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HOMOMORPHISMS AND STRONG HOMOMORPHISMS OF RELATIONAL STRUCTURES

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Dedicated to the memory of Professor Otakar Borůvka

Abstract. In this paper, we present a construction of all homomorphisms of an n-ary relational structure into another n-ary relational structure. This construction may be used if constructing continuous transformations of a totally additive closure space into another space of the same type.

In the present paper we describe, among others, a construction of all homomorphisms of an n-ary relational structure into another structure of the same type. This construction belongs to a series of constructions of homomorphisms between various structures. Construction of all strong homomorphisms of an $n+1$-ary relational structure into another structure of the same type was reduced to the construction of certain homomorphisms between algebras with one operation of arity n (see [5], [6], [7]). Construction of all homomorphisms of an algebra with one operation of arity n into an algebra of the same type may be reduced to the construction of certain homomorphisms of one mono-unary algebra into another one (cf. [8]). In the present paper we reduce the construction of all homomorphisms of an n-ary relational structure into another structure of the same type to the construction of all strong homomorphisms between suitable n-ary relational structures. Thus, the construction investigated here may be successively reduced to the construction of suitable homomorphisms of a mono-unary algebra into another algebra of the same type. All homomorphisms of a mono-unary algebra into another one were found in [2], [3] which was a solution of a problem formulated by O. Borůvka about 1950.

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Let A be a set, $n \geq 2$ an integer. By an *n-ary relation* on A we mean a set $r \subseteq A^n = A \times \cdots \times A$ where A appears n times in the last Cartesian product. The ordered pair (A, r) is said to be an *n-ary relational structure*.

Let (A, r), (A', r') be n-ary relational structures, h a mapping of A into A' such that for any $(x_1, \ldots, x_n) \in A^n$ the condition $(x_1, \ldots, x_n) \in r$ implies $(h(x_1), \ldots, h(x_n)) \in r'$. Then the mapping h is called a *homomorphism* of (A, r) into (A', r').

A mapping h of A into A' is said to be a *strong homomorphism* of (A, r) into (A', r') if it has the following property. For any $x_1, \ldots, x_{n-1}$ in A and any x'_n in A' the condition $(h(x_1), \ldots, h(x_{n-1}), x'_n) \in r'$ holds if and only if there exists an element x_n in A such that $(x_1, \ldots, x_{n-1}, x_n) \in r$ and $h(x_n) = x'_n$.

It is easy to see that any strong homomorphism is a homomorphism. The relationship between these types of homomorphisms is described by the following theorem.

Theorem 1. *Let $n \geq 2$ be an integer, (A, r), (A', r') n-ary relational structures, h a mapping of A into A'. Then the following assertions are equivalent.*

(i) *h is a homomorphism of (A, r) into (A', r').*
(ii) *There exist an n-ary relation $\overline{r} \supseteq r$ on A and an n-ary relation $\underline{r'} \subseteq r'$ on A' such that h is a strong homomorphism of $(A, \overline{r})$ into $(A', \underline{r'})$.*

Proof. Let (i) hold. Put $\underline{r'} = \{(h(x_1), \ldots, h(x_n)) \in (A')^n;\ (x_1, \ldots, x_n) \in r\}$, $\overline{r} = \{(x_1, \ldots, x_n) \in A^n;\ (h(x_1), \ldots, h(x_n)) \in \underline{r'}\}$.

If $(x'_1, \ldots, x'_n) \in \underline{r'}$, there exist $x_1, \ldots, x_n$ in A such that $(x_1, \ldots, x_n) \in r$ and $h(x_i) = x'_i$ for any i with $1 \leq i \leq n$. It follows that $(x'_1, \ldots, x'_n) = (h(x_1), \ldots, h(x_n)) \in r'$. Thus, $\underline{r'} \subseteq r'$.

If $(x_1, \ldots, x_n) \in r$, then $(h(x_1), \ldots, h(x_n)) \in \underline{r'}$ which implies that $(x_1, \ldots, x_n) \in \overline{r}$. Hence $r \subseteq \overline{r}$.

