ARCHIVUM MATHEMATICUM (BRNO)
Tomus 33 (1997), 121 – 126

ON A PERTURBED NONLINEAR THIRD ORDER DIFFERENTIAL EQUATION

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Dedicated to the memory of Professor Otakar Borůvka

Abstract. In this paper we will study some asymptotic properties of a nonlinear third order differential equation viewed as a perturbation of a simpler nonlinear equation investigated recently by the authors in [4].

In a recent paper [4], the authors obtained some results on the asymptotic behaviour of solutions of a third order nonlinear differential equation of the form

$$u''' + q(t)u' + p(t)h(u) = 0 \tag{1}$$

and in doing so, generalized some results of [5] and [6] on the third order linear differential equations and supplemented some results of [1] for the case $q(t) \equiv 0$ and $p(t) < 0$.

Here we consider the third order nonlinear differential equation

$$y''' + q(t)y' + p(t)\,h(y) = g(t, y, y', y'') \tag{N}$$

viewed as a perturbation of (1).

The asymptotic behaviour of solutions of a differential equation similar to (N) was studied, for example, by Ezeilo [2]. He considered the equation

$$x''' + ax'' + bx' + f(x) = p(t, x, x', x'')\,,$$

where a, b are positive constants. The operator on the left-hand side can be transformed into the trinomial differential operator and so the last equation is a particular case of equation (N).

In this paper we will suppose that q, q' and p are continuous function of $t \in (a, \infty)$, $-\infty < a < \infty$, h is continuous functions of $u \in R$, and g is continuous on $(a, \infty) \times R^3$, $R = (-\infty, \infty)$.

1991 *Mathematics Subject Classification*: 34C10, 34C15.
Key words and phrases: asymptotic properties, nonlinear equation, perturbed third order differential equation.

The further suppositions on the functions h and g, used in this paper, are:

(i) $h(u)u > 0$ for $u \neq 0$,

(ii) $\lim_{u\to 0} \dfrac{h(u)}{u} = \theta$, $0 < \theta < \infty$ and $H(u) = \begin{cases} \dfrac{h(u)}{u} & \text{for } u \neq 0 \\ \theta & \text{for } u = 0 \end{cases}$,

(iii) $g(t, x_1, x_2, x_3)x_1 > 0$ for $x_1 \neq 0$.

Under the solution of equation (1) or (N) we will understand a function u or y defined on $[t_0, \infty)$ for some $t_0 > a$ that fulfills equation (1) or (N) on this interval. A solution of (1) or (N) defined on $[t_0, \infty)$, nontrivial in a neighborhood of infinity, will be called oscillatory on $[t_0, \infty)$ if it has infinite number of zeros on this interval with the limit point at infinity. Otherwise the solution is called nonoscillatory on $[t_0, \infty)$.

In the first part of the paper, we will interest ourselves in the case $p(t) < 0$ for $t \in (a, \infty)$ and for asymptotic properties of solutions of (N) we will use some results of [4] concerning equation (1) and we will obtain some results for nonoscillatory solutions of equation (N).

In the second part we will be interested in the asymptotic properties of oscillatory solutions of (N). Similar problems were posed and solved in the famous school of professor Borůvka, by M. Zlámal [6] and by M. Ráb [5] for the linear third order differential equation.

1. At the beginning of this section we introduce one auxiliary statement on linear equation and one of the results concerning equation (1), given in [4], as tools for obtaining main results.

Lemma 1. *Let* (i), (ii) *hold. Let* $p(t) < 0$, $q'(t) \geq 0$ *for* $t \in (a, \infty)$ *and let* $y_1 \in C((a, \infty))$. *Let further* u *be a solution of equation*

$$u''' + q(t)u' + p(t)H(y_1)u = 0 \tag{2}$$

defined on $[t_0, \infty)$ *with the property* $u(t_0) = u'(t_0) = 0$, $u''(t_0) > 0$. *Then* $u(t) > 0$ *for* $t > t_0$, $t_0 \in (a, \infty)$.

Proof. Multiply equation (2) with u and integrate it from t_0 to t. We obtain the integral identity

$$u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) + \int_{t_0}^{t} [p(\tau)H(y_1(\tau)) - \frac{1}{2}q'(\tau)]u^2(\tau)\, d\tau = c\,, \tag{3}$$

where $c = 0$ for the above solution u.

If we suppose that $u(t_1) = 0$ for some $t_1 > t_0$ we obtain the contradiction to the identity, because the expression under the integral sign is negative. □

Theorem A [4, Theorem 1]**.** *Let* (i), (ii) *hold for* $t \in (a, \infty)$ *and let* $q(t) \geq 0$, $q'(t) \geq 0$, $p(t) < 0$ *for* $t \in (a, \infty)$. *Let further* $H(u) \geq \theta > 0$ *for all* $u \in (-\infty, \infty)$ *and let* $\int_{t_0}^{\infty} \left[\frac{1}{2}q'(t) - \theta\, p(t)\right] dt = \infty$, $t_0 > a$. *Also, let* f *be a nonnegative* *function having a continuous third derivative in* (a,∞) *satisfying* $f'''(t)+q(t)f'(t)+p(t)\,\theta\,f(t)\le 0$ *for* $t\in[t_0,\infty)$. *Then given any solution* u_1 *of the differential equation* (1) *without any zero in* (α,∞), $t_0\le\alpha<\infty$, *there exist numbers* $k>0$, $\xi\ge\alpha$ *such that* $|u_1(t)|-kf(t)>0$ *for all* $t\in(\xi,\infty)$.

