ARCHIVUM MATHEMATICUM (BRNO)
Tomus 33 (1997), 147 – 155

DISPERSIONS FOR LINEAR DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

František Neuman

Dedicated to the memory of Professor Otakar Borůvka

Abstract. For linear differential equations of the second order in the Jacobi form

$$y'' + p(x)y = 0$$

O. Borůvka introduced a notion of dispersion. Here we generalize this notion to certain classes of linear differential equations of arbitrary order. Connection with Abel's functional equation is derived. Relations between asymptotic behaviour of solutions of these equations and distribution of zeros of their solutions are also investigated.

I. Motivation

For a linear differential equation of the second order in the Jacobi form

$$(p) \qquad y'' + p(x)y = 0, \quad p \in C^0(I), I = (a, b), -\infty \le a < b \le \infty$$

O. Borůvka [2] introduced the notions of a *phase* and the *dispersion* as follows. Consider two linearly independent solutions y_1 and y_2 of (p). A phase of (p) corresponding to the pair y_1, y_2 is a continuous function $\alpha : I \longrightarrow \mathbb{R}$ satisfying the relation

$$\tan \alpha(x) = y_1(x)/y_2(x)$$

wherever $y_2(x) \neq 0$. The continuity of α implies $\alpha \in C^3(I)$ with $\alpha'(x) \neq 0$ on I, because

$$\alpha'(x) = \frac{c}{y_1^2(x) + y_2^2(x)}, \quad c = \text{const.} \quad \neq 0.$$

1991 *Mathematics Subject Classification*: 34A30, 34C10, 34C11, 39B22.

Key words and phrases: linear differential equations, distribution of zeros, asymptotic behaviour, Abel's functional equation.

Research supported by the grants # 201/96/0410 of the Czech Republic and # A 119404 of the Academy of Sciences of the Czech Republic.

Moreover, the general solution of (p) can be written in the form

$$y(t; c_1, c_2) = \frac{c_1}{\sqrt{|\alpha'(x)|}} \sin(\alpha(x) + c_2).$$

If the equation (p) is oscillatory for $x \to b_-$, then $\lim_{x\to b_-} |\alpha(x)| = \infty$.

The dispersion φ of (p) is defined [2] as follows. For arbitrary $x_0 \in (a, b)$, let y be a nontrivial solution of (p) vanishing at x_0, i.e. $y(x_0) = 0$. If there exists a zero of this solution y to the right of x_0, then the first zero of them is denoted by $\varphi(x_0)$. Evidently φ is defined on (a, b) if (p) is oscillatory for $x \to b_-$. Borůvka has shown that $\varphi \in C^3$, $\varphi'(x) > 0$ and $\varphi(x) > x$ and the following Abel's functional equation holds:

$$\alpha(\varphi(x)) = \alpha(x) + \pi \cdot \mathrm{sign}\ \alpha'$$

wherever φ is defined. This functional equation was intensively studied by B. Choczewski [3], see also M. Kuczma [4], and in connection with the second order differential equations by E. Barvínek [1]. Important connections between distribution of zeros of solutions of oscillatory second-order equations (p) and their asymptotic properties were studied in [5], see also [6]. Here we generalize these results to linear differential equations of the n-th order.

II. Preliminary results

Consider a linear differential equation of the form

$$(P) \qquad y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_0(x)y = 0 \qquad \text{on} \qquad I,$$

I being an open interval of the reals, p_i are real-valued continuous functions defined on I for $i = 0, 1, \dots, n-1$, i.e. $p_i \in C^0(I), p_i : I \to \mathbb{R}$.

Take functions $f : J \to \mathbb{R}$ and $h : J \to I$ such that

$$f \in C^n(J), f(t) \neq 0 \quad \text{for each} \quad t \in J, \quad \text{and}$$

$$h \in C^n(J), h'(t) \neq 0 \quad \text{for each} \quad t \in J, \quad \text{and} \quad h(J) = I.$$

For each solution y of equation (P) the function z defined as

$$(f, h) \qquad z : J \to \mathbb{R}, \quad z(t) := f(t) \cdot y(h(t)), \quad t \in J,$$

satisfies again a differential equation of the same form

$$(Q) \qquad z^{(n)} + q_{n-1}(t)z^{(n-1)} + \dots + q_0(t)z = 0 \qquad \text{on} \qquad J.$$

Since h is a C^n-diffeomorphism of J onto I, solutions y are transformed into solutions z on their whole intervals of definition. This is why we also speak about a *global* transformation of equation (P) into equation (Q).

