $(H3)$ $((H4))$ we get (10). Then (9) gives (13) and (11) implies (15). In view of (12), inequalities (13) lead to (14) and inequalities (15) to (16).

Remark 3. If y satisfies (12) with $y_0^{n-1} > 0$ and $(H3)$ $((H4))$ holds, then by Lemma 3 we have

$$y^{(n-1)}(t) \geq y_0^{n-1} \quad (y^{(n-1)}(t) \geq y_0^{n-1}(1 - \int_t^{t_0} [-p_1(s) + \sum_{k=2}^{n} \int_t^s p_k(u) \frac{(u-s)^{k-2}}{(k-2)!} du]\, ds),\ a \leq t \leq t_0, \tag{17}$$

and

$$(-1)^{i+1} y^{(n-i)}(t) > 0, \quad a \leq t < t_0,\ i = 2, \ldots, n \ (\text{if } n \geq 2). \tag{18}$$

Under condition $(H1')$ or condition $(H3')$ stronger results can be proved.

Lemma 3'. *Suppose that $(H1')$ holds and let y be the solution of (1) satisfying at $t_0 \in I$ the initial conditions*

$$y^{(i)}(t_0) = y_0^i \geq 0, \quad i = 0, \ldots, n-1.$$

Then

$$y^{(i)}(t) \geq y_0^i \quad \text{for all } t \geq t_0,\ i = 0, \ldots, n-1. \tag{19}$$

Moreover, if p_n is not identically zero in any subinterval of I and y is a nontrivial solution of (1), then

$$y^{(i)}(t) > y_0^i \quad \text{for all } t > t_0,\ i = 0, \ldots, n-1. \tag{20}$$

Proof. Since $g(t) \geq 0$ for all $t \geq t_0$ and $A(t,s) \geq 0$ for $t_0 \leq s \leq t$, by Lemma 2 we have $y^{(n-1)}(t) \geq g(t) \geq y_0^{(n-1)}$, $t \geq t_0$. Hence (19) is true. In view of (1), the second statement easily follows.

The following lemma can be similarly proved.

Lemma 3". *Suppose that* $(H3')$ *holds,* $n = 2m+1$ *(*$n = 2m$*) and* $t_0 > a$*. Let* y *be the solution of (1) satisfying the initial conditions*

$$(-1)^i y^{(i)}(t_0) = (-1)^i y_0^i \le 0 \quad ((-1)^i y^{(i)}(t_0) = (-1)^i y_0^i \ge 0), \quad i = 0, \dots, n-1.$$

Then

$$(-1)^i y^{(i)}(t) \le (-1)^i y_0^i \quad ((-1)^i y^{(i)}(t) \ge (-1)^i y_0^i) \quad \text{for all } t,$$
$$a \le t \le t_0,\ i = 0, \dots, n-1. \tag{21}$$

Moreover, if p_n *is not identically zero in any subinterval of* I *and* y *is a nontrivial solution of (1), then*

$$(-1)^i y^{(i)}(t) < (-1)^i y_0^i \quad ((-1)^i y^{(i)}(t) > (-1)^i y_0^i) \quad \text{for all } t,$$
$$a \le t < t_0,\ i = 0, \dots, n-1. \tag{22}$$

Proof. By the initial conditions $(-1)^i y_0^i \le 0$ $((-1)^i y_0^i \ge 0)$ and $(H3')$ it follows that $g(t) = y_0^{n-1} + \sum_{j=0}^{n-2} y_0^j \sum_{k=n-j}^{n} \int_t^{t_0} p_k(s)(-1)^{j-n+k} \frac{(t_0-s)^{j-n+k}}{(j-n+k)!} ds \le 0$ in $[a, t_0]$ and $A(t,s) \le 0$ for $a \le t \le s \le t_0$. Then Lemma 2 implies that $y^{(n-1)}(t) \le g(t) \le y_0^{n-1}$, $a \le t \le t_0$ and (21) is true. The second statement follows from the fact that in view of (1), $y^{(n-1)}$ is strictly increasing in $[a, t_0]$.

3. Main results

Similarly as in [13], pp. 356–358, we can prove theorems on the existence of monotonic solutions.

