ARCHIVUM MATHEMATICUM (BRNO)
Tomus 33 (1997), 127 – 138

ON THE VALLÉE—POUSSIN PROBLEM FOR SINGULAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Dedicated to the memory of Professor Otakar Borůvka

Abstract. For the differential equation

$$u^{(n)}(t) = f(t, u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))),$$

where the vector function $f :]a, b[\times \mathbb{R}^{kn} \to \mathbb{R}^k$ has nonintegrable singularities with respect to the first argument, sufficient conditions for existence and uniqueness of the Vallée–Poussin problem are established.

§1. Formulation of the Existence and Uniqueness Theorems

In the present paper for the vector differential equation with deviating arguments

$$u^{(n)}(t) = f(t, u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))) \tag{1.1}$$

we consider the multi-point boundary value problem of Vallée-Poussin

$$u^{(j-1)}(t_i) = 0 \qquad (j = 1, \dots, n_i; i = 1, \dots, m), \tag{1.2}$$

where $k \geq 1$, $n \geq m \geq 2$,

$$n_i \in \{1, \dots, n-1\}, \qquad \sum_{i=1}^{m} n_i = n, \qquad a = t_1 < t_2 < \dots < t_m = b,$$

$$\tau_j : [a, b] \to [a, b] \qquad (j = 1, \dots, n) \text{ are measurable,} \tag{1.3}$$

1991 *Mathematics Subject Classification*: 34K10.

Key words and phrases: singular differential equation with deviating arguments, the Valée-Poussin problem, existence theorem, uniqueness theorem.

Supported by the grant 201/96/0410 of the Grant Agency of the Czech Republic (Praque) and by the grant 619/1996 of the Development Fund of Czech Universities.

$$f(\cdot, x_1, \dots, x_n) :]a,b[\to \mathbb{R}^k \text{ is measurable for every } x_j \in \mathbb{R}^k$$
$$(j = 1, \dots, n) \quad \text{and } f(t, \cdot, \dots, \cdot) : \mathbb{R}^{nk} \to \mathbb{R}^k \text{ is continuous} \tag{1.4}$$
$$\text{for almost every} \quad t \in]a,b[.$$

Equation (1.1) is called *regular*, if $f(\cdot, x_1, \dots, x_n)$ is summable in $]a,b[$ for arbitrary fixed $x_j \in \mathbb{R}^k$ $(j = 1, \dots, n)$ and — *singular* otherwise.

Previously, the Vallée–Poussin problem was mostly considered for equations of the form

$$u^{(n)}(t) = f(t, u(t), u'(t), \dots, u^{(n-1)}(t))$$

both in the regular [2],[9]-[13] and singular [1], [3]-[8], [14] cases. As for equation (1.1), with $\tau_k(t) \not\equiv t$, $(k = 1, \dots, n)$ the problem was essentially left unexplored. In this paper we attempt to fill in this gap in a certain way.

We are interested mainly in the case when (1.1) is singular, although the results stated below are new also in the regular case.

The following notation is used throughout the paper:

$$n_{ij} = \begin{cases} n_i + 1 - j & \text{for} \quad j < n_i + 1 \\ 0 & \text{for} \quad j \geq n_i + 1 \end{cases};$$

$$\lambda_j(t) = \prod_{i=1}^{m} |t - t_i|^{n_{ij}} \qquad (j = 1, \dots, n);$$

if $\alpha \in [0, n - n_1]$, $\beta \in [0, n - n_m]$, then

$$n_{1j\alpha} = \begin{cases} n - \alpha - j & \text{for} \quad j \leq n_1, \ \alpha > n - n_1 - 1 \\ n_1 + 1 - j & \text{for} \quad j \leq n_1, \ \alpha \leq n - n_1 - 1 \\ 0 & \text{for} \quad n_1 < j \leq n - \alpha \\ n - \alpha - j & \text{for} \quad j > n - \alpha \end{cases},$$

$$n_{mj\beta} = \begin{cases} n - \beta - j & \text{for} \quad j \leq n_m, \ \beta > n - n_m - 1 \\ n_m + 1 - j & \text{for} \quad j \leq n_m, \ \beta \leq n - n_m - 1 \\ 0 & \text{for} \quad n_m < j \leq n - \beta \\ n - \beta - j & \text{for} \quad j > n - \beta \end{cases};$$

$$\lambda_{j\alpha\beta}(t) = \begin{cases} (t-a)^{n_{1j\alpha}}(b-t)^{n_{mj\beta}} \prod\limits_{i=2}^{m-1} |t - t_j|^{n_{ij}} & \text{for} \quad m > 2 \\ (t-a)^{n_{1j\alpha}}(b-t)^{n_{2j\beta}} & \text{for} \quad m = 2 \end{cases};$$

