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# ON VERONESE-BORŮVKA SPHERES

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*Dedicated to the memory of Professor Otakar Borůvka*

Abstract. In this paper, history of reserches for minimal immersions from constant Gaussian curvature 2-manifolds into space forms is explained with special emphasis of works of O. Borůvka. Then recent results for the corresponding probrem to classify minimal immersions of such surfaces in complex space forms are discussed.

Let $S^n[K]$ be an $n$ dimensional sphere in $R^{n+1}$ of constant sectional curvature $K$. The Veronese surface in this talk means the mapping from $R^3$ into $R^5$ defined by

$$\begin{aligned} u^1 &= \frac{1}{\sqrt{3}}yz,\ u^2 = \frac{1}{\sqrt{3}}zx,\ u^3 = \frac{1}{\sqrt{3}}xy,\ u^4 = \frac{1}{2\sqrt{3}}(x^2-y^2), \\ u^5 &= \frac{1}{6}(x^2+y^2-2z^2),\ \text{where } (x,y,z)\in R^3. \end{aligned}$$

This gives an isometric minimal immersion from $S^2[\frac{1}{3}]$ into $S^4[1]$. For any positive integer $n$, the set of spherical harmonics with degree $n$ of three variables is a $(2n+1)$ dimensional vector space. Considering the unit sphere $S^{2n}[1]$ in the vector space for the standard inner product, we get an isometric minimal immersion :

$$S^2\left[\frac{2}{n(n+1)}\right] \longrightarrow S^{2n}[1].$$

In the early 1930's, Borůvka has studied structures of the second, third and higher order fundamental forms of this minimal surface in detail [3], [4], [5]. It is called now the Veronese-Borůvka sphere. After about 40 years of these Borůvka's reserches, Calabi [7] and Simons [17] revived the Borůvka spheres; in fact they reminded us that minimal submanifolds in spheres are important to study singularities of minimal varieties in a Euclidean space. The Veronese-Borůvka spheres

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show us concrete examples of minimal surfaces in the spheres. We know now many results characterizing them, see for example [15].

Minimal surfaces with constant Gaussian curvature in $S^n[1]$ are classified by Kenmotsu [11] when $n \leq 4$ and by Bryant [6] for all $n$ even locally.

**Theorem 1.** *Let $M^2[K]$ be a two dimensional Riemannian manifold of constant curvature $K$ and let $X : M^2[K] \longrightarrow S^n[1]$ be an isometric minimal immersion of $M^2[K]$ into $S^n[1]$. Then, when $K > 0$, the image is a part of the Veronese-Borůvka sphere in a totally geodesic $S^{2m}[1] \subset S^n[1]$; when $K = 0$, the image is a part of the generalized Clifford surface [10] in a totally geodesic $S^{2m+1}[1] \subset S^n[1]$; there is no minimal surface with $K < 0$ in a sphere.*

Wallach [18] has also proved this when $K > 0$. Theorem 1 says that if a minimal surface in a unit sphere has "simple" Riemannian structure, then the shape of the surface is explicitly determined.

Let us consider whether a Kaehler version of these results holds. $CP^n$ denotes an n dimensional complex projective space endowed with the Fubini-Study metric of constant holomorphic sectional curvature $4\rho$.

When $K$ is one of some positive rational numbers, Bando and Ohnita [1] constructed a family of isometric minimal immersions of $M^2[K]$ into $CP^n$. We call them complex Borůvka spheres. Bolton, Jensen, Rigoli and Woodward [2] and Chi and Zheng [8] also found out the same surfaces independently.

When $K = 0$, Ludden, Okumura and Yano [14] found out a flat minimal torus, $CT$, in $CP^2$ :

$$CT = \left\{(z_0, z_1, z_2) \in CP^2 : |z_0|^2 = |z_1|^2 = |z_2|^2\right\}$$

This is called the complex Clifford torus.

Later Kenmotsu [12] has classified all isometric and totally real minimal immersions of the flat Euclidean plane into $CP^n$; these are called the generalized complex Clifford surfaces.

It is not known untill now whether there are minimal surfaces with constant negative curvature in $CP^n$.

A minimal surface in a Kaehler manifold has an invariant of the first order, called the Kaehler angle $\alpha$ of the minimal surface; it is defined by $\cos\alpha = \langle Je_1, e_2\rangle$, where $\{e_1, e_2\}$ is an orthonormal basis on the surface in a Kaehler manifold and $J$ denotes the complex structure of the ambient space. The Kaehler angle can not be considered for surfaces in a sphere. All known minimal surfaces with constant Gaussian curvature in $CP^n$ have constant Kaehler angles.

