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SOME REMARKS ON THE Ω-STABILITY FOR FAMILIES OF POLYNOMIALS

Jean Mawhin

Dedicated to the memory of Professor Otakar Borůvka

Abstract. Using Brouwer degree, we prove a more general version of the zero exclusion principle for families of polynomials and apply it to obtain very simple proofs of extensions of recent results on the Routh-Hurwitz and Schur-Cohn stability of families of polynomials.

1. Introduction

It is well know that the problem of the stability of a linear difference or differential system with constant coefficients reduces to the question of locating the roots of its characteristic polynomial in a suitable region of the complex plane. The Routh-Hurwitz and Schur-Cohn tests, which respectively correspond to the open left half-space and the open unit ball, are well known in this respect [2, 3].

In a recent work [9], Zahreddine has considered the following problem: given a path-wise connected region Ω in the complex plane and a set S of polynomials of the same degree, find conditions under which all polynomials of S have their zeros inside Ω. In the special case where S is made of all the convex combinations of two polynomials which are stable in the Routh-Hurwitz or the Schur-Cohn sense, Zahreddine has found necessary and sufficient conditions for this set S to have the same stability. His approach is algebraic and based upon some properties of resultants and standard Routh-Hurwitz or Schur-Cohn stability conditions for a complex polynomial [2, 3].

The aim of this note is to show that a very simple proof of more general version of this result can be obtained by using the elementary properties of the Brouwer degree [4]. We first use this technique to prove a more general version of the standard *zero exclusion principle* [1, 6] and then apply it to the proof of the Zahreddine's results.

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2. A zero exclusion principle

Let $\Omega \subset \mathbb{C}$ be open and $p : \mathbb{C} \to \mathbb{C}$ be a polynomial.

Definition 2.1. *We say that p is Ω-stable if $p(z) \neq 0$ for $z \notin \Omega$.*

When $\Omega = \{z \in \mathbb{C} : \Re z < 0\}$, the Ω-stability will be called the *Routh-Hurwitz stability;* when Ω is the unit open ball $B(1)$ in $\mathbb{C}$, the Ω-stability will be called the *Schur-Cohn stability.*

Let $\Lambda \subset \mathbb{R}^m$ and $\{p(\cdot, \lambda)\}_{\lambda \in \Lambda}$ be a continuous family of polynomials on $\mathbb{C}$. This means that $p : \mathbb{C} \times \Lambda \to \mathbb{C}$ is continuous and, for each $\lambda \in \Lambda$, $p(\cdot, \lambda)$ is a polynomial on $\mathbb{C}$.

Definition 2.2. *We say that $\{p(\cdot, \lambda)\}_{\lambda \in \Lambda}$ is Ω-stable if, for each $\lambda \in \Lambda$, $p(\cdot, \lambda)$ is Ω-stable, i.e. if $p(z, \lambda) \neq 0$ for $z \notin \Omega$ and $\lambda \in \Lambda$.*

We now state and prove a more general version of the *zero exclusion principle.*

Theorem 2.1. *Let $\Lambda \subset \mathbb{R}^m$ and $\{p(\cdot, \lambda)\}_{\lambda \in \Lambda}$ be a continuous family of polynomials on $\mathbb{C}$. Assume that the following conditions hold:*

(1) *Λ is connected and compact;*
(2) *$p(\cdot, \lambda)$ has degree $d \geq 1$ for each $\lambda \in \Lambda$;*
(3) *$p(\cdot, \lambda_0)$ is Ω-stable for some $\lambda_0 \in \Lambda$;*
(4) *$p(z, \lambda) \neq 0$ for each $\lambda \in \partial\Omega$ and each $\lambda \in \Lambda$.*

Then $\{p(\cdot, \lambda)\}_{\lambda \in \Lambda}$ is Ω-stable.

Proof. By assumption 2, we can assume, without loss of generality, that

$$p(z, \lambda) = z^d + \sum_{k=1}^{d} a_k(\lambda) z^{d-k},$$

where the $a_k : \Lambda \to \mathbb{C}$ are continuous $(1 \leq k \leq d)$. Hence, if $p(z, \lambda) = 0$ for some $z \in \mathbb{C}$ and $\lambda \in \Lambda$, we have

$$|z|^d \leq \sum_{k=1}^{d} \alpha_k |z|^{d-k},$$

where $\alpha_k = \max_{\lambda \in \Lambda} |a_k(\lambda)|$, $(1 \leq k \leq d)$. Consequently, $|z| \leq R$, where R is the positive root of the equation

$$R^d - \sum_{k=1}^{d} \alpha_k R^{d-k} = 0.$$

Let $\Omega_R = \Omega \cap B(R+1)$, with $B(R+1) \subset \mathbb{C}$ the open ball of centre 0 and radius $R+1$. By the above result and assumption 4, we have

$$p(z, \lambda) \neq 0 \text{ for each } (z, \lambda) \in \partial\Omega_R \times \Lambda.$$