This result, in the case $f(t)>0$ for $t\in(a,\infty)$, will be used to prove our first theorem.

Theorem 1. *Suppose that the hypotheses of Theorem A are satisfied and* (iii) *holds. Then:*

a) *If* y_1 *is a solution of equation* (N) *defined on* $[T,\infty)$, $T>a$, *with the property* $y_1(t_0)=f(t_0)>0$, $y_1'(t_0)=f'(t_0)$, $y_1''(t_0)=f''(t_0)$, $t_0>T$, *then* $y_1(t)-f(t)>0$ *for all* $t>t_0$.

b) *If* $y_1(t_0)-f(t_0)=u(t_0)>0$, $y_1'(t_0)-f'(t_0)=u'(t_0)$, $y_1''(t_0)-f''(t_0)=u''(t_0)$ *and*

$$u(t_0)u''(t_0)-\frac{1}{2}u'^2(t_0)+\frac{1}{2}q(t_0)u^2(t_0)>0\,, \tag{4}$$

then $y_1(t)-f(t)>0$ *for all* $t>t_0$.

Proof.

a) Let y_1 fulfill the prescribed initial conditions. Then the function $u=y_1-f$ is the solution of the equation

$$\begin{aligned} u'''+q(t)u'+p(t)\,H(y_1(t))\,u &= g(t,y_1,y_1',y_1'')-\\ &\quad-[f'''+q(t)f'+p(t)\,H(y_1(t))f] \end{aligned} \tag{5}$$

and by the method of variation of constants we obtain for u the relation

$$u(t)= \tag{6}$$
$$=\int_{t_0}^{t}[g(\tau,y_1(\tau),y_1'(\tau),y_1''(\tau))-(f'''(\tau)+q(\tau)f'(\tau)+H(y_1(\tau))\,f(\tau))]\,W(t,\tau)\,d\tau$$

where $W(t,\tau)=\begin{vmatrix} u_1(t) & u_2(t) & u_3(t)\\ u_1(\tau) & u_2(\tau) & u_3(\tau)\\ u_1'(\tau) & u_2'(\tau) & u_3'(\tau)\end{vmatrix}$

and u_1, u_2, u_3 form the fundamental set of solutions of equation (2), where y_1 is the solution of (N) on $[T,\infty)$ and the wronskian of u_1, u_2, u_3 is equal to 1.

$W(\tau,t)>0$ for $t_0\le\tau<t$ by Lemma 1 and therefore the assertion of Theorem 1 in case a) follows from (6).

b) Let y_1 be a solution of (N) with the prescribed initial conditions and with condition (4).

Let $\bar u$ be the solution of (2) (where y_1 is the above solution of (N)) with the initial conditions $u(t_0)$, $u'(t_0)$, $u''(t_0)$. The condition (4) and the integral identity (3), where $c=u(t_0)u''(t_0)-\frac{1}{2}u'^2(t_0)+\frac{1}{2}q(t_0)u^2(t_0)\ge 0$ imply, that $\bar u(t)$ is positive for $t > t_0$. Denote by $u(t) = y_1(t) - f(t)$. Clearly $u(t)$ is the solution of (5) and by method of variation of constants, as in the case a) it can be written

$$u(t) = \bar{u}(t) + \\ + \int_{t_0}^{t} [g(\tau, y_1(\tau), y_1'(\tau), y_1''(\tau)) - (f'''(\tau) + q(\tau)f'(\tau) + H(y_1(\tau))f(\tau))]\, W(t,\tau)\, d\tau$$

and from this identity there follows the assertion of the case b) and the theorem is proved. □

Remark 1. From the proof of Theorem A [4] there follows that the suppositions of Theorem A are suppositions on the coefficients of the linear differential equation (2) and this fact will be directly used to prove our second theorem.