$g_0 : [a,b] \times [a,b] \to \mathbb{R}$ is the Green's function of the differential equation

$$u^{(n)} = 0 \tag{1.5}$$

with boundary conditions (1.2);

$$(1.6)\quad \gamma_{j\alpha\beta} = \sup\left\{(s-a)^{-\alpha}(b-s)^{-\beta}\lambda^{-1}_{j\alpha\beta}(t)\left|\frac{\partial^{j-1}g_0(t,s)}{\partial t^{j-1}}\right| : a < t, s < b\right\}$$
$$(j=1,\dots,n)^{1)};$$

$\mathbb{R}^k$ — space of k–dimensional column vectors $x = (\xi_j)_{j=1}^k$ with elements $\xi_j \in \mathbb{R}$ $(j = 1,\dots,k)$ and the norm

$$\|x\| = \sum_{j=1}^k |\xi_j|;$$

$\mathbb{R}^{k\times k}$ — the space of $k\times k$ matrices $X = (\xi_{ij})_{i,j=1}^k$ with elements $\xi_{ij} \in \mathbb{R}$ $(i,j = 1,\dots,k)$ and the norm

$$\|X\| = \sum_{ij=1}^k |\xi_{ij}|;$$

$\mathbb{R}^k_+ = \big\{(\xi_j)_{j=1}^k \in \mathbb{R}^k : \xi_j \ge 0\ (j = 1,\dots,k)\big\}$;
$\mathbb{R}^{k\times k}_+ = \big\{(\xi_{ij})_{i,j=1}^k \in \mathbb{R}^{k\times k} : \xi_{ij} \ge 0\ (i,j = 1,\dots,k)\big\}$;
if $x,\, y \in \mathbb{R}^k$ and $X,\, Y \in \mathbb{R}^{k\times k}$ then

$$x \le y \Leftrightarrow y - x \in \mathbb{R}^k_+;\ X \le Y \Leftrightarrow Y - X \in \mathbb{R}^{k\times k}_+;$$

if $x = (\xi_i)_{i=1}^k \in \mathbb{R}^k$ and $X = (\xi_{ij})_{i,j=1}^k \in \mathbb{R}^{k\times k}$ then

$$|x| = (|\xi_i|)_{i=1}^k,\ |X| = (|\xi_{ij}|)_{i,j=1}^k;$$

$\mathrm{r}(X)$ — spectral radius of the matrix X;
$\widetilde{C}^{n-1}_{\alpha\beta}(\,]a,b[\,;\mathbb{R}^k)$ — the space of vector functions $u : \,]a,b[\, \to \mathbb{R}^k$ absolutely continuous[2)] together with their derivatives up to order $n-1$ inclusive on any segment contained in $]a,b[$ and

$$\int_a^b (t-a)^\alpha (b-t)^\beta \|u^{(n)}(t)\|\mathrm{d}t < +\infty;$$

$\widetilde{C}^{n-1}([a,b];\mathbb{R}^k)$ — the space of vector functions $u : [a,b] \to \mathbb{R}^k$ absolutely continuous on $[a,b]$ together with their derivatives up to order $n-1$ inclusive.

[1)] By Lemma 2.5 in the monography [4], $\gamma_{j\alpha\beta} < +\infty$ for $0 \le \alpha \le n - n_1$, $0 \le \beta \le n - n_m$ $(j = 1,\dots,n)$.
[2)] Vector or matrix function is said to be absolutely continuous, measurable, etc. if all its components have such a property.

Clearly, we can consider $\widetilde{C}_{00}^{n-1}(\,]a,b[\,;\mathbb{R}^k)$ and $\widetilde{C}^{n-1}([a,b];\mathbb{R}^k)$ to be identical, since every element of $\widetilde{C}_{00}^{n-1}(\,]a,b[\,;\mathbb{R}^k)$ is a restriction to $]a,b[$ of some element of $\widetilde{C}^{n-1}([a,b];\mathbb{R}^k)$.