Hence the Brouwer degree $\deg[p(\cdot, \lambda), \Omega_R, 0]$ [4] is well defined for each $\lambda \in \Lambda$, and its value is independent of λ. On the other hand, it is well known (see [4]) that $\deg[p(\cdot, \lambda), \Omega_R, 0]$ is equal to the number of zeros of $p(\cdot, \lambda)$ in Ω_R, counted with their multiplicities. By assumption 3 and the definition of Ω-stability, every possible zero of $p(\cdot,\lambda_0)$ lies in Ω, and hence in Ω_R. Consequently, $\deg[p(\cdot,\lambda_0),\Omega_R,0]=d$, and, by the homotopy invariance of Brouwer degree, $\deg[p(\cdot,\lambda),\Omega_R,0]=d$ for each $\lambda\in\Lambda$. This implies that all the zeros of $p(\cdot,\lambda)$ are in Ω_R, hence in Ω, and each $p(\cdot,\lambda)$ is Ω-stable.

Remark 2.1. It is easy to get rid of the assumption of compactness for Λ in Theorem 2.1.

3. The first Schur transform of a polynomial and its properties

To a polynomial $p(z)$ on $\mathbb{C}$, one can associate, with Schur [8], the polynomial on $\mathbb{C}$

$$p^*(z) := \overline{p(-\overline{z})},$$

that Zahreddine [9] calls the *paraconjugate* of p and that we will call the *first Schur transform* of p. Notice that

$$p^{**}(z) := (p^*)^*(z) = \overline{p^*(-\overline{z})} = \overline{\overline{p(z)}} = p(z),$$

and that, for $c\in\mathbb{C}$ and another polynomial q over $\mathbb{C}$, one has

$$(cp)^*(z) = \overline{c}p^*(z), \quad (p+q)^*(z) = p^*(z) + q^*(z).$$

Define, with Zahreddine [9],

$$N(z) = \frac{1}{2}\left[p(z) + (-1)^d p^*(z)\right], \quad D(z) = \frac{1}{2}\left[p(z) - (-1)^d p^*(z)\right], \tag{1}$$

so that

$$p(z) = N(z) + D(z), \tag{2}$$

$$p^*(z) = (-1)^d[N(z) - D(z)], \tag{3}$$

$$N^*(z) = \frac{1}{2}\left[p^*(z) + (-1)^d p(z)\right] = (-1)^d N(z),$$

$$D^*(z) = \frac{1}{2}\left[p^*(z) - (-1)^d p(z)\right] = (-1)^{d+1} D(z).$$

Lemma 3.1. *z_0 is a common zero to N and D if and only if z_0 and $-\overline{z_0}$ are zeros of p.*

Proof. If z_0 is a common zero to N and D, then, by (2) and (3), we have $p(z_0) = 0 = p(-\overline{z_0})$. If z_0 and $-\overline{z_0}$ are zeros of p, then $p(z_0) = 0 = p^*(z_0)$, and, by (1), we have $N(z_0) = D(z_0) = 0$.

Corollary 3.1. *Any zero of p which lies on the imaginary axis is a common zero of N and D.*

Proof. For such a zero z_0, one has $z_0 = -\overline{z_0}$.

The resultant [5] of two polynomials p and q over $\mathbb{C}$ will be denoted by $R[p,q]$.

4. The Routh-Hurwitz stability of a family of polynomials

Let $\Lambda \subset \mathbb{R}^m$ be compact and connected and let $\{p(\cdot,\lambda)\}_{\lambda\in\Lambda}$ be a continuous family of polynomials on $\mathbb{C}$ such that $p(\cdot,\lambda)$ has degree d for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, let

$$p(z,\lambda) = N(z,\lambda) + D(z,\lambda)$$

be the decomposition defined by (2). The following result generalizes, with a simpler proof, Theorem 3.1 of [9].