Let $\mathbf{y}(x) = (y_1(x), \dots, y_n(x))^T$ denote an n-tuple of linearly independent solutions of the equation (P) considered as a column vector function or as a curve in n-dimensional Euclidean space $\mathbb{E}_n$ with the independent variable x as the parameter and $y_1(x), \dots, y_n(x)$ as its coordinate functions; M^T denotes the transpose of the matrix M.

If $\mathbf{z}(t) = (z_1(t), \dots, z_n(t)^T$ denotes an n-tuple of linearly independent solutions of the equation (Q), then the global transformation (f, h) can be equivalently written as

$$\mathbf{z}(t) = f(t) \cdot \mathbf{y}(h(x))$$

or, for an arbitrary regular constant $n \times n$ matrix A ,

$$\mathbf{z}(t) = Af(t) \cdot \mathbf{y}(h(x))$$

expressing only the fact that another n-tuple of linearly independent solutions of the *same* equation (Q) is taken.

To emphasize this situation, let us denote by $(P_\mathbf{y})$ and $(Q_\mathbf{z})$ the equations (P) and (Q), respectively. Capital P refers to the coefficients p_i of the equation $(P_\mathbf{y})$, subscript $\mathbf{y}$ expresses a particular choice of an n-tuple of linearly independent solutions. Similarly for $(Q_\mathbf{z})$ and other equations considered here.

Denote by $W[\mathbf{y}](x)$ the Wronski determinant of $\mathbf{y}$, i.e.
$\det(\mathbf{y}(x), \mathbf{y}'(x), \dots, \mathbf{y}^{(n-1)}(x))$.

The coefficient p_{n-1} in $(P_\mathbf{y})$ is given as

$$p_{n-1}(x) = -(\ln |W[\mathbf{y}](x)|)'.$$

We have $p_{n-1} \equiv 0$ exactly when $W[\mathbf{y}](x) = \text{const.} \neq 0$. Since

$$W[f.\mathbf{y}(h)](t) = (f(t))^n.(h'(t))^{\frac{n(n-1)}{2}}.W[\mathbf{y}](h(t)),$$

for the coefficient q_{n-1} in $(Q_\mathbf{z})$ we have

$$q_{n-1}(t) = -n\frac{f'(t)}{f(t)} - \frac{n(n-1)}{2}\frac{h''(t)}{h'(t)} + p_{n-1}(h(t)).h'(t). \tag{1}$$

Namely, if $p_{n-1} \equiv 0$ then $q_{n-1} \equiv 0$ occurs exactly when

$$f(t) = c.|h'(t)|^{\frac{1-n}{2}}, \quad c = \text{const.} \neq 0. \tag{2}$$

Since the factor f belongs to $C^n(J)$, we have $h \in C^{n+1}(J)$.

III. Notation and basic properties

Let all solutions of an equation $(R_\mathbf{u})$ be periodic or half-periodic with a period $d, d > 0$:

$$\begin{aligned} \mathbf{u}(x+d) &= \mathbf{u}(x), \quad \text{or} \\ \mathbf{u}(x+d) &= -\mathbf{u}(x) \quad \text{on} \quad \mathbb{R}. \end{aligned}$$

Then all coefficients r_i of $(R_\mathbf{u})$ are periodic, $r_i(x+d) = r_i(x)$ on $\mathbb{R}$.

Lemma 1. *There is no equation $(R_{\mathbf{u}})$ of odd order n with all half-periodic solutions.*

Proof. Consider $W[\mathbf{u}](x)$ for an equation $(R_{\mathbf{u}})$ and its n-tuple $\mathbf{u}$ of linearly independent solutions. For $\mathbf{u}(x+d) = -\mathbf{u}(x)$ we would have

$$W[\mathbf{u}](x+d) = W[-\mathbf{u}](x) = -W[\mathbf{u}](x),$$

because n is odd. Since $W[\mathbf{u}]$ is nonvanishing and continuous, this is a contradiction. $\square$

Consider an equation $(S_{\mathbf{v}})$ of the same order that can be transformed into the equation $(R_{\mathbf{u}})$:

$$\mathbf{u}(x) = f(x).\mathbf{v}(h(x)),$$

h being a C^n-diffeomorphism of $\mathbb{R}$ onto J, the interval of definition of $(S_{\mathbf{v}})$.

