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TRANSFORMATION THEORY OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS – FROM LOCAL TO GLOBAL INVESTIGATIONS

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Dedicated to the memory of Professor Otakar Borůvka

Abstract. A survey of investigations of linear differential equations from the point of view of transformations is described. These investigations started in the middle of the last century and continued till the present time. Essential step was done in the fifties by O. Borůvka, who started global investigations of the second order equations.

The early beginning of the study of differential equations is closely connected with the discovery and development of infinitesimal calculus by G. W. Leibnitz (1646 – 1716) and I. Newton (1643 – 1727) at the end of the 17th century. Then the theory of differential equations was developed at the same time as other parts of mathematics. The integration factor and the method of variation of parameters were introduced by J. Bernoulli in 1691 and 1693, the special equation

$$y' = a(x)y^2 + b(x)y + c(x)$$

was studied by J. P. Riccati in 1724, solutions of linear equations with constant coefficients were discovered by L. Euler in 1750. Among significant contributors of this period there were Ch. Huygens, J. L. d'Alembert, A. C. Clairaut, J. Wallis, B. Taylor, J. Stirling, C. MacLaurin, P. S. Laplace, J. L. Lagrange, G. Monge, J. and D. Bernoulli, J. Liouville, E. Weyr, A. Cauchy, and S. Lie. Their names occur in the titles of many celebrated methods and theorems, not only in the theory of differential equations. Multidisciplinarity at least in the rank of "pure mathematics" was not of so rare occurrence at that time. The study of differential equations was often naturally connected with problems in physics, astronomy, engineering and, of course, also with the development of the other parts of mathematics, especially geometry. Let us mention just one example from many

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others: hundreds of applications of Hill's equation (linear second order equation with periodic coefficients) in mechanics, astronomy, the theory of electric circuits, of the electric conductivity of metals, and of the cyclotron.

In each area of mathematics there is a significant step consisting in investigation not only particular, single, individual objects (matrices, triangles, curves, surfaces, ..., differential equations), but in considering connections among these objects, such as transformations, motions, deformations of the objects one into another.

For linear differential equations this step was done in 1834 by E. E. Kummer [3], who was the first who considered a transformation, a substitution of the form

$$z(t) = f(t)y(h(t)) \tag{1}$$

converting solutions $y = y(x)$ of a second order linear differential equation

$$y'' + p_1(x)y' + p_0(x)y = 0$$

into solutions $z = z(t)$ of another equation of the same kind,

$$z'' + q_1(t)z' + q_0(t)z = 0.$$

Nonlinear 3rd order equations expressing the relations among the coefficients of these equations and involving functions f and h from the transformation are now called the Kummer equations as well as the transformation itself.

Then also higher order linear differential equations, their invariants and canonical forms were studied by F. Brioschi, A. R. Forsyth, E. Laguerre, E. Forsyth, just to mention only some of them. They considered the transformation (1) still involving two functions as already introduced by Kummer. Perhaps the best known result from the second half of the last century is the so-called Laguerre-Forsyth canonical form of linear differential equations of the n-th order

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_0(x)y = 0$$

characterized by the vanishing of the coefficients of the $(n-1)$st and $(n-2)$nd derivatives of the independent variables, i.e. equations of the form

$$y^{(n)} + p_{n-3}(t)y^{(n-3)} + \cdots + p_0(t)y = 0.$$

However, it was not until 1893 that P. Stächel [8], and independently S. Lie [4] one year later, proved that the transformation considered by Kummer and all his successors is in fact the most general pointwise transformation that under a differentiability condition converts any linear homogeneous differential equation of the nth order, $n \geq 2$, into an equation of the same kind.

Only their result gave the justification to all the previous investigations because, basically, Kummer and others exploited only the fact that linearity and homogeneity of equations are preserved. The 1st order equations admit a wider class of transformations. However, it is not so important, because they can be solved explicitly in a "closed form", by "quadratures" (formulas involving coefficients, their derivatives and anti-derivatives in finite compositions of "known" functions, like addition, multiplication, $x \mapsto x^n$, $x \mapsto e^x$, $x \mapsto \sin x$, etc.)

But, still differentiability conditions remained in assumptions after Stächel and Lie and it posed an open question: Do they exist transformations not smooth enough?