Theorem 1. *Suppose that* $(H1)$ *or* $(H2)$ *holds. Then there exists a solution* y *of (1) such that*

$$y^{(i)}(t) > 0 \quad \text{for all } t > a,\ i = 0, \dots, n-1. \tag{23}$$

Theorem 2. *Suppose that* $(H3)$ *or* $(H4)$ *holds. Then there exists a solution* z *of (1) such that either*

$$(-1)^i z^{(i)}(t) > 0 \quad \text{for all } t \in I,\ i = 0, \dots, n-2,$$
$$(-1)^{n-1} z^{(n-1)}(t) \ge 0 \quad \text{in } I, \tag{24}$$

or

$$z(t) > 0 \quad \text{for all } t \in I \text{ and there exists a } t_0 \in I \text{ such that}$$
$$z^{(i)}(t) \equiv 0 \quad \text{for all } t \ge t_0,\ i = 1, \dots, n-1.$$

Remark 4. If $p_n(t) \not\equiv 0$ in any neighbourhood of ∞, then in Theorem 2 only the first statement can hold.

Remark 5. Theorem 2 extends the statement of Corollary 2.2 in [6], p. 594.

The fundamental property of equation (1) under assumptions $(H1)$–$(H4)$ is given by the following theorem which can be proved in a similar way as Theorem 3 in [13], pp. 358–359.

Theorem 3. *Let $t_0 \in I$ and let $u(t)$, $v(t) \in C^n(I)$ be two functions such that*

$$\begin{aligned} & u^{(i)}(t_0) = v^{(i)}(t_0), \quad i = 0, \dots, n-1, \quad \text{and} \\ (25) \qquad & L(u)(t) \geq L(v)(t) \quad \text{for all } t \in I. \end{aligned}$$

Then the following statements hold:

(i) If assumptions $(H1)$ or $(H2)$ are satisfied, then

$$u^{(i)}(t) \geq v^{(i)}(t) \quad \text{for all } t \geq t_0,\ i = 0, \dots, n-1.$$

(ii) If assumptions $(H3)$ or $(H4)$ are fulfilled, then

$$(-1)^{n-i} u^{(i)}(t) \geq (-1)^{(n-i)} v^{(i)}(t) \quad \text{for all } t,\ a \leq t \leq t_0,\ i = 0, \dots, n-1.$$

Moreover, if there is a t_1, $t_0 \leq t_1$ in case (i) ($a < t_1 \leq t_0$ in case (ii)) such that $L(u)(t_1) > L(v)(t_1)$, then

$$\begin{aligned} u^{(i)}(t) > v^{(i)}(t) \quad & \text{for all } t > t_1,\ i = 0, \dots, n-1 \\ ((-1)^{n-i} u^{(i)}(t) > (-1)^{n-i} v^{(i)}(t) \quad & \text{for all } t,\ a \leq t < t_1,\ i = 0, \dots, n-1). \end{aligned}$$

The proof of the following theorem is based on Lemma 3' and can be proceeded in the same way as that of Theorem 3' in [13], pp. 359–360.

Theorem 3'. *Suppose that $(H1')$ holds. Let $t_0 \in I$ and let u, $v \in C^n(I)$ be two functions such that*

$$\begin{aligned} & u^{(i)}(t_0) \geq v^{(i)}(t_0), \quad i = 0, \dots, n-1, \quad \text{and} \\ (25') \qquad & L(u)(t) \geq L(v)(t) \quad \text{for all } t \geq t_0. \end{aligned}$$

Then

$$u^{(i)}(t) \geq v^{(i)}(t) \quad \text{for all } t \geq t_0,\ i = 0, \dots, n-1.$$

If we apply Lemma 3" instead of Lemma 3', we get the following theorem.

Theorem 3". *Suppose that assumption $(H3')$ is fulfilled, $n = 2m+1$ ($n = 2m$) and $t_0 > a$. Let u, $v \in C^n(I)$ be two functions such that*

$$\begin{aligned} & (-1)^i u^{(i)}(t_0) \leq (-1)^i v^{(i)}(t_0) \quad ((-1)^i u^{(i)}(t_0) \geq (-1)^i v^{(i)}(t_0)), \quad i = 0, \dots, n-1, \\ (25") & \\ & \text{and} \quad L(u)(t) \geq L(v)(t) \quad \text{for all } t,\ a \leq t \leq t_0. \end{aligned}$$

Then

$$\begin{aligned} (-1)^i u^{(i)}(t) \leq (-1)^i v^{(i)}(t) \quad & ((-1)^i u^{(i)}(t) \geq (-1)^i v^{(i)}(t)) \\ & \text{for all } t,\ a \leq t \leq t_0,\ i = 0, \dots, n-1. \end{aligned}$$

Let $t_0 \in I$. Denote by $y_0, \dots, y_{n-1}$ the solutions of (1) defined on I which are determined by the initial conditions

$$y_i^{(j)}(t_0) = \delta_{ij} \text{ (the Kronecker symbol)}, \quad i, j = 0, \dots, n-1.$$

It is clear that for each $j \in \{0, \dots, n-1\}$ each solution y of (1) such that $y^{(j)}(t_0) = 0$ is a linear combination $\sum_{\substack{k=0 \\ k \neq j}}^{n-1} c_k y_k$. The set of all such solutions will be called the *band of solutions of (1) of the j-th kind at the point* t_0. If the wronskian $W(y_0, \dots, y_{j-1}, y_{j+1}, \dots, y_{n-1})$ does not vanish on a subinterval $J \subset I$, then we say that this band is *regular on* J.