Take $f(x) = ||\mathbf{u}(x)|| := \sqrt{u_1^2(x) + \cdots + u_n^2(x)}$, $||\cdot||$ denoting the Euclidean norm of an n-dimensional vector. Evidently $f \in C^n(\mathbb{R}), f(x) > 0$, and $f(x+d) = f(x)$ on $\mathbb{R}$ and $||\mathbf{v}(t)|| = ||\mathbf{v}(h(x))|| = ||\mathbf{u}(x)||/f(x) = 1$.

Let h be chosen so that

$$r_{n-1}(x) = -n\frac{f'(x)}{f(x)} - \frac{n(n-1)}{2}.\frac{h''(x)}{h'(x)} \quad , \tag{3}$$

i.e. $h(x) := c \cdot \int_{x_0}^{x}[(f(\sigma))^{\frac{2}{1-n}} \cdot \exp\{\frac{-2}{n(n-1)} \cdot \int_{\sigma_0}^{\sigma} r_{n-1}(\tau)d\tau\}]d\sigma + k$.

For $r_{n-1} \in C^{n-2}(\mathbb{R})$, we have $h \in C^n(\mathbb{R}), h'(x) > 0, h(x+d) = h(x)+p$ because of d-periodicity of f and r_{n-1}, $h(\mathbb{R}) = \mathbb{R}$. Select c so that $p = d$. Due to relation (1) with respect to (3) where r_{n-1} stands for q_{n-1} and $s_{n-1} \equiv 0$ for p_{n-1}, we see that

(i) the coefficient s_{n-1} in equation $(S_{\mathbf{v}})$ is identically zero;

(ii) all solutions of $(S_{\mathbf{v}})$ are periodic or half-periodic with the period d, and

(iii) $||\mathbf{v}(t)|| = 1$.

Choose $t_0 \in \mathbb{R}$ arbitrarily. Let v be a nontrivial solution of $(S_{\mathbf{v}})$ with the zero of multiplicity $(n-1)$ at t_0, i.e. satisfying

$$v(t_0) = v'(t_0) = \cdots = v^{(n-2)}(t_0) = 0, v^{(n-1)}(t_0) \neq 0.$$

Up to a constant multiplier, v is determined uniquely.

Lemma 2. *For the above solution v, the points $t_0 + kd, k \in \mathbb{Z}$, are zeros of multiplicity $n-1$.*

Proof follows from the periodicity or half-periodicity of all solutions of $(S_{\mathbf{v}})$. $\square$

Now suppose that in addition to the above properties of equation $(S_{\mathbf{v}})$, the following one is also satisfied:

(iv) for each $t_0 \in \mathbb{R}$, any solution having a zero of multiplicity $n-1$ at t_0 has the point $t_0 + d$ as its first zero of the same multiplicity to the right of t_0.

Remark 1. Property (iv) implies that d is the smallest positive period for which $\mathbf{v}$ of $(S_{\mathbf{v}})$ is periodic or half-periodic on the whole $\mathbb{R}$.

Notation 1. Let $\mathcal{S}$ denote the set of all linear differential equations satisfying the properties (i), (ii), (iii), and (iv). Furthermore, let $\mathcal{P}$ be the set of all linear differential equations that can be obtained from equations in $\mathcal{S}$ by all global transformations (f, h).

IV. Dispersions

Definition. Let an equation $(P_{\mathbf{y}})$ of the order n belong to $\mathcal{P}$. Take an arbitrary x_0 from its interval of definition I and consider a nontrivial solution y having a zero of multiplicity $n-1$ at x_0. Denote by $\varphi(x_0)$ the first zero of the same multiplicity of this solution y to the right of x_0. Call this function φ the *dispersion* of the equation $(P_{\mathbf{y}})$.

Theorem 1. *Let (f, h) be the transformation that transforms an equation $(P_{\mathbf{y}})$ of the n-th order from $\mathcal{P}$ into an equation from $\mathcal{S}$. The dispersion φ of $(P_{\mathbf{y}})$ is well-defined on the whole interval of definition I of this equation and satisfies Abel's functional equation*

$$h(\varphi(x)) = h(x) + d \cdot \operatorname{sign} h' \quad , \qquad x \in I. \tag{4}$$

Futhermore,

$$\varphi \in C^n(I), \quad \varphi(x) > x, \quad \varphi'(x) > 0, \quad \varphi(I) = I, \quad \text{and}$$

$$\lim_{i \to -\infty} \varphi^{[i]}(x_0) = a, \lim_{i \to \infty} \varphi^{[i]}(x_0) = b, \quad \text{for each} \quad x_0 \in I = (a, b),$$

where $\varphi^{[i]}$ denotes the i-th iterate of φ, i.e. $\varphi^{[1]} = \varphi, \varphi^{[i+1]} = \varphi \circ \varphi^{[i]}$.