At the beginning of this century, in 1910, G. D. Birkhoff presented an example of a third order equation not transformable into the Laguerre-Forsyth canonical form on its whole interval of definition. By this example he pointed out that the previous investigations were of local character as a whole. This was not very encouraging, since many important questions required global investigations. Local methods and results are not sufficient when studying problems of a global nature, such as boundedness, periodicity, asymptotic and oscillatory behavior, nonvanishing solutions, and consequently the factorization of linear differential operators, as well as many other questions.

In connection with local investigations of differential equations in the 19th century, the following remark might explain why perhaps neither the problem of the global character of results was posed. I think that the mathematicians of that time were preoccupied by analytic functions due to their very successful applications in several areas. The following note of J. Hadamard (1865-1963) in his article in Encyclopédie Française [2] from 1937 may illustrate the situation in the case of the existence theorem for differential equations: "Au temps de mes études, la méthode de Cauchy-Lipshitz (celle de Picard n'avoit pas été créée) ne nous avait même pas été signalée. Lorsqu'un hasard mit quelques-uns d'entre nous en présence de l'exposé de Lipschitz, nous nous y intéressâmes comme à, une démonstration nouvelle, mais sans nous rende compte qu'il y avait là, un résultat différent de celui que nous connaissions."

Of course, there were isolated results of a global character, like e.g., the Sturm Separation and Comparison Theorems on zeros of solutions of the second order linear differential equations, and others. However, there was not a unified theory offering sufficiently general methods and dealing systematically with global behavior of solutions. To demonstrate it, let us mention that G. Sansone's [7] example of a third order linear differential equation with all oscillatory solutions occurred as late as in 1948. It was 17 years after G. Mammana [5] in 1931 showed how the non-existence of such an equation would have been a basic (sufficient and necessary) condition for factorization of linear differential operators (of the third order).

It sometimes happens after finding an interesting achievement, that one can understand better the thought descriptions and methods in papers or books of mathematicians of the previous periods, and discovers that what he considers as a completely new, or at least partially new, was already known. Just the style of writing was different from the present one. (Of course, I have not in mind the absence of conditions of non-zero denominators or a sufficient smoothness of functions whose derivatives are required.)

However, on two examples, namely on the canonical equations and on the description of distribution of zeros of solutions, we want to show that there was indeed an absence of solutions of some global problems concerning linear differential equations, whose answers are not hidden in old papers.

In the fifties O. Borůvka started the systematic study of global properties of the second order linear differential equation,

$$y'' + p(x)y = 0, \qquad p \in C^0(a,b), \qquad -\infty \le a < b \le \infty,$$

the equation in some sense the first one from those whose solutions are not available in a "closed form"; on the other hand, the equation with an extensive literature. He carried out an in-depth investigation and summarized his original methods and results in his monograph [1] published in 1967 in Berlin, and in extended form in 1971 in London.

In the last 40 years, starting from Borůvka's methods and results for the second order equations, an intensive research of linear differential equations of an arbitrary order was carried out what resulted in developing sufficiently general methods and results describing global properties of these equations. It is important to mention that not only analytic methods were involved in those investigations. Also algebraic, topological and geometrical tools, including differential geometrical ones, together with methods and results of the theory of dynamical systems, and especially of functional equations made it possible to deal with problems of a global nature in contrast to the previous local investigations of isolated results or examples.

Consider the n-th order linear homogeneous differential equation, $n \ge 2$,

$$P \equiv y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0$$

with continuous coefficients $p_i \in C^0(I)$ on an open interval $I = (a,b)$ (bounded or unbounded), $-\infty \le a < b \le \infty$.

We again consider pointwise transformations of solutions of equation P into an equation Q

$$Q \equiv z^{(n)} + q_{n-1}(t)z^{(n-1)} + \cdots + q_0(t)z = 0$$
$$q_i \in C^0(J),\ J = (c,b), \quad -\infty \le c < d \le \infty,$$

of the form

$$z(t) = f(t)y(h(t))$$

converting solutions y of P into solutions z of Q. Here $f, h \in C^n(J), f(t) \ne 0$ on J and $h'(t) \ne 0$ on J (this is not necessary to suppose, as Stächel and Lie did; the general form of the transformations and the smoothness of the functions follow from the requirement of homeomorphism of the pointwise transformation of solutions).

However, in addition, we require

$$h(J) = I,$$

i.e., h is a C^n-diffeomorphism of the (whole) interval J onto the (whole) interval I. It means that we want to transform solutions on their whole interval of definition into solutions of the transformed equation again on the whole interval. With this requirement we call α to be a global transformation of equation P into Q.