Along with (1.1) we will need to examine the vector differential inequality

$$|u^{(n)}(t)| \le \sum_{j=1}^{n} H_j(t)|u^{(j-1)}(\tau_j(t))|, \tag{1.7}$$

where $H_j :]a,b[\to \mathbb{R}_+^{k\times k}$ $(j = 1,\dots,n)$ are measurable matrix functions.

We say that problem (1.1), (1.2) (problem (1.7), (1.2)) has a solution in the space $\widetilde{C}_{\alpha\beta}^{n-1}(\,]a,b[\,;\mathbb{R}^k)$, if there is a vector function $u \in \widetilde{C}_{\alpha\beta}^{n-1}(\,]a,b[\,;\mathbb{R}^k)$, satisfying boundary conditions (1.2), which satisfies equation (1.1) (inequality (1.7)) almost everywhere in $]a,b[$.

Note that by $u^{(j-1)}(a)$ (by $u^{(j-1)}(b)$) in (1.2) we mean the right (left) limit of $u^{(j-1)}$ at a (at b).

Theorem 1.1. *Let the inequalities*

$$\tau_i(t) > a \quad \text{for} \quad i > n-\alpha, \qquad \tau_i(t) < b \quad \text{for} \quad i > n-\beta \ ^{3)} \tag{1.8}$$

and

$$|f(t,x_1,\dots,x_n)| \le \sum_{j=1}^{n} H_j(t)|x_j| + h(t) \tag{1.9}$$

hold in $]a,b[$ *and in* $]a,b[\,\times\mathbb{R}^{kn}$, *where* $H_j :]a,b[\to \mathbb{R}_+^{k\times k}$ $(j = 1,\dots,n)$ *and* $h :]a,b[\to \mathbb{R}_+^k$, *respectively, are measurable matrix and vector functions, which for some* $\alpha \in [0, n-n_1]$ *and* $\beta \in [0, n-n_m]$ *satisfy*

$$\int_a^b (t-a)^{\alpha}(b-t)^{\beta}\lambda_{j\alpha\beta}(\tau_j(t))\|H_j(t)\|dt < +\infty \qquad (j = 1,\dots,n), \tag{1.10}$$

$$\int_a^b (t-a)^{\alpha}(b-t)^{\beta}\|h(t)\|dt < +\infty. \tag{1.11}$$

Furthermore, let the problem (1.7), (1.2) admit only the trivial solution in the space $\widetilde{C}_{\alpha\beta}^{n-1}(\,]a,b[\,;\mathbb{R}^k)$. *Then the problem (1.1), (1.2) has at least one solution* $u \in \widetilde{C}_{\alpha\beta}^{n-1}(\,]a,b[\,;\mathbb{R}^k)$.

[3)] If $\alpha = 0$ $(\beta = 0)$ then the first (the second) inequality drops out.

Corollary 1.1. *Let the inequalities (1.8) hold in* $]a,b[$ *and the inequality (1.9) hold in* $]a,b[\times\mathbb{R}^n$, *where* $H_j :]a,b[\to\mathbb{R}_+^{k\times k}$ $(j=1,\dots,n)$ *and* $h :]a,b[\to\mathbb{R}_+^k$, *respectively, are measurable matrix and vector functions, which for some* $\alpha\in[0,n-n_1]$ *and* $\beta\in[0,n-n_m]$ *satisfy conditions (1.10) and (1.11). If, furthermore,*

$$r\left(\sum_{j=1}^{n}\gamma_{j\alpha\beta}\int_a^b (t-a)^\alpha(b-t)^\beta\lambda_{j\alpha\beta}(\tau_j(t))H_j(t)dt\right)<1, \tag{1.12}$$

then the problem (1.1), (1.2) has at least one solution $u\in\widetilde{C}^{n-1}_{\alpha\beta}(]a,b[;\mathbb{R}^k)$.

