ARCHIVUM MATHEMATICUM (BRNO)
Tomus 33 (1997), 41 – 56

ON SUPERMINIMAL SURFACES

Thomas Friedrich

Dedicated to the memory of Professor Otakar Borůvka

Abstract. Using the Cartan method O. Borůvka (see [B1], [B2]) studied superminimal surfaces in four-dimensional space forms. In particular, he described locally the family of all superminimal surfaces and classified all of them with a constant radius of the indicatrix. We discuss the mentioned results from the point of view of the twistor theory, providing some new proofs. It turns out that the superminimal surfaces investigated by geometers at the beginning of this century as well as by O. Borůvka have a holomorphic and horizontal lift into the twistor space. Global results concerning superminimal surfaces have been obtained during the last 15 years. In this paper we investigate superminimal surfaces in the hyperbolic four-spaces.

1. The Indicatrix of a Surface in a Four-Space.

Let (X^4, g) be a Riemannian manifold and consider an isometric immersion of a surface M^2 into X^4, $f : M^2 \to X^4$. We denote by $T(M^2)$ and $N(M^2)$ the tangent bundle and the normal bundle of the surface M^2, respectively. The bundle S of all symmetric (1,1)-tensors

$$A : TM^2 \to TM^2$$

is a 3-dimensional Euclidean vector bundle over M^2 with the inner product

$$< A, B >= Tr(A \cdot B).$$

The second fundamental form $\mathrm{II}(\vec{n}) : T_m(M^2) \to T_m(M^2)$ depending on a normal vector $\vec{n} \in N_m(M^2)$ is an element of the space S_m. For a fixed point $m \in M^2$ of the surface we define the indicatrix of the normal curvature by

$$I(m) = \{\mathrm{II}(\vec{n}) : \vec{n} \in N_m(M^2), ||\vec{n}|| = 1\}.$$

$I(m)$ is a closed curve contained in the 3-dimensional Euclidean space S_m, and it is the analogue of the Dupin indicatrix of an ordinary surface in the Euclidean

1991 *Mathematics Subject Classification*: 53C42, 53A35.
Key words and phrases: minimal surfaces, hyperbolic spaces.
Supported by the SFB 288 of the DFG.

3-space.

Proposition 1 (Kommerell 1897). *Let $f : M^2