Theorem 4.1. *The family $\{p(\cdot,\lambda)\}_{\lambda\in\Lambda}$ is Routh-Hurwitz-stable if and only if $p(\cdot,\lambda_0)$ is Routh-Hurwitz-stable for some $\lambda_0 \in \Lambda$ and $R[N(\cdot,\lambda), D(\cdot,\lambda)] \neq 0$ for each $\lambda \in \Lambda$.*

Proof. *Necessity.* If $R[N(\cdot,\lambda_0), D(\cdot,\lambda_0)] = 0$ for some $\lambda_0 \in \Lambda$, then, by a classical result [5], $N(\cdot,\lambda_0)$ and $D(\cdot,\lambda_0)$ have a common zero z_0, so that, by Lemma 3.1,

$$p(z_0,\lambda_0) = 0 = p(-\overline{z_0},\lambda_0).$$

Now $\Re z_0 \geq 0$ if and only if $\Re(-\overline{z_0}) \leq 0$, and hence $p(\cdot,\lambda_0)$ is not Routh-Hurwitz-stable.

Sufficiency. Assume now that $R[N(\cdot,\lambda), D(\cdot,\lambda)] \neq 0$ for each $\lambda \in \Lambda$. Then, for each $\lambda \in \Lambda$, $N(\cdot,\lambda)$ and $D(\cdot,\lambda)$ have no common zeros, which, by Corollary 3.1, implies that $p(\cdot,\lambda)$ has no zero on the imaginary axis, which is the boundary of $\{z \in \mathbb{C} : \Re z < 0\}$. The conclusion follows then from Theorem 2.1.

We obtain immediately as a special case the Theorem 3.1 of Zahreddine. Let

$$p_0(z) = z^d + \sum_{k=1}^{d} a_k^0 z^{d-k}, \quad p_1(z) = z^d + \sum_{k=1}^{d} a_k^1 z^{d-k}$$

be two monic Routh-Hurwitz-stable polynomials. Let

$$p_j(z) = N_j(z) + D_j(z), \quad (j = 0,1)$$

be their respective decompositions (2) and let

$$p(z,\lambda) = (1-\lambda)p_0 + \lambda p_1, \quad (0 \leq \lambda \leq 1),$$

be their convex combinations. It is immediate to check that if, for each $\lambda \in [0,1]$,

$$p(z,\lambda) = N(z,\lambda) + D(z,\lambda)$$

is the decomposition (2) of $p(\cdot,\lambda)$, then

$$N(z,\lambda) = (1-\lambda)N_0(z) + \lambda N_1(z), \quad D(z,\lambda) = (1-\lambda)D_0(z) + \lambda D_1(z), \quad (0 \leq \lambda \leq 1).$$

Corollary 4.1. *Assume that p and q are Routh-Hurwitz-stable. Then their convex combinations $(1-\lambda)p + \lambda q$, $(\lambda \in [0,1])$, are Routh-Hurwitz-stable if and only if $R[(1-\lambda)N_0 + \lambda N_1, (1-\lambda)D_0 + \lambda D_1] \neq 0$ for each $\lambda \in]0,1[$.*

5. The second Schur transform of a polynomial and its properties

To a monic polynomial

$$p(z) = z^d + \sum_{k=1}^{d} a_k z^{d-k}$$

on $\mathbb{C}$, one can associate, with Schur [7], the polynomial on $\mathbb{C}$

$$p^{\#}(z) := z^d \overline{p\left(\frac{1}{\overline{z}}\right)},$$

that we may call the *second Schur transform* of p. Notice that

$$p^{\#\#}(z) := (p^{\#})^{\#}(z) = z^d \overline{p^{\#}\left(\frac{1}{\overline{z}}\right)} = z^d \overline{\frac{1}{\overline{z}^d}\overline{p(z)}} = p(z),$$

that $p^{\#}(0) = 1$ and that, for $c \in \mathbb{C}$ and another monic polynomial q over $\mathbb{C}$, one has

$$(cp)^{\#}(z) = \overline{c}p^{\#}(z), \quad (p+q)^{\#}(z) = p^{\#}(z) + q^{\#}(z).$$

Define, with Zahreddine [9],

$$H(z) = \frac{1}{2}\left[p(z) + p^{\#}(z)\right], \quad K(z) = \frac{1}{2}\left[p(z) - p^{\#}(z)\right], \tag{4}$$

so that

$$p(z) = H(z) + K(z), \tag{5}$$

$$p^{\#}(z) = H(z) - K(z), \tag{6}$$

$$H^{\#}(z) = \frac{1}{2}\left[p^{\#}(z) + p(z)\right] = H(z),$$

$$K^{\#}(z) = \frac{1}{2}\left[p^{\#}(z) - p(z)\right] = -K(z).$$

Lemma 5.1. *If z_0 is a common zero to H and K then $z_0 \neq 0$ and z_0 and $\frac{1}{\overline{z}_0}$ are zeros of p. If $z_0 \neq 0$ and $\frac{1}{\overline{z}_0}$ are zeros of p, then z_0 is a common zero to H and K.*

Proof. If z_0 is a common zero to H and K, then, by (5) and (6), we have $0 = p(z_0) = p^{\#}(z_0)$, so that $z_0 \neq 0$ and $p\left(\frac{1}{\overline{z}_0}\right) = 0$. If $z_0 \neq 0$ and $\frac{1}{\overline{z}_0}$ are zeros of p, then $p(z_0) = 0 = p^{\#}(z_0)$, and, by (4), we have $H(z_0) = K(z_0) = 0$.