Theorem 2. *Suppose that the hypotheses of Theorem A are satisfied and* (iii) *holds. Then given any solution y_1 of the differential equation* (N) *without zeros in* (α, ∞), $t_0 \le \alpha < \infty$, *there exist numbers $k > 0$, $\xi \ge \alpha$ such that $|y_1(t)| - kf(t) > 0$ for all $t \in (\xi, \infty)$.*

Proof. Let y_1 be a solution of (N) defined on $[t_0, \infty)$ and let $y_1 \ne 0$ on (α, ∞), $\alpha \ge t_0$. Then it is also a solution of the third order linear differential equation

$$y''' + q(t)y' + p(t)\, H(y_1)y = g(t, y_1, y_1', y_1''),$$

which can be written for $t \in (\alpha, \infty)$ in the form

$$y''' + q(t)y' + \left[p(t)\, H(y_1) - \frac{g(t, y_1, y_1', y_1'')}{y_1}\right] y = 0\,. \tag{7}$$

This equation has the form (2) and the coefficients of equation (7) fulfill the suppositions of Theorem A and the proof of Theorem 2 follows directly from Theorem A. □

Corollary 1. Let the suppositions of Theorem 1 or 2 be satisfied and let $f(t) = e^t$, i.e. $1 + q(t) + p(t)\,\theta \le 0$ for $t \in (\alpha, \infty)$. Then the assertion of Theorem 1 or 2 holds for the case $f(t) = e^t$.

Example 1. Let us consider the differential equation

$$y''' - 2y = 2y(1 + y^2 + y'^2 + y''^2)\,.$$

This equation fulfills the suppositions of Corollary 1 and therefore certain its solutions without zeros (Theorem 1) diverge to infinity more rapidly as e^t and as it follows from Theorem 2 the absolute value of every its solution without zeros (if exists) diverges to infinity more rapidly than e^t.

2. In this section we derive sufficient conditions for the solutions of (N) to be nonoscillatory and the necessary condition for the solutions of (N) to be oscillatory, moreover, we derive asymptotic properties of oscillatory solutions of equation (N).

Let (ii) hold. Multiply equation (N) by y and integrate it from t_0 to t, $t_0, t > a$. We obtain

$$(8)\quad yy'' - \frac{1}{2}y'^2 + \frac{1}{2}qy^2 + \int_{t_0}^{t} [p(\tau)\, H(y(\tau)) - \frac{1}{2}q'(\tau)]\, y^2(\tau)\, d\tau = \\ = \int_{t_0}^{t} g(\tau, y(\tau), y'(\tau), y''(\tau))\, y(\tau)\, d\tau + k\,.$$

Theorem 3. *Let* (i), (ii), (iii) *hold and let* $q'(t) \geq 0$ *and* $p(t) < 0$ *for* $t \in (a, \infty)$. *Let* y *be a solution of* (N) *defined on* $[t_0, \infty)$ *with the property*

$$(9)\qquad y(t_0)\, y''(t_0) - \frac{1}{2}y'^2(t_0) + \frac{1}{2}q(t_0)\, y^2(t_0) \geq 0\,.$$

Then $y(t) \neq 0$ *for* $t > t_0$ *and every oscillatory solution of* (N) *has only simple zeros.*

Proof. The first assertion of Theorem 3 follows immediately from identity (8). From this assertion there follows that the necessary condition for the solution y of (N) defined on $[T, \infty)$, $T > a$ to be oscillatory is the condition

$$(10)\qquad y(t)y''(t) - \frac{1}{2}y'^2(t) + \frac{1}{2}q(t)\, y^2(t) < 0 \quad \text{for all} \quad t \geq T\,.$$

If $t_1 \geq T$ is a zero of y from the inequalities (10) and (9) there follows that $-\frac{1}{2}y'^2(t_1)$ must be negative and the Theorem 3 is proved. □

Remark 2. From the proof of Theorem 3 there follows that the condition (10) is necessary for the solution y of (N) to be oscillatory on $[T, \infty)$.

Note, that a function f defined on (a, ∞) belongs to the class L^2 if

$$\int_{\alpha}^{\infty} f^2(t)\, dt < \infty\,.$$

Then the following theorem holds.

Theorem 4. *Let* (i), (ii), (iii) *hold and let* $H(u) \geq \theta > 0$ *for all* $u \in R$. *Let further* $q(t) \leq 0$, $p(t) \leq m < 0$ *for* $t \in (a, \infty)$ *and* $\frac{1}{2}q'(t) - p(t)\,\theta \geq d > 0$ *for all* $t \in (a, \infty)$. *Then every oscillatory solution* y *of (N) defined on* $[t_0, \infty)$, $t_0 > a$, *belongs to the class* L^2 *and moreover* $\int_{t_0}^{\infty} g(t, y(t), y'(t), y''(t))\, y(t)\, dt < \infty$.

Proof. Let y be an oscillatory solution of (N) on $[t_0, \infty)$. Then in the integral identity (8) for this solution there is $k < 0$ and therefore from the necessary condition (10) and from this identity there follows, that the integrals

$$\int_{t_0}^{\infty} [p(\tau)\, H(y(\tau)) - \frac{1}{2}q'(\tau)] y^2(\tau)\, d\tau \quad \text{and} \quad \int_{t_0}^{\infty} g(\tau, y(\tau), y'(\tau), y''(\tau))\, y(\tau)\, d\tau$$

exist and from the supposition on p, H and q' there follows that $\int_{t_0}^{\infty} y^2(\tau)\, d\tau < \infty$ and the theorem is proved. □

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