The following lemma can be proved in a similar way as Theorem 4 in [13], pp. 360–361.

Lemma 4. *Let $t_0 \in I$ ($t_0 > a$), $j \in \{0, \dots, n-1\}$. Then the band of solutions of (1) of the j-th kind at t_0 is regular in (t_0, ∞) (in $[a, t_0)$) if and only if for each $t_1 > t_0$ (for each t_1, $a \le t_1 < t_0$) the solution y of the initial value problem (1),*

$$\text{(12')} \qquad y^{(i)}(t_1) = 0, \quad i = 0, \dots, n-2 \text{ (if } n \ge 2\text{) and } y^{(n-1)}(t_1) = y_1^{n-1} \neq 0$$

is such that $y^{(j)}(t_0) \neq 0$.

By this lemma the following theorem holds.

Theorem 4. *Suppose that $(H3)$ or $(H4)$ ($(H1)$ or $(H2)$) holds. Then for each point $t_0 \in I$ (for each point $t_0 > a$), each $j \in \{0, \dots, n-1\}$, the band of solutions of (1) of the j-th kind at t_0 is regular in (t_0, ∞) (in $[a, t_0)$) and hence the functions $y_0, \dots, y_{j-1}, y_{j+1}, \dots, y_{n-1}$ form a fundamental system of solutions for a certain homogeneous linear differential equation of the $(n-1)$-st order in (t_0, ∞) (in $[a, t_0)$).*

Remark 6. The notion of the band of solutions has been shown especially fruitful in the theory of the third order linear differential equations, see [5].

References

[1] O. Borůvka, *Lineare Differentialtransformationen 2.Ordnung*, VEB, Berlin, 1967; *Linear Differential Transformations of the Second Order*, The English Univ. Press, London, 1971.

[2] M. Gera, *Bedingungen der Nichtoszillationsfähigkeit für die lineare Differentialgleichung dritter Ordnung* $y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0$, Acta F. R. N. Univ. Comen.-Mathematica **XXIII** (1969), 13–34.

[3] M. Gera, *Bedingungen der Nicht-oszillationsfähigkeit und der Oszillationsfähigkeit für die lineare Differentialgleichung dritter Ordnung*, Mat. časop. **21** (1971), 65–80.

[4] M. Gera, *Einige oszillatorische Eigenschaften der Lösungen der Differentialgleichung dritter Ordnung* $y''' + p(x)y' + q(x)y = 0$, Scripta Fac. Sci. Nat. UJEP Brunensis, Arch. Math. **VII** (1971), 65–76.

[5] M. Greguš, *Third Order Linear Differential Equations*, D. Reidel Publ. Co., Dordrecht, 1987.

[6] Ph. Hartman, *Ordinary Differential Equations*, J. Wiley and Sons, New York, 1964; Russian translation, Mir, Moscow, 1970.
[7] I. T. Kiguradze, T. A. Čanturija, *Asymptotical Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Nauka, Moscow, 1990. (Russian)
[8] M. A. Krasnoseľskij, *Approximate Solution of Operator Equations*, Nauka, Moscow, 1969. (Russian)
[9] F. Neuman, *Global Properties of Linear Ordinary Differential Equations*, Academia, Praha, 1991.
[10] R. Rabczuk, *Foundations of Differential Inequalities*, Pan. Wydav. Nauk., Warsaw, 1976. (Polish)
[11] J. Regenda, *Oscillatory and Nonoscillatory Properties of Solutions of the Differential Equation* $y^{(4)} + P(t)y" + Q(t)y = 0$, Math. Slovaca **28** (1978), 329–342.
[12] E. Rovderová, *Existence of a Monotone Solution of a Nonlinear Differential Equation*, J. Math. Anal. Appl. **192** (1995), 1–15.
[13] V. Šeda, *On a Class of Linear n-th Order Differential Equations*, Czech. Math. J. **39(114)** (1989), 350–369.

Department of Mathematical Analysis
Comenius University
Mlynská dolina
842 15 Bratislava, SLOVAKIA
E-mail: seda@fmph.uniba.sk