Ohnita [16] proved that

**Theorem 2.** *Let $X : M^2[K] \longrightarrow CP^n$ be an isometric minimal immersion with constant Kaehler angle. Then, when $K > 0$, the image is a part of the complex Borůvka sphere in a totally geodesic $CP^m \subset CP^n$; when $K = 0$, the image is a part of the generalized complex Clifford surface in a totally geodesic $CP^m \subset CP^n$; when $K < 0$, there is no such an immersion even locally.*

The main purpose of this talk is to explain our recent result [13], in which we can classify minimal surfaces with constant Gaussian curvature in $CP^2$ without any other global or local assumptions of the surfaces [9], [8]. Namely we proved that

**Theorem 3.** *Let $X : M^2[K] \longrightarrow CP^2$ be an isometric minimal immersion. Then the Kaehler angle of $X$ is constant.*

Outline of the proof: Under the condition that $K$ is constant, we determine the covariant derivatives of the second fundamental form of $X$ explicitly. Using these, we see that the Kaehler angle $\alpha$ is an isoparametric function on the surface. That is, $\Delta\alpha$ and $|d\alpha|^2$ are functions of $\alpha$. These give an overdetermined system for $\alpha$. From its integrability condition, we see that there is a function of one variable which satisfies two ordinary differential equations. It shall be remarked that the coefficients of these ODE's are written in a precise way by elementary functions, although they are different from the sign of $K$.

Let us explain the system of these ODE's when $K > 0$. If there exists an isometric minimal immersion $X : M^2[K] \longrightarrow CP^2$ such that the Kaehler angle is not constant, then we have a non-constant function $y = y(x)$ of one variable which satisfies the following two ODE's on an interval of $(0, \pi)$:

$$\begin{aligned} &y''(x) + \cot x \cdot y'(x) - \cot y(x) \cdot y'(x)^2 + \frac{3\rho}{K} \sin 2x \cdot y'(x)^3 = 0\ , \\ &y''(x) - \cot x \cdot y'(x) - F_1(y(x)) y'(x)^2 + \frac{2}{K}(4\rho - K - 6\sin^2 x) \cot x \cdot y'(x)^3 \\ &\qquad\qquad + \frac{2}{K}(4\rho - K - 3\sin^2 x) F_1(y(x)) y'(x)^4 = 0, \end{aligned}$$

where we set

$$F_1(y) = \frac{c_2 + 3\sqrt{c_1}\cos y}{\sqrt{c_1}\sin y}$$

and $c_1,\ c_2$ are real constants.

For the case of $K = 0$, we have the following system:

$$\begin{aligned} &y''(x) + \cot x \cdot y'(x) + \frac{3\rho}{c_1} \sin 2x \cdot e^{2y(x)} y'(x)^3 = 0\ , \\ &y''(x) - \cot x \cdot y'(x) - (2 + \frac{c_2}{\sqrt{c_1}}) y'(x)^2 + \frac{4\rho}{c_1}(2 - \sin^2 x) \cot x \cdot e^{2y(x)} y'(x)^3 \\ &\quad + 2\rho \frac{(c_2 + 3\sqrt{c_1})}{c_1\sqrt{c_1}} (4 - \sin^2 x) e^{2y(x)} y'(x)^4 = 0. \end{aligned}$$

For the case of $K < 0$, we have the following system:

$$\begin{aligned} &y''(x) + \cot x \cdot y'(x) - \coth y(x) \cdot y'(x)^2 + \frac{3\rho}{L} \sin 2x \cdot y'(x)^3 = 0\ , \\ &y''(x) - \cot x \cdot y'(x) - F_2(y(x)) y'(x)^2 + \frac{2(4\rho + L - 6\rho\sin^2 x)}{L} \cot x \cdot y'(x)^3 \\ &\quad + \frac{2(4\rho + L - 3\rho\sin^2 x)}{L} F_2(y(x)) y'(x)^4 = 0\ , \end{aligned}$$

where we set

$$F_2(y) = \frac{(c_2 + 3\sqrt{c_1}\cosh y)}{\sqrt{c_1}\sinh y} .$$

We studied these systems in detail and proved that they did not have common solution except constants [13]. As a corollary of Theorem 2 and 3 , we proved that *there is no minimal surface with constant negative Gaussian curvature in* $CP^2$ *even locally.*

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