To overcome the difficulty that we have no "formula" for solutions based on coefficients, we proceed in a similar way as in algebra: for the nth order polynomials we denote by $\lambda_1, \dots, \lambda_n$ their zeros and work with them. Here we "identify" or "represent" an equation P by (any of) its n-tuple of linearly independent solutions $y_1, \dots, y_n$ (and we exploit the fact that each such an n-tuple of functions determines the equation P uniquely and at the same time it is characterized by continuous derivations up to the n-th order with the nonvanishing Wronskian, $W(x) \neq 0$ for each $x \in I$). We write briefly this n-tuple as the column vector

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in C^n(I), \qquad W[\mathbf{y}](x) \neq 0.$$

At this moment, the transformation α applied to each of the solutions $y_1, \dots, y_n$ can be written as

$$(\alpha) \qquad \mathbf{z}(t) = A \cdot f(t)\mathbf{y}(h(t)),$$

a nonsingular constant n by n matrix A expresses only another choice of an n-tuple of independent solutions of the *same* equation Q. This is because the solution space of the n-dimensional vector space whose bases are just the n-tuples with nonvanishing Wronskians.

Two aspects in our further considerations seem to be important:

Algebraic, when equations $P, Q, \dots$ (together with their intervals of definition, because we want to handle the situation globally) are considered as objects, and transformations $\alpha, \beta, \dots$ are morphisms of a category.

Geometrical, when $\mathbf{y}$ being a representative of an equation P is considered as a curve in the n-dimensional vector space $\mathbb{V}_n$, the independent variable x viewed as its parameter.

An equation P is (globally) transformable into Q if a (global) transformation α exists, converting solutions of P into solutions of Q in the sense of formula (α), briefly:

$$P\alpha = Q.$$

This relation of transformability is an equivalence relation, and the set of all linear differential equations for each (or all) $n \geq 2$ is decomposed into the classes of equivalent equations. Moreover, if we define the composition of transformations, morphisms, $\alpha \circ \beta$ by

$$(P\alpha)\beta = P(\alpha \circ \beta),$$

the category of linear differential equations becomes the Ehresmann groupoid (i.e. α^{-1} always exists), and the classes of equivalent equations (together with global transformations) become the Brandt groupoids ("connected" components). From this algebraic points of view we immediately see the importance of the so-called stationary group of an equation P, the group of all (global) transformations of the equation into itself. These stationary groups completely determinate the structure of all transformations between two equations. It is also evident that a global canonical form is a special form of representative equations available in each class of equivalence. Moreover, we know that conditions on a particular selection of such "special representatives", "canonical forms" depended on us if only each class of equivalence admits at least one equation of this type (the less the better).

However, besides these two already mentioned important tasks:

characterization of stationary groups, and

global canonical forms,

there is also another one, namely, to find (sufficient and necessary condition, hopefully "effective") for two given equations to be globally transformable:

criterion of global equivalence.

For the second order equations the answers to these questions were done by O. Borůvka [1]. He gave a complete characterization of all stationary groups, and his canonical form for the second order equations was

$$y'' + y = 0 \text{ on } I,$$

where I runs through the denumerable set of intervals

$$\{(0, \pi/2), (0, \pi), \dots, (0, k\pi/2), \dots, (0, \infty), (\infty, \infty)\}.$$

Let us note, that we have in fact a denumerable set of equations in this canonical form, since each equation is considered globally, i.e. together with its interval of definition. Borůvka's criterion of global equivalence can be roughly formulated as follows:

Two second-order linear homogeneous equations are globally equivalent if and only if their solutions have the same number of zeros.

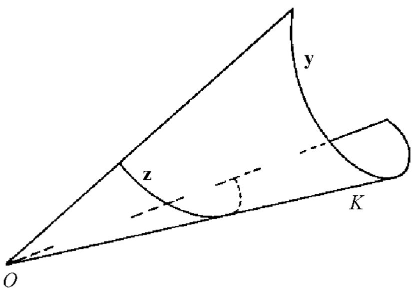
In particular, a both-side oscillatory equation is globally equivalent just only to any other again both-side oscillatory equation (canonical form of this class is $y'' + y = 0$ on $(-\infty, \infty)$), a one-side oscillatory equation just to any other one-(either left or right, but not both)-side oscillatory equation ($y'' + y = 0$ on $(0, \infty)$ is canonical), etc., see above intervals for canonical forms.

Let us mention that this criterion is not "effective" in general, we have no explicit formula involving coefficients to know the number of zeros of solutions (only sometimes some sufficient conditions, say for equations to be oscillatory).