Proof. Take $x_0 \in I$ arbitrarily, and denote by y a nontrivial solution of $(P_{\mathbf{y}})$ having x_0 as its zero of multiplicity $n-1$. Writing $y(x) = \mathbf{c}^T \cdot \mathbf{y}(x)$, where $\mathbf{c}$ is a suitable constant vector and '$\cdot$' denotes the dot product, we have

$$\mathbf{c}^T \cdot \mathbf{y}(x_0) = \mathbf{c}^T \cdot \mathbf{y}'(x_0) = \ldots = \mathbf{c}^T \cdot \mathbf{y}^{(n-2)}(x_0) = 0, \quad \mathbf{c}^T \cdot \mathbf{y}^{(n-1)}(x_0) \neq 0.$$

Hence

$$\begin{aligned} 0 &= \mathbf{c}^T \cdot f(x_0) \cdot \mathbf{v}(h(x_0)), \\ 0 &= \mathbf{c}^T \cdot [f(x_0) \cdot \mathbf{v}'(h(x_0)) \cdot h'(x_0) + f'(x_0) \cdot \mathbf{v}(h(x_0))], \quad \ldots \quad , \\ 0 &= \mathbf{c}^T \cdot [f(x_0) \cdot \mathbf{v}^{(n-2)}(h(x_0)) \cdot (h'(x_0))^{n-2} + L(n-3)], \\ 0 &\neq \mathbf{c}^T \cdot [f(x_0) \cdot \mathbf{v}^{(n-1)}(h(x_0)) \cdot (h'(x_0))^{n-1} + L(n-2)], \end{aligned} \tag{5}$$

where $L(i)$ is a linear combination of the vectors $\mathbf{v}(h(x_0)), \mathbf{v}'(h(x_0)), \ldots, \mathbf{v}^{(i)}(h(x_0))$ with some scalar functions as coefficients.

Since $f(x_0)$ and $h'(x_0) \neq 0$, the solution $v(t) = \mathbf{c}^T \cdot \mathbf{v}(t)$ of equation $(S_{\mathbf{v}})$ has a zero of multiplicity $n-1$ at $t_0 = h(x_0)$. Due to the property (iv), the first zero of this solution of the same multiplicity to the right of t_0 is $t_0 + d$, to the left of t_0 is $t_0 - d$. Hence

$$h(\varphi(x_0)) = t_0 + d = h(x_0) + d \quad \text{for increasing } h \text{, and}$$

$$h(\varphi(x_0)) = t_0 - d = h(x_0) - d \quad \text{for decreasing } h.$$

Since $x_0 \in I$ was arbitrary, Abel's equation (4) holds.

Now, h is a C^n-diffeomorphism of I onto $\mathbb{R}$, thus

$$\varphi(x) = h^{-1}(h(x) + d.\mathrm{sign}h') \tag{6}$$

is defined for all $x \in I, \varphi(x) > x$, and $\varphi'(x) > 0$ on I because $h'(\varphi(x)).\varphi'(x) = h'(x)$. Moreover

$$\varphi^{[i]}(x) = h^{-1}(h(x) + i.d.\mathrm{sign}h'),$$

hence

$$\lim_{i\to-\infty} \varphi^{[i]}(x_0) = a \quad \text{and} \quad \lim_{i\to\infty} \varphi^{[i]}(x_0) = b.$$

□

Lemma 3. *For each equation $(P_{\mathbf{y}}) \in \mathcal{P}$ and its dispersion φ the following is true: $\mathbf{y}(x_0)$ is parallel to $\mathbf{y}(\varphi(x_0))$, and $\varphi(x_0)$ is the first parameter to the right of x_0 when it happens, i.e.*

$$\varphi(x_0) = \min_{x > x_0} \{ \mathit{rank}\ (\mathbf{y}(x_0), \mathbf{y}(x)) = 1 \}.$$

Proof follows from the definition of the dispersion, the system (5) and the property (iv). □

V. Asymptotic behaviour

Since the factor f in the global transformation (f, h) is in general independent on the function h, we cannot expect a relation between asymptotic behaviour of solutions (depending on f) and the distribution of their zeros (depending on φ and hence on h). However, for linear differential equations of the n-th order with the vanishing coefficients by the $(n-1)$-st derivative we have the relation (2). Thus let us consider the following class of equations.