Suppose that $x_1, \ldots, x_{n-1} \in A$, $x'_n \in A'$ are arbitrary.

(a) If $(h(x_1), \ldots, h(x_{n-1}), x'_n) \in \underline{r'}$, there exist $y_1, \ldots, y_n$ in A such that $(y_1, \ldots, y_n) \in r$ and $h(y_1) = h(x_1), \ldots, h(y_{n-1}) = h(x_{n-1}), h(y_n) = x'_n$. Since $(h(x_1), \ldots, h(x_{n-1}), h(y_n)) \in \underline{r'}$, we obtain $(x_1, \ldots, x_{n-1}, y_n) \in \overline{r}$.

(b) If there exists $x_n \in A$ such that $(x_1, \ldots, x_{n-1}, x_n) \in \overline{r}$ and $h(x_n) = x'_n$, we have $(h(x_1), \ldots, h(x_{n-1}), x'_n) = (h(x_1), \ldots, h(x_{n-1}), h(x_n)) \in \underline{r'}$.

Thus, h is a strong homomorphism of $(A, \overline{r})$ into $(A', \underline{r'})$ and (ii) holds.

Let (ii) hold. Suppose that $x_1, \ldots, x_n$ in A are arbitrary. If $(x_1, \ldots, x_n) \in r$, then $(x_1, \ldots, x_n) \in \overline{r}$. Put $x'_n = h(x_n)$. Since h is a strong homomorphism of $(A, \overline{r})$ into $(A', \underline{r'})$, we obtain $(h(x_1), \ldots, h(x_{n-1}), h(x_n)) = (h(x_1), \ldots, h(x_{n-1}), x'_n) \in \underline{r'} \subseteq r'$. Thus, h is a homomorphism of (A, r) into (A', r') and (i) holds. □

As a consequence we obtain

Construction of all homomorphisms

Let $n \geq 2$ be an integer, let n-ary relational structures (A, r), (A', r') be given.
Choose an n-ary relation $\overline{r} \supseteq r$ on A and an n-ary relation $\underline{r'} \subseteq r'$ on A'.
Construct all strong homomorphisms of $(A, \overline{r})$ into $(A', \underline{r'})$ using [7].

Any of them is a homomorphism of (A, r) into (A', r') and any homomorphism of (A, r) into (A', r') may be constructed in this way by a suitable choice of $\overline{r}$ and $\underline{r'}$.

In examples 1 and 2 we meet the construction of all strong homomorphisms of a binary relational structure (A, t) into a structure (A', t') of the same type. By [5] we construct $(\boldsymbol{P}(A), P[t])$ where $\boldsymbol{P}(A)$ is the power set of A and $P[t](X) = \{y \in A;\ \textit{there exists}\ x \in X\ \textit{with}\ (x, y) \in t\}$ for any $X \in \boldsymbol{P}(A)$. Clearly, $(\boldsymbol{P}(A), P[t])$ is a mono-unary algebra. Similarly, we construct the mono-unary algebra $(\boldsymbol{P}(A'), P[t'])$. The construction of all strong homomorphisms of (A, t) into (A', t') means to construct all totally additive and atom-preserving homomorphisms of $(\boldsymbol{P}(A), P[t])$ into $(\boldsymbol{P}(A'), P[t'])$.

A mapping H of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$ is called *totally additive* if $H(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} H(X_i)$ for any system of sets $(X_i)_{i \in I}$ where $X_i \in \boldsymbol{P}(A)$ for any $i \in I$. A mapping H of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$ is referred to as *atom-preserving* if for any $x \in A$ there exists $x' \in A'$ such that $H(\{x\}) = \{x'\}$.