Corollary 1.2. *Let the inequality (1.9) hold in* $]a,b[\times\mathbb{R}^{kn}$, *where* $H_j :]a,b[\to\mathbb{R}_+^{k\times k}$ $(j=1,\dots,n)$ *are measurable matrix functions and* $h :]a,b[\to\mathbb{R}_+^k$ *is a summable vector function. Furthermore, let* $\lambda_j H_j$ $(j=1,\dots,n)$ *be summable and*

$$r\left(\sum_{j=1}^{n}\gamma_j\int_a^b \lambda_j(\tau_j(t))H_j(t)dt\right)<1,$$

where

$$\gamma_j=\frac{1}{(n-j)!}2^{n_{0j}}(b-a)^{n-j-\sum_{i=1}^m n_{ij}},\ n_{0j}=\min\{1,n_{1j}+n_{mj}\}\ (j=1,\dots,n).$$

Then the problem (1.1), (1.2) has at least one solution $u\in\widetilde{C}^{n-1}([a,b];\mathbb{R}^k)$.

Corollary 1.3. *Let the inequality (1.9) hold in* $]a,b[\times\mathbb{R}^{kn}$, *where* $H_j :]a,b[\to\mathbb{R}_+^{k\times k}$ $(j=1,\dots,n)$ *are measurable matrix functions and* $h :]a,b[\to\mathbb{R}_+^k$ *is a summable vector function. Furthermore, let there exist* $H_0\in\mathbb{R}_+^{k\times k}$ *such that* $r(H_0)<1$ *and the inequality*

$$\sum_{j=1}^{n}\gamma_j^*\lambda_j(\tau_j(t))H_j(t)\le H_0$$

hold in $]a,b[$, *where*

$$\gamma_j^*=\frac{1}{(n+1-j)!}(b-a)^{n+1-j-\sum_{i=1}^m n_{ij}}\qquad (j=1,\dots,n).$$

Then the problem (1.1), (1.2) has at least one solution $u\in\widetilde{C}^{n-1}([a,b];\mathbb{R}^k)$.

Theorem 1.2. *Let the inequalities (1.8) hold in* $]a,b[$ *and let the inequality*

$$|f(t,x_1,\dots,x_n)-f(t,y_1,\dots,y_n)|\le\sum_{j=1}^{n}H_j(t)|x_j-y_j| \tag{1.13}$$

hold in $]a,b[\times\mathbb{R}^{kn}$, *where* $H_j :]a,b[\to\mathbb{R}_+^{k\times k}$ $(j=1,\dots,n)$ *are measurable matrix functions satisfying the conditions of Theorem 1.1. Let further*

$$\int_a^b (t-a)^\alpha(b-t)^\beta\|f(t,0,\dots,0)\|dt<+\infty\,. \tag{1.14}$$

Then the problem (1.1), (1.2) has a unique solution in the space $\widetilde{C}^{n-1}_{\alpha\beta}(]a,b[;\mathbb{R}^k)$.

Corollary 1.4. *Let the inequalities (1.8) hold in* $]a,b[$ *and the inequality (1.13) hold in* $]a,b[\times\mathbb{R}^{kn}$, *where* $H_j :]a,b[\to\mathbb{R}_+^{k\times k}$ $(j=1,\dots,n)$ *are measurable matrix functions satisfying the conditions of Corollary 1.1. Let further the condition (1.14) be fulfilled. Then the problem (1.1), (1.2) has a unique solution in the space* $\widetilde{C}^{n-1}_{\alpha\beta}(]a,b[;\mathbb{R}^k)$.

Corollary 1.5. *Let*

$$\int_a^b \|f(t,0,\dots,0)\|dt < +\infty$$

and let the inequality (1.13) hold in $]a,b[\times\mathbb{R}^{kn}$, *where* $H_j :]a,b[\to\mathbb{R}_+^{k\times k}$ $(j=1,\dots,n)$ *are measurable matrix functions satisfying either the conditions of Corollary 1.2 or the conditions of Corollary 1.3. Then the problem (1.1), (1.2) has a unique solution in the space* $\widetilde{C}^{n-1}([a,b];\mathbb{R}^k)$.