Corollary 5.1. *Any zero z_0 of p which lies on the unit circle is a common zero of H and K.*

Proof. For such a zero z_0, one has $0 \neq z_0 = \frac{1}{\overline{z}_0}$.

6. The Schur-Cohn-stability of a family of polynomials

Let $\Lambda \subset \mathbb{R}^m$ be compact and connected and let $\{p(\cdot,\lambda)\}_{\lambda\in\Lambda}$ be a continuous family of polynomials on $\mathbb{C}$ such that $p(\cdot,\lambda)$ has degree d for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, let

$$p(z,\lambda) = H(z,\lambda) + K(z,\lambda)$$

be the decomposition defined by (5). The following result generalizes, with a simpler proof, Theorem 4.1 of [9].

Theorem 6.1. *The family $\{p(\cdot,\lambda)\}_{\lambda\in\Lambda}$ is Schur-Cohn-stable if and only if $p(\cdot,\lambda_0)$ is Schur-Cohn-stable for some $\lambda_0 \in \Lambda$ and $R[H(\cdot,\lambda),K(\cdot,\lambda)] \neq 0$ for each $\lambda \in \Lambda$.*

Proof. *Necessity.* If $R[H(\cdot,\lambda_0),K(\cdot,\lambda_0)] = 0$ for some $\lambda_0 \in \Lambda$, then, by a classical result [5], $H(\cdot,\lambda_0)$ and $K(\cdot,\lambda_0)$ have a common zero z_0, so that by Lemma 5.1, $z_0 \neq 0$ and

$$p(z_0,\lambda_0) = 0 = p\left(\frac{1}{\overline{z_0}},\lambda_0\right).$$

Now $|z_0| \geq 1$ if and only if $\left|\frac{1}{\overline{z_0}}\right| \leq 1$, and hence $p(\cdot,\lambda_0)$ is not Schur-Cohn-stable.

Sufficiency. Assume now that $R[H(\cdot,\lambda),K(\cdot,\lambda)] \neq 0$ for each $\lambda \in \Lambda$. Then, for each $\lambda \in \Lambda$, $K(\cdot,\lambda)$ and $H(\cdot,\lambda)$ have no common zeros, which, by Corollary 5.1, implies that $p(\cdot,\lambda)$ has no zero on the unit circle, which is the boundary of $B(1)$. The conclusion follows then from Theorem 2.1.

We obtain immediately as a special case the Theorem 4.1 of Zahreddine. Let

$$p_0(z) = z^d + \sum_{k=1}^{d} a_k^0 z^{d-k}, \quad p_1(z) = z^d + \sum_{k=1}^{d} a_k^1 z^{d-k}$$

be two monic Schur-Cohn-stable polynomials. Let

$$p_j(z) = H_j(z) + K_j(z), \quad (j = 0,1)$$

be their respective decompositions (5) and let

$$p(z,\lambda) = (1-\lambda)p_0 + \lambda p_1, \quad (0 \leq \lambda \leq 1),$$

be their convex combinations. It is immediate to check that if, for each $\lambda \in [0,1]$,

$$p(z,\lambda) = H(z,\lambda) + K(z,\lambda)$$

is the decomposition (5) of $p(\cdot,\lambda)$, then

$$H(z,\lambda) = (1-\lambda)H_0(z) + \lambda H_1(z), \quad K(z,\lambda) = (1-\lambda)K_0(z) + \lambda K_1(z), \quad (0 \leq \lambda \leq 1).$$

Corollary 6.1. *Assume that p and q are Schur-Cohn-stable. Then their convex combinations $(1-\lambda)p + \lambda q$, $(\lambda \in [0,1])$, are Schur-Cohn-stable if and only if $R[(1-\lambda)H_0 + \lambda H_1, (1-\lambda)K_0 + \lambda K_1] \neq 0$ for each $\lambda \in]0,1[$.*

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Université Catholique de Louvain
Institut mathématique, chemin du cyclotron, 2
B-1348 Louvain-la-Neuve, BELGIUM
E-mail: MAWHIN@AMM.UCL.AC.BE