Thus, we construct all homomorphisms of $(\boldsymbol{P}(A), P[t])$ into $(\boldsymbol{P}(A'), P[t'])$ according to [2] and [3] and reject all of them that are not totally additive and atom-preserving. If H is a totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), P[t])$ into $(\boldsymbol{P}(A'), P[t'])$, then we put $h(x) = H(\{x\})$ for any $x \in A$. The mapping h is a strong homomorphism of (A, t) into (A', t') and any strong homomorphism of (A, t) into (A', t') may be constructed in this way. For the details see [5].

Example 1. Let us have two binary relational structures (A, r), (A', r') where $A = \{a, b, c\}$, $A' = \{a', b', c'\}$ and the relations r, r' are given by the following tables.

r	a	b	c
a	1	1	0
b	0	1	0
c	0	1	1

r'	a'	b'	c'
a'	1	1	1
b'	0	1	1
c'	0	0	1

We now define the relations $\overline{r}$, $\underline{r'}$ by the following tables.

$\overline{r}$	a	b	c
a	1	1	1
b	0	1	1
c	0	1	1

$\underline{r'}$	a'	b'	c'
a'	1	0	1
b'	0	0	0
c'	0	0	1

We construct the mono-unary algebras $(\boldsymbol{P}(A), P[\overline{r}])$, $(\boldsymbol{P}(A'), P[\underline{r'}])$ (see Fig. 1). For the operations $P[\overline{r}]$, $P[\underline{r'}]$ we obtain the following tables.

X	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a, b, c\}$
$P[\overline{r}](X)$	$\emptyset$	$\{a, b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{b, c\}$	$\{a, b, c\}$

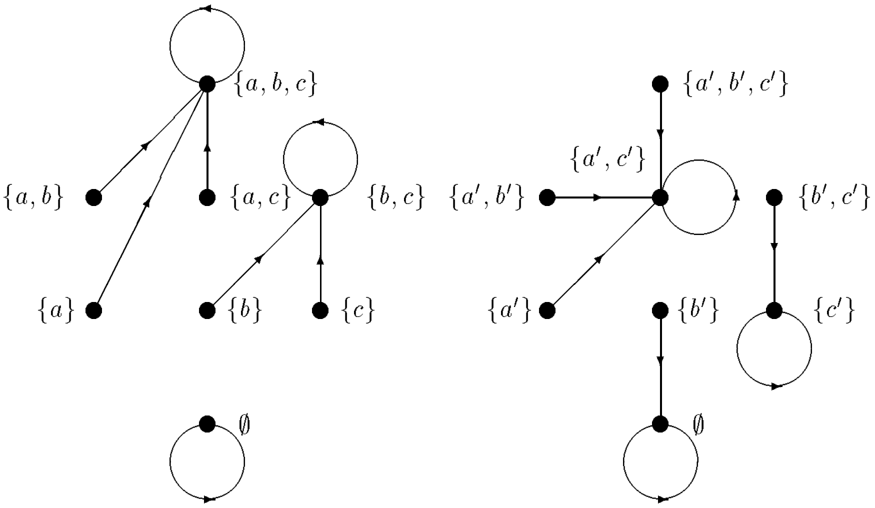
X	$\emptyset$	$\{a'\}$	$\{b'\}$	$\{c'\}$	$\{a', b'\}$	$\{a', c'\}$	$\{b', c'\}$	$\{a', b', c'\}$
$P[\underline{r'}](X)$	$\emptyset$	$\{a', c'\}$	$\emptyset$	$\{c'\}$	$\{a', c'\}$	$\{a', c'\}$	$\{c'\}$	$\{a', c'\}$

Let us choose an arbitrary totally additive atom-preserving homomorphism H of the mono-unary algebra $(\boldsymbol{P}(A), P[\overline{r}])$ into $(\boldsymbol{P}(A'), P[\underline{r'}])$. It is easy to see that we may take $H(\{a,b,c\}) = \{a',c'\}$, $H(\{b,c\}) = \{c'\}$ because H assigns elements of cycles in the second algebra to elements of cycles in the first algebra. Since H is atom-preserving, we obtain $H(\{b\}) = \{c'\}$, $H(\{c\}) = \{c'\}$, $H(\{a\}) = \{a'\}$. It is easy to see that H may be extended to a totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), P[\overline{r}])$ into $(\boldsymbol{P}(A'), P[\underline{r'}])$. Putting $H(\{x\}) = h(x)$ for any $x \in A$ we obtain a strong homomorphism of the structure $(A, \overline{r})$ into $(A', \underline{r'})$ which is a homomorphism of the structure (A, r) into (A', r'). We have $h(a) = a'$, $h(b) = c' = h(c)$.