Example. Consider the differential equation

$$u^{(n)}(t) = n!\prod_{i=1}^{m}(t_0-t_i)^{-n_i}u(t_0)+c_0, \tag{1.15}$$

where $t_0\in[a,b]\setminus\{t_1,\dots,t_m\}$, and $c_0\in\mathbb{R}^k$ is a nonzero vector. For (1.15) all conditions of Corollary 1.3 are satisfied, except the condition $\mathrm{r}(H_0)<1$, because in this case H_0 is the unit matrix. We will show that the problems (1.15), (1.2) have no solution. Assume, to the contrary, that there is a solution u. Since the right hand side of (1.15) is independent on t, u has the form

$$u(t) = c\prod_{i=1}^{m}(t-t_i)^{n_i},$$

where $c\in\mathbb{R}^k$. Substituting this into (1.15), we get $n!c = n!c+c_0$, which is impossible, because c_0 is a nonzero vector.

This example shows that the condition $\mathrm{r}(H_0)<1$ in Corollary 1.3 is optimal and cannot be replaced by $\mathrm{r}(H_0)\le 1$.

§2. Auxiliary Results

2.1. Lemmas on Apriori Estimates. For arbitrary $j\in\{1,\dots,n\}$ and $u\in\widetilde{C}^{n-1}_{\alpha\beta}(]a,b[;\mathbb{R}^k)$ we set

$$\alpha_{nj} = \begin{cases} \alpha-n+j & \text{for} \quad j>n-\alpha \\ 0 & \text{for} \quad j\le n-\alpha \end{cases},$$

$$\beta_{nj} = \begin{cases} \beta-n+j & \text{for} \quad j>n-\beta \\ 0 & \text{for} \quad j\le n-\beta \end{cases},$$

$$\nu(u)(t) = \sum_{j=1}^{n}(t-a)^{\alpha_{nj}}(b-t)^{\beta_{nj}}\|u^{(j-1)}(t)\| \qquad \text{for} \quad a<t<b,$$

$$\nu(u)(a) = \lim_{t\to a}\nu(u)(t), \qquad \nu(u)(b) = \lim_{t\to b}\nu(u)(t).$$

Note that $\nu(u) \in C([a,b];\mathbb{R}_+)$ for every $u \in \widetilde{C}^{n-1}_{\alpha\beta}(\,]a,b[\,;\mathbb{R}^k)$. Hence there exists

$$\nu^*(u) = \max\{\nu(u)(t) : a \le t \le b\}.$$

Using the definition of $\widetilde{C}^{n-1}_{\alpha\beta}(\,]a,b[\,;\mathbb{R}^k)$, it is easy to verify the following two lemmas.

Lemma 2.1. *Let $\alpha,\ \beta \in [0,n]$, $\rho_0 > 0$, and $h_0 : \,]a,b[\, \to \mathbb{R}_+$ be a measurable function, satisfying*

$$\int_a^b (t-a)^\alpha (b-t)^\beta h_0(t)dt < +\infty. \tag{2.1}$$

Then the operator ν maps the set

$$\Big\{u \in \widetilde{C}^{n-1}_{\alpha\beta}(\,]a,b[\,;\mathbb{R}^k) : \nu^*(u) \le \rho_0, \|u^{(n)}(t)\| \le h_0(t) \quad \text{for almost all} \quad t \in \,]a,b[\,\Big\}$$

into a compact subset of $C([a,b];\mathbb{R})$.

Lemma 2.2. *There exists a positive constant η, depending only on α, β, t_j and n_j $(j = 1,\dots,m)$, such that for any $\alpha \in [0, n-n_1]$, $\beta \in [0, n-n_m]$ and any vector function $u \in \widetilde{C}^{n-1}_{\alpha\beta}(\,]a,b[\,;\mathbb{R}^k)$, satisfying boundary condition (1.2), the following estimations hold:*

$$|u^{(j-1)}(t)| \le \eta\nu^*(u)\lambda_{j\,\alpha\beta}(t) \qquad \text{for} \quad a < t < b \quad (j = 1,\dots,n).$$

The following lemma is crucial in the proof of Theorem 1.1.