Notation 2. A linear differential equation of an order n belongs to the subset $\mathcal{P}_0$ of $\mathcal{P}$ if its coefficient by the $(n-1)$-st derivative is identically zero.

Remark 2. Evidently $\mathcal{S} \subset \mathcal{P}_0$ and $h \in C^{n+1}(I)$. Then, due to (6), the dispersion φ of each equation from $\mathcal{P}_0$ is also in $C^{n+1}(I)$.

Theorem 2. *Let an equation* $(P_{\mathbf{y}})$ *of the n-th order belong to* $\mathcal{P}_0$ *and let* $\varphi : I \to I$ *denote its dispersion. If*

a) $\varphi(x) - x$ *is a nondecreasing function, or*

b) $\varphi(x) - x$ *is a nonincreasing function, or*

c) $\varphi(x) - x = \delta = const. > 0$,

then

a') maxima of absolute values of each solution of $(P_{\mathbf{y}})$ *on consecutive intervals* $[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]$, $i = 0, 1, 2, \dots$, *form a nondecreasing sequence, or*

b') those maxima form a nonincreasing sequence, or

c') each solution of $(P_{\mathbf{y}})$ *is periodic or half-periodic with the period* δ, *respectively.*

Remark 3. Conditions a) - c) mean that the distances between consecutive zeros of multiplicity $n-1$ of each solution of $(P_{\mathbf{y}})$ are a) nondecreasing, b) nonincreasing, c) the same.

Proof of Theorem 2. Since $(P_{\mathbf{y}}) \in \mathcal{P}_0$,

$$\mathbf{y}(x) = |h'(x)|^{\frac{1-n}{2}} \mathbf{v}(h(x)), \quad x \in I, \quad h(I) = \mathbb{R} \tag{7}$$

for some $(S_{\mathbf{v}}) \in \mathcal{S}$. For the dispersion φ of $(P_{\mathbf{y}})$ we have

$$h(\varphi(x)) = h(x) + d \cdot \operatorname{sign}\, h'$$

and

$$h'(\varphi(x)) \cdot \varphi'(x) = h'(x), \quad x \in I. \tag{8}$$

Each solution y of $(P_{\mathbf{y}})$ can be written as $\mathbf{c}^T \cdot \mathbf{y}(x)$ for a suitable constant vector $\mathbf{c}$. Choose $x_0 \in I$. Let M_i be the maximum of $|y|$ on the interval $[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]$, i.e.

$$M_i := \max_{[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]} |\mathbf{c}^T \cdot \mathbf{y}(x)| =$$

$$\max_{[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]} ||h'(x)|^{\frac{1-n}{2}} \cdot \mathbf{c}^T \cdot \mathbf{v}(h(x))|.$$

Now, due to (8), we have

$$M_{i+1} = \max_{[\varphi^{[i+1]}(x_0), \varphi^{[i+2]}(x_0)]} ||h'(x)|^{\frac{1-n}{2}} \cdot \mathbf{c}^T \cdot \mathbf{v}(h(x))| =$$

$$\max_{[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]} ||h'(\varphi(x))|^{\frac{1-n}{2}} \cdot \mathbf{c}^T \cdot \mathbf{v}(h(\varphi(x)))| =$$

$$\max_{[\varphi^{[i]}(x_0), \varphi^{[i+1]}(x_0)]} \left| |(h'(x)|^{\frac{1-n}{2}} \cdot (\varphi'(x))^{\frac{n-1}{2}} \cdot \mathbf{c}^T \cdot \mathbf{v}(h(x) \pm d) \right| =$$

$$\max_{[\varphi^{[i]}(x_0),\varphi^{[i+1]}(x_0)]}\{(\varphi'(x))^{\frac{n-1}{2}}\cdot|h'(x)|^{\frac{1-n}{2}}\cdot\mathbf{c}^T\cdot\mathbf{v}(h(x)+d)\}\geq$$

$$\min_{[\varphi^{[i]}(x_0),\varphi^{[i+1]}(x_0)]}(\varphi'(x))^{\frac{n-1}{2}}\cdot\max_{[\varphi^{[i]}(x_0),\varphi^{[i+1]}(x_0)]}|\pm|h'(x)|^{\frac{1-n}{2}}\cdot\mathbf{c}^T\cdot\mathbf{v}(h(x))|\}=$$

$$\min_{[\varphi^{[i]}(x_0),\varphi^{[i+1]}(x_0)]}(\varphi'(x))^{\frac{n-1}{2}}\cdot M_i.$$