Example 2. Let two binary relational structures (A, r), (A', r') be given where $A = \{a, b\}$, $A' = \{a', b'\}$ and the relations r, r' are defined by the following tables.

r	a	b
a	0	1
b	1	0

r'	a'	b'
a'	1	1
b'	0	1



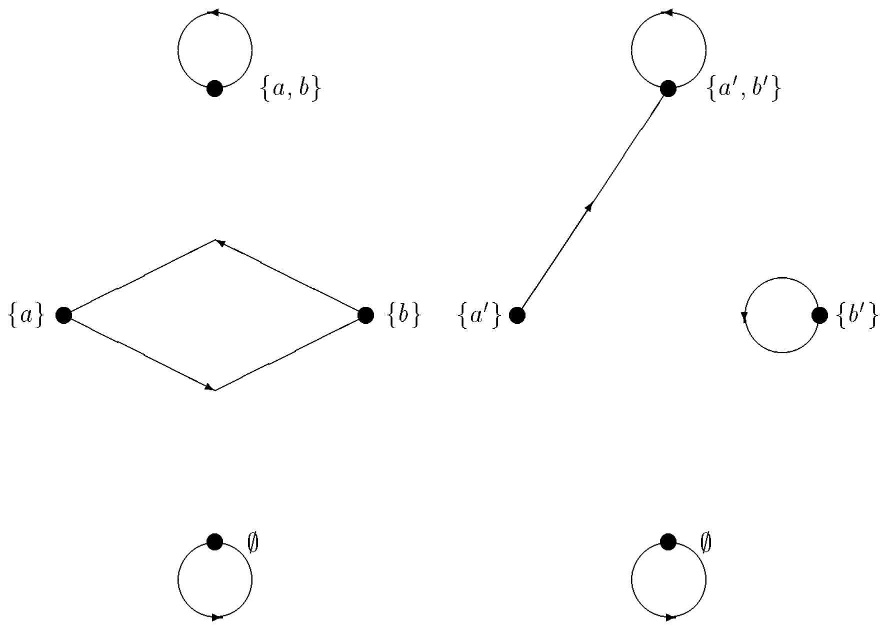
$(\boldsymbol{P}(A), P[\overline{r}])$ $(\boldsymbol{P}(A'), P[\underline{r'}])$

Fig. 1

We put $\overline{r} = r$, $\underline{r'} = r'$. Then the operations $P[\overline{r}]$, $P[\underline{r'}]$ have the following tables (see Fig. 2).

X	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$P[\overline{r}](X)$	$\emptyset$	$\{b\}$	$\{a\}$	$\{a,b\}$

X	$\emptyset$	$\{a'\}$	$\{b'\}$	$\{a',b'\}$
$P[\underline{r'}](X)$	$\emptyset$	$\{a',b'\}$	$\{b'\}$	$\{a',b'\}$