Lemma 2.3. *Let $\alpha \in [0, n-n_1]$, $\beta \in [0, n-n_m]$, $H_j : \,]a,b[\, \to \mathbb{R}^{k\times k}_+$ $(j = 1,\dots,n)$ and $h : \,]a,b[\, \to \mathbb{R}^k_+$ be measurable matrix and vector functions, satisfying conditions (1.10) and (1.11). Furthermore, let (1.8) hold in $]a,b[$ and let the problem (1.7), (1.2) have only the trivial solution in $\widetilde{C}^{n-1}_{\alpha\beta}(\,]a,b[\,;\mathbb{R}^k)$. Then there exists a positive number ρ such that any solution $u \in \widetilde{C}^{n-1}_{\alpha\beta}(]a,b[;\mathbb{R}^k)$ of the differential inequality*

$$|u^{(n)}(t)| \le \sum_{j=1}^n H_j(t)|u^{(j-1)}(\tau_j(t))| + h(t) \tag{2.2}$$

with boundary conditions (1.2) satisfies the estimations

$$\|u^{(j-1)}(t)\| \le \rho\lambda_{j\,\alpha\beta}(t) \qquad \text{for } a < t < b \quad (j = 1,\dots,n). \tag{2.3}$$

Proof. By Lemma 2.2, it is enough to prove that there is a positive number ρ_0 such that any solution $u \in \widetilde{C}^{n-1}_{\alpha\beta}(\,]a,b[\,;\mathbb{R}^k)$ of the problem (1.1), (1.2) satisfies the estimation

$$\nu^*(u) < \rho_0.$$

Assume there is no such ρ_0. Then for any natural number l there is a solution $u_l \in \widetilde{C}^{n-1}_{\alpha\beta}(\,]a,b[\,;\mathbb{R}^k)$ of the problem (1.1), (1.2) such that

$$\nu^*(u_l) > l\,.$$

Put

$$v_l(t) = [\nu^*(u_l)]^{-1} u_l(t)\,.$$

Then

$$\nu^*(v_l) = 1\,, \tag{2.4}$$

$$|v_l^{(n)}(t)| \le \sum_{j=1}^n H_j(t)|v_l^{(j-1)}(\tau_j(t))| + \frac{1}{l}h(t)\,. \tag{2.5}$$

On the other hand, by Lemma 2.2,

$$\|v_l^{(j-1)}(t)\| \le \eta\lambda_{j\,\alpha\beta}(t) \qquad \text{for} \quad a<t<b \quad (j=1,\dots,n). \tag{2.6}$$

Taking into account this and condition (1.8), we get from (2.5)

$$\|v_l^{(n)}(t)\| \le h_0(t) \qquad \text{for almost all} \quad t \in\,]a,b[, \tag{2.7}$$

where

$$h_0(t) = \eta\sum_{j=1}^n \lambda_{j\,\alpha\beta}(\tau_j(t))\|H_j(t)\| + \|h(t)\|$$

and h_0 satisfies (2.1), as follows from (1.10) and (1.11).

By (2.1), (2.6) and (2.7), the sequences $(v_l^{(j-1)})_{l=1}^\infty$ $(j=1,\dots,n)$ are uniformly bounded and equicontinuous on any segment contained in $]a,b[$. Therefore, by the lemma of Arcela–Ascoli, we can assume, without any loss of generality, that the sequences converge uniformely on any such segment.

Denote by

$$\lim_{l\to+\infty} v_l(t) = u(t) \qquad \text{for} \quad a<t<b\,.$$

Then

$$\lim_{l\to\infty} v_l^{(j-1)}(t) = u^{(j-1)}(t) \quad \text{uniformly on any segment in} \quad]a,b[. \tag{2.8}$$

By Lemma 2.1, from (2.4), (2.6)–(2.8) it follows that

$$\lim_{l\to\infty} \nu(v_l)(t) = \nu(u)(t) \quad \text{uniformly on } [a,b],$$

u satisfies the boundary conditions (1.2) and

$$\nu^*(u) = 1\,. \tag{2.9}$$

It follows from (2.5) that for any $s,\, t\in\,]a,b[$ we have

$$|v_l^{(n-1)}(t)-v_l^{(n-1)}(s)|\le\sum_{j=1}^n\left|\int_s^t H_j(\xi)|v_l^{(j-1)}(\tau_j(\xi))|\mathrm{d}\xi\right|+\frac{1}{l}\left|\int_s^t h(\xi)\,\mathrm{d}\xi\right|.$$