For the case a), when $\varphi(x)-x$ is increasing, i.e. $\varphi'(x)\geq 1$ everywhere, we get $M_{i+1}\geq M_i$, hence the consecutive maxima cannot decrease. Analogously in the case when b) $\varphi'(x)\leq 1$, we have

$$M_{i+1}=\max_{[\varphi^{[i]}(x_0),\varphi^{[i+1]}(x_0)]}\{(\varphi'(x))^{\frac{n-1}{2}}|\pm|h'(x)|^{\frac{1-n}{2}}\cdot\mathbf{c}^T\cdot\mathbf{v}(h(x))|\}\leq$$

$$\max_{[\varphi^{[i]}(x_0),\varphi^{[i+1]}(x_0)]}(\varphi'(x))^{\frac{n-1}{2}}\cdot M_i\leq M_i,$$

and the consecutive maxima cannot increase.

For $\varphi(x)-x=\delta>0$ we have $\varphi(x)=x+\delta$, $\varphi'(x)=1$, $h(\varphi(x))=h(x)\pm d$, and also $h'(\varphi(x))=h'(x)$. Hence

$$\begin{aligned}\mathbf{y}(\varphi(x))&=\mathbf{y}(x+\delta)\ \text{ and also}\\&=|h'(\varphi(x))|^{\frac{1-n}{2}}.\mathbf{v}(h(\varphi(x)))=|h'(x)|^{\frac{1-n}{2}}.\mathbf{v}(h(x)\pm d)\\&=\pm|h'(x)|^{\frac{1-n}{2}}.\mathbf{v}(h(x))=\pm\mathbf{y}(x).\square\end{aligned}$$

VI. Final remarks

The dispersion just introduced for the n-th order linear differential equations is a proper generalization of this notion introduced by O. Borůvka for the second order equations of the Jacobi form. The role of the set of equations $\mathcal{S}$ in his case is played by a single equation

$$v''+v=0\quad\text{on }\ \mathbb{R}$$

that admits half-periodic solutions with the period π:

$$\mathbf{v}(t)=\begin{pmatrix}\sin t\\ \cos t\end{pmatrix},\qquad\mathbf{v}(t+\pi)=-\mathbf{v}(t).$$

The subset of the second order equations from $\mathcal{P}$ is formed by all both side oscillatory equations in the general form, the set $\mathcal{P}_0$ consists from all both-side oscillatory equations in the Jacobi form, (p). The function h here is the (first) phase and φ coincides with the dispersion introduced by O. Borůvka for equation (p). This φ in fact generalizes the dispersion even for the second order equations because it is defined for equations in the general form: $y''+p_1(x)y'+p_0(x)y=0$.

References

[1] E. Barvínek, *O rozložení nulových bodů řešení lineární diferenciální rovnice* $y'' = Q(t)y$ *a jejich derivací*, Acta F. R. N. Univ. Comenian **5** (1961), 465–474.
[2] O. Borůvka, *Linear Differential Transformations of the Second Order*, The English Univ. Press, London, 1971.
[3] B. Choczewski, *On differentiable solutions of a functional equation*, Ann. Polon. Math. **13** (1963), 133–138.
[4] M. Kuczma, *Functional Equations in a Single Variable*, PWN, Warszawa, 1968.
[5] F. Neuman, *Distribution of zeros of solutions of* $y'' = q(t)y$ *in relation to their behaviour in large*, Studia Sci. Math. Hungar **8** (1973), 177–185.
[6] F. Neuman, *Global Properties of Linear Ordinary Differential Equations*, Mathematics and Its Applications (East European Series) **52**, Kluwer Acad. Publ., Dordrecht-Boston-London, 1991, ISBN 0-7923-1269-4.

Mathematical Institute
Academy of Sciences, branch Brno
Žižkova 22
616 62 Brno, CZECH REPUBLIC
E-mail: neuman@ipm.cz