$(\boldsymbol{P}(A), P[\overline{r}])$ $(\boldsymbol{P}(A'), P[\underline{r'}])$

Fig. 2

An atom-preserving homomorphism H of $(\boldsymbol{P}(A), P[\overline{r}])$ into $(\boldsymbol{P}(A'), P[\underline{r'}])$ assigns to any atom $\{x\} \in \boldsymbol{P}(A), x \in A$ an atom $\{x'\} \in \boldsymbol{P}(A')$, $x' \in A'$. Since the elements $\{a\}$, $\{b\}$ form a cycle, the elements $H(\{a\})$, $H(\{b\})$ form a cycle, too. Clearly $H(\{a\}) = \emptyset = H(\{b\})$ contradicts the hypothesis that H is atom-preserving. For the same reason the case $H(\{a\}) = \{a', b'\} = H(\{b\})$ is impossible. Thus we have $H(\{a\}) = \{b'\} = H(\{b\})$, $H(\emptyset) = \emptyset$, $H(\{a,b\}) = \{b'\}$. Then H is a totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), P[\overline{r}])$ into $(\boldsymbol{P}(A'), P[\underline{r'}])$. It follows that the mapping h defined by $h(a) = b' = h(b)$ is a strong homomorphism of $(A, \overline{r})$ into $(A', \underline{r'})$, i.e., a homomorphism of (A, r) into (A', r').

Example 3. Let $A \neq \emptyset$ be a set, r a ternary relation on A such that $(x, y, z) \in r$ implies $x = y = z$. Let $A' \neq \emptyset$ be a set and r' a ternary relation on A' with the following property: If $x' \in A'$, $y' \in A'$, then $(x', y', y') \in r'$. In particular, $(x', x', x') \in r'$ holds for any $x' \in A'$.

We put $\overline{r} = \{(x, x, x);\ x \in A\}$, $\underline{r'} = \{(x', x', x');\ x' \in A'\}$. Then, clearly, $r \subseteq \overline{r}$, $\underline{r'} \subseteq r'$. It is easy to see that a mapping h of A into A' is a strong homomorphism of $(A, \overline{r})$ into $(A', \underline{r'})$ if and only if it is injective; for the details see Example 3 of [1]. It follows that any injective mapping of A into A' is a homomorphism of (A, r) into (A', r').

On the other hand, it is easy to see that any mapping h of A into A' is a homomorphism of the structure (A, r) into (A', r'). The corresponding ternary relations $\overline{r}$ and $\underline{r'}$ are described in the proof of Theorem 1. We have $\underline{r'} = \{(h(x), h(x), h(x));\ (x, x, x) \in r\}$, $\overline{r} = \{(x, y, z);\ (h(x), h(y), h(z)) \in \underline{r'}\}$. Clearly, h is a strong homomorphism of $(A, \overline{r})$ into $(A', \underline{r'})$.

Example 4. Let $m \geq 1$, $n \geq 1$ be integers such that n divides m, suppose $A = \{a_1, \ldots, a_m\}$, $A' = \{a'_1, \ldots, a'_n\}$ where $a_i \neq a_j$ for any i, j with $1 \leq i < j \leq m$ and $a'_i \neq a'_j$ for any i, j satisfying $1 \leq i < j \leq n$. Let us have $f(a_i) = a_{i+1}$ for any i with $1 \leq i < m$, $f(a_m) = a_1$, $f'(a'_i) = a'_{i+1}$ for any i with $1 \leq i < n$, $f'(a'_n) = a'_1$. Then (A, f), (A', f') are mono-unary algebras that can be regarded as binary relational structures. If putting $h(a_i) = a'_j$ where $i \equiv j\ (mod\ n)$ we obtain a homomorphism of the algebra (A, f) onto (A', f') that may be regarded as a strong homomorphism of the binary relational structure (A, f) onto (A', f'). If choosing an arbitrary binary relation r on A such that $r \subseteq f$ and an arbitrary binary relation r' on A' such that $f' \subseteq r'$, then h is a homomorphism of (A, r) onto (A', r').

The presented examples are intended to demonstrate our Construction in a transparent way. For this reason the sets appearing in Example 1 and 2 have small cardinalities. Besides, these examples present the construction of one homomorphism using one possible choice of $\overline{r}$ and $\underline{r'}$; the remaining cases can be solved in a similar way. Naturally, all homomorphisms of a relational structure (A, r) into (A', r') may be constructed simply by testing all mappings of A into A' and by rejecting all that are not homomorphisms. Our Construction offers another way for solving this problem. Example 4 presents another application of Theorem 1: construction of some pairs of binary relational structures with a prescribed homomorphism.