Passing to the limit for $l\to+\infty$ in these inequalities, then by conditions (1.10), (1.11), (2.6)–(2.8) and by the Lebesque dominance theorem we obtained

$$|u^{(n-1)}(t)-u^{(n-1)}(s)|\le\sum_{j=1}^n\left|\int_s^t H_j(\xi)|u^{(j-1)}(\tau_j(\xi))|\mathrm{d}\xi\right|,$$

$$\|u^{(n-1)}(t)-u^{(n-1)}(s)\|\le\|\int_s^t h_0(\xi)\mathrm{d}\xi\|.$$

Since s and t are arbitrary and (2.1) holds, it follows from these inequalities that $u\in\widetilde{C}^{n-1}_{\alpha\beta}(\,]a,b[\,;\mathbb{R}^k)$ and satisfies the vector differential inequality (1.7). On the other hand, as we have already observed, u satisfies conditions (1.2) and (2.9). But that is impossible, for the problem (1.7), (1.2) has no nontrivial solution in the space $\widetilde{C}^{n-1}_{\alpha\beta}(]a,b[;\mathbb{R}^n)$. This contradiction proves the lemma. □

2.2. Lemmas on Unique Solvability of the Problem (1.7), (1.2).

Lemma 2.4. *If the matrix functions $H_j:\,]a,b[\,\to\mathbb{R}^{k\times k}$ $(j=1,\dots,n)$ satisfy conditions of Corollary 1.1, then the problem (1.7), (1.2) has only the trivial solution in the space $\widetilde{C}^{n-1}(\,]a,b[\,;\mathbb{R}^k)$.*

Proof. Let $u\in\widetilde{C}^{n-1}_{\alpha\beta}(\,]a,b[\,;\mathbb{R}^k)$ be an arbitrary solution of the problem (1.7), (1.2). Put

$$v=\int_a^b(t-a)^\alpha(b-t)^\beta|u^{(n)}(t)|\mathrm{d}t.$$

Then by Lemma 8.4 and 8.5 from monography [4] and by (1.6)

$$u(t)=\int_a^b g_0(t,s)u^{(n)}(s)\mathrm{d}s$$

and

$$|u^{(j-1)}(t)|\le\gamma_{j\,\alpha\beta}\lambda_{j\,\alpha\beta}(t)v\quad\text{for}\quad a<t<b\quad(j=1,\dots,n).$$

From these estimates we find by (1.7)

$$v\le\left(\sum_{j=1}^n\gamma_{j\,\alpha\beta}\int_a^b(t-a)^\alpha(b-t)^\beta\lambda_{j\,\alpha\beta}(\tau_j(t))H_j(t)\mathrm{d}t\right)v$$

which implies, by (1.12), that $v=0$, i.e. $u(t)\equiv0$. □

Applying Lemma 8.7 and 8.8 from monography [4], one can easily verify the following

Lemma 2.5. *If the matrix functions $H_j:\,]a,b[\,\to\mathbb{R}^{k\times k}$ $(j=1,\dots,n)$ satisfy either conditions of Corollary 1.2 or conditions of Corollary 1.3, then the problem (1.7), (1.2) has only the trivial solution in the space $\widetilde{C}^{n-1}([a,b]:\mathbb{R}^k)$.*

§3. Proofs of the Existence and Uniqueness Theorems

Proof of Theorem 1.1. Let ρ be a positive constant for which conclusion of Lemma 2.3 holds. For arbitrary $t \in]a,b[$, $x_j \in \mathbb{R}^k$, $j \in \{1,\dots,n\}$ and a natural number l denote

$$\sigma_j(t,x_j) = \begin{cases} x_j & \text{for} \quad \|x_j\| \le \rho\lambda_{j\alpha\beta}(\tau_j(t)) \\ \frac{\rho}{\|x_j\|}\lambda_{j\alpha\beta}(\tau_j(t))x_j & \text{for} \quad \|x_j\| > \rho\lambda_{j\alpha\beta}(t) \end{cases}, \tag{3.1}$$

$$\delta_l(t) = \begin{cases} 1 & \text{for} \quad t \in \left[a+\frac{b-a}{2l}, b-\frac{b-a}{2l}\right] \\ 0 & \text{for} \quad t \notin \left[a+\frac{b-a}{2l}, b-\frac{b-a}{2l}\right] \end{cases}, \tag{3.2}$$

$$f_l(t,x_1,\dots,x_n) = \delta_l(t)f(t,\sigma_1(t,x_1),\dots,\sigma_n(t,x_n)) \tag{3.3}$$

and consider the differential equation

$$u^{(n)}(t) = f_l(t,u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t))) \tag{3.4}$$

with boundary conditions (1.2).