For the n-th order equations the list of all possible stationary groups was obtained with their complete characterization [6]. There are, up to conjugacy, 10 different types (some types involving a denumerable set of subtypes), ranging from the (maximal) three parametric group of (increasing and decreasing) diffeomorphisms of $\mathbb{R}$ onto $\mathbb{R}$ to the (minimal) trivial group consisting from the identity on $\mathbb{R}$ only.

Global canonical forms of linear differential equations of the n-th order were derived under several (different) requirements by using either geometrical methods (including Cartan's moving-frame-of-reference method) or combination of the analytic calculation and the theory of functional equations. For a geometrical approach it is important how we can "see" the whole class of equations globally equivalent to a given one, say P, with its n-tuple of linearly independent solutions $\mathbf{y}$, now a curve $\mathbf{y}$ in $\mathbb{V}_n$.

$\mathbf{z} = f \cdot \mathbf{y}(h)$



Considering the transformation (α), the change of the independent variable $x = h(t)$ cannot be seen "geometrically" on the set of points of $\mathbf{y}$, this is an (admissible) reparameterization. And multiplication by a nonzero factor f gives a section $f \cdot \mathbf{y}$ on the cone K in $\mathbb{V}_n$ formed by the half-lines going from the origin and passing the points of the curve $\mathbf{y}$. The matrix A in (α) does not change an equation, it selects only a certain n-tuple of solutions. All equations globally equivalent to a given equation P are obtained when only f and h run through all admissible functions, the matrix A may be fixed, say the unit matrix. From this geometrical point of view, to select a "special", "canonical" form means to choose a "special" section (by f) on a fixed cone K (given by P) and its "special" parameterization (by h). If these "special" requirements are arbitrarily chosen, however in such a manner that they can be applied to all curves $\mathbf{y}$, we come to special n-tuples and the corresponding equations may be announced as "canonical".

One of those choices (after making the vector space $\mathbb{V}_n$ an Euclidean space $\mathbb{E}_n$) is the central projection of the curve $\mathbf{y}$ onto the unit sphere $\mathbb{S}_{n-1}$ ($f = 1/|\mathbf{y}|$) and then introducing the length parameterization into this projection ($\mathbf{y}/|\mathbf{y}| := \mathbf{z}; |\mathbf{z}'(t)| = 1$). Since this can be always done without any additional requirements, the corresponding equations may be called globally canonical (others may be obtained, if we prefer other than length parameterization, e.g. $|\mathbf{z}'(t)| = 1 + t^2$).

Another choice of conditions, based on finding all covariant functors of certain subcategories of linear differential equations obtained by functional equations, leads us to introduce

$$y^{(n)} + y^{(n-2)} + p_{n-3}(x)y^{(n-3)} + \cdots + p_0(x)y = 0$$

on some intervals $I \subset \mathbb{R}$ as global canonical forms. This gives the following global canonical forms:

$$y'' + y = 0 \quad \text{(cf. Borůvka [1])}$$

for the second order,

$$y''' + y' + p_0(x)y = 0$$

for the third order, and

$$y^{(IV)} + y'' + p_1(x)y' + p_0(x)y = 0$$

for the forth order equations on certain intervals, etc.

We could see that if Laguerre and Forsyth had taken our

$$1 \quad 0 \quad 1$$

instead of their

$$1 \quad 0 \quad 0$$

as the first three coefficients, they would have got global forms instead of their local ones. To impose some two conditions on coefficients is correct, since two (rather arbitrary) functions f and h occur in global transformations. However, there is a question, on which coefficients, and whether the zero is always the best choice.

For linear differential equations of the nth order, $n \geq 3$, we can in general (i.e. with exception of one special type of equations) decide from the coefficients whether two given equations are or are not globally equivalent (without solving these equations). That means that on contrast to the second order equations, for higher order equations in general we have an effective criterion of the global equivalence.

The methods developed enable us to find new interesting global invariants of the n-th order equations, involving e.g. the order of smoothness (differentiability) of coefficients, the invariants that can occur only for sufficiently large order of equations.

Answers to questions concerning global behavior of solutions and global transformations of linear differential equations required combination of various techniques and results from different areas of mathematics. Some of them are easier, some sophisticated, some involving not easy calculations of an analytic nature. However, there are several of them that are interesting for their simplicity and transparency. These concern namely questions about possible distribution of zeros of solutions.