Construction of all strong homomorphisms of one n-ary relational structure into another one that appears as a step in our Construction is transformed in construction of all homomorphisms of one mono-$n-1$-ary algebra into another algebra of the same type in [5], [6], [7]. By [8] the construction of all homomorphisms of one mono-$n-1$-ary algebra into another one may be reduced to construction of all so called decomposable homomorphisms of one mono-unary algebra of a particular class into another one (cf. [2], [3], [4]). These constructions demonstrate the fundamental meaning of mono-unary algebras in some problems concerning homomorphisms of algebraic structures.

We now present an application of homomorphisms between relational structures.

Let A be a set. A mapping R of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A)$ is said to be a *closure* on A if it is extensive, monotone, and idempotent. An ordered pair $(\boldsymbol{P}(A), R)$ is referred to as a *closure space* if R is a closure on A. This closure space is called *totally additive* if so is R. By Lemma 4, Theorem 3 and Theorem 4 of [5], R is a totally additive closure on A if and only if $R = P[r]$ holds for some preordering r on A. This preordering r may be obtained from R by means of an operator Q that is defined as follows.

Let A be a set, R a mapping of $\boldsymbol{P}(A)$ into itself. We put $Q[R] = \{(x, y) \in A \times A;\ y \in R(\{x\})\}$. By Theorem 3 of [5], $Q[R]$ is a preordering for any closure space (A, R). By Lemma 4 of [5], $R = P[Q[R]]$ holds if and only if R is totally additive.

Let $(\boldsymbol{P}(A), R)$, $(\boldsymbol{P}(A'), R')$ be totally additive closure spaces. A mapping h of A into A' is called a *continuous transformation* of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$ if $P[h](R(X)) \subseteq R'(P[h](X))$ holds for any $X \in \boldsymbol{P}(A)$. A mapping h of A into A' is referred to as a *continuous and closed transformation* of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$ if $P[h](R(X)) = R'(P[h](X))$ is satisfied for any $X \in \boldsymbol{P}(A)$. By Theorem 6 of [5] a mapping h of A into A' is a continuous and closed transformation of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$ if and only if it is a strong homomorphism of $(A, Q[R])$ into $(A', Q[R'])$.

Theorem 2. *Let $(\boldsymbol{P}(A), R)$, $(\boldsymbol{P}(A'), R')$ be totally additive closure spaces, h a mapping of A into A'. Then the following assertions are equivalent.*

(i) *h is a continuous transformation of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$.*
(ii) *h is a homomorphism of $(A, Q[R])$ into $(A', Q[R'])$.*

Proof. Put $r = Q[R]$, $r' = Q[R']$.

Let (i) hold and suppose that $(x, y) \in r$. Then $y \in P[r](\{x\})$ which implies that $h(y) \in P[h](P[r](\{x\})) = P[h](P[Q[R]](\{x\})) = P[h](R(\{x\})) \subseteq R'(P[h](\{x\})) = P[Q[R']](P[h](\{x\})) = P[r'](\{h(x)\})$ and, therefore, $(h(x), h(y)) \in r'$. Hence (ii) holds.

Let (ii) hold and suppose that $y' \in P[h](R(X))$ where $X \in \boldsymbol{P}(A)$ is arbitrary. Since $R = P[r]$, there exists $y \in P[r](X)$ such that $h(y) = y'$. Thus there exists $x \in X$ such that $(x, y) \in r$ which implies $(h(x), h(y)) \in r'$. Since $h(x) \in P[h](X)$, we obtain $y' = h(y) \in P[r'](P[h](X))$. Thus, $P[h](R(X)) = P[h](P[r](X)) \subseteq P[r'](P[h](X)) = R'(P[h](X))$ holds for any $X \in \boldsymbol{P}(A)$ and (i) is satisfied. $\square$

It follows that constructions of continuous transformations of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$ may be reduced to constructions of homomorphisms of $(A, Q[R])$ into $(A', Q[R'])$ where $Q[R]$, $Q[R']$ are preorderings.

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