By (1.3), (1.4), (1.8)–(1.11) and (3.1) the vector function $f_l : [a,b]\times\mathbb{R}^{nk} \to \mathbb{R}^k$ belongs to the Caratheodory class and satisfies in $]a,b[\,\times\mathbb{R}^{nk}$ the inequalities

$$|f_l(t,x_1,\dots,x_n)| \le \sum_{j=1}^n H_j(t)|x_j| + h(t)\,, \tag{3.5}$$

$$\|f_l(t,x_1,\dots,x_n)\| \le h_l(t) \le h_0(t)\,, \tag{3.6}$$

where

$$h_0(t) = \rho\sum_{j=1}^n \lambda_{j\alpha\beta}(\tau_j(t))\|H_j(t)\| + \|h(t)\|\,, h_l(t) = \delta_l(t)h_0(t)\,,$$

h_0 satisfies condition (2.1) and h_l is summable in $[a,b]$. Applying the Schauder's principle and condition (3.6), it becomes clear the problem (3.4), (1.2) has a solution $u_l \in \widetilde{C}^{n-1}([a,b];\mathbb{R}^k)$ and

$$u_l(t) = \int_a^b g_0(t,s)f_l(s,u_l(\tau_1(s)),\dots,u_l^{(n-1)}(\tau_n(s)))ds. \tag{3.7}$$

Since $\widetilde{C}^{n-1}([a,b];\mathbb{R}^k) \subset \widetilde{C}^{n-1}_{\alpha\beta}(]a,b[;\mathbb{R}^k)$ we have $u_l \in \widetilde{C}^{n-1}_{\alpha\beta}(]a,b[;\mathbb{R}^k)$. On the other hand, by (3.5) and (3.6), for almost every $t \in]a,b[$ we have the inequality

$$\|u_l^{(n)}(t)\| = \|f_l(t,u_l(\tau_1(t)),\dots,u_l^{(n-1)}(\tau_n(t)))\| \le h_0(t) \tag{3.8}$$

and u_l is a solution to the problem (2.2), (1.2). Consequently, by the choice of ρ, we get the estimates

$$\|u_l^{(j-1)}(t)\| \le \rho\lambda_{j\,\alpha\beta}(t) \qquad \text{for} \quad a < t < b \quad (j = 1, \dots, n). \tag{3.9}$$

By (3.1), (3.8), (3.9) and the lemma of Arcela–Ascoli we can assume, without any loss of generality, that the sequences $(u_l^{(j-1)})_{l=1}^{\infty}$ $(j = 1, \dots, n)$ are uniformly convergent on every segment in $]a, b[$. Denote

$$u(t) = \lim_{l\to\infty} u_l(t). \tag{3.10}$$

Then by (3.1)–(3.3) and (3.9) for almost every $t \in\,]a, b[$ we have

$$\lim_{l\to\infty} f_l(t, u_l(\tau_1(t)), \dots, u_l^{(n-1)}(\tau_n(t))) = f(t, u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))). \tag{3.11}$$

On the other hand, by (1.6) and (3.8)

$$\|g_0(t,s) f_l(t, u_l(\tau_1(s)), \dots, u_l^{(n-1)}(\tau_n(s)))\| \le \gamma_{1\alpha\beta}(t)(s-a)^{\alpha}(b-s)^{\beta} h_0(s). \tag{3.12}$$

From conditions (3.1), (3.10)–(3.12) and from Lebesque's dominance theorem we obtain that

$$u(t) = \int_a^b g_0(t,s) f(s, u(\tau_1(s)), \dots, u^{(n-1)}(\tau_n(s)))\mathrm{d}s$$

and $\|u^{(n)}(t)\| \le h_0(t)$ for almost all $t \in\,]a, b[$. Consequently, $u \in \widetilde{C}^{n-1}_{\alpha\beta}(\,]a, b[\,; \mathbb{R}^k)$ is a solution of the problem (1.1), (1.2). □

In order to verify Theorem 1.2, it is enough to note that (1.13) and (1.14) implies (1.9) and (1.11), where $h(t) = f(t, 0, \dots, 0)$.

By Lemmas 2.4 and 2.5, Theorems 1.1, 1.2 imply Corollaries 1.1–1.5.

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