Hence, consider again an equation P and its n-tuple of linearly independent solutions $\mathbf{y}$ as a curve in the n-dimensional space $\mathbb{V}_n$. The relation

$$c_1 y_1(x_0) + \cdots + c_n y_n(x_0) = 0$$

can be read in two equivalent ways:

a) the solution $c_1 y_1(x) + \cdots + c_n y_n(x)$ of P has a zero at x_0,

b) the curve $\mathbf{y}$ intersects the hyperplane $c_1\xi_1 + \cdots + c_n\xi_n = 0$ (passing the origin) at the point $\mathbf{y}(x_0)$ of the parameter x_0.

The equivalence of a) and b) (fixed P, arbitrary $c_1, \ldots, c_n$) gives the theorem on the geometric representation of zeros of solutions. For $\mathbb{V}_n$ to be Euclidean we consider instead of $\mathbf{y}$, its central projection $\mathbf{y}/|\mathbf{y}|$ on the unit sphere $\mathbb{S}_{n-1}$ (now, without the change of parameterization). Then b) is valid for $\mathbf{y}/|\mathbf{y}|$ instead of $\mathbf{y}$ which in now on the (compact) $\mathbb{S}_{n-1}$ and great circles on $\mathbb{S}_{n-1}$ represent hyperplanes; the multiple zeros correspond to the contacts of higher orders.

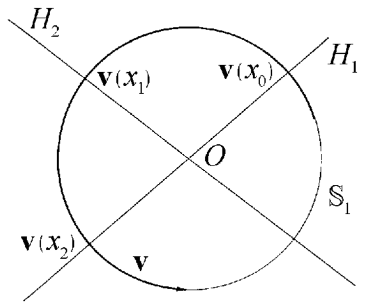
By using this approach we can sometimes see without lengthy and tiresome calculations, simply by drawing a curve on the unit sphere (at least for $\mathbb{S}_2$), what is possible and what is impossible in distribution of zeros. We are not saying that this is a proof, but certainly it gives hints on how to proceed with the proof. This method also makes some complicated constructions or proofs easily understandable and gives suggestions concerning possible results or investigations of open problems in this area (sometimes it may even discover an inaccuracy in lengthy $\varepsilon - \delta$ proofs).

Let us illustrate our approach on two examples.

n=2:

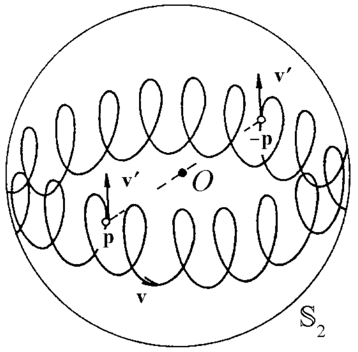
Due to the nonvanishing determinant, $\det(\mathbf{y}, \mathbf{y}') \neq 0$ the radius vector $\mathbf{y}$ and the tangent vector $\mathbf{y}'$ are linearly independent, hence the curve $\mathbf{y}$ encircles (in positive or negative direction) the origin (never passing it). Its central projection on the unit sphere $\mathbb{S}_1$, the unit circle, is an arc $\mathbf{v} = \mathbf{y}/|\mathbf{y}|$ on it (possibly encircling origin several, even countable many times). Hyperplanes are straight lines. We may observe:

between any two consecutive intersections of the line H_1 with the arc $\mathbf{v}$ there is just one intersection of $\mathbf{v}$ with the line H_2, $x_0 < x_1 < x_2$.



In our interpretation it gives exactly the separation theorem for second order equations.

n=3:



A "prolonged cycloid" $\mathbf{v} = \mathbf{v}(x)$ goes periodically infinitely many times around the equator of the unit sphere $\mathbb{S}_2$ in $\mathbb{E}_3$ as its parameter x runs from $-\infty$ to $+\infty$. This curve is sufficiently smooth, of the class $C^3(-\infty, \infty)$, and without points of inflexions, i.e. $\mathbf{v}, \mathbf{v}'$, and $\mathbf{v}''$ are not colinear, or $\det(\mathbf{v}, \mathbf{v}', \mathbf{v}'') \neq 0$. Each great circle on $\mathbb{S}_2$ intersects $\mathbf{v}$ at points with an infinite sequence of parameters (both for $x \to -\infty$ and $x \to \infty$). In our interpretation, each solution of the corresponding 3rd order linear differential equation is oscillatory (to both sides), another very evident example of an equation demonstrating the impossibility of factorization of all linear differential operators of the third order.

The details can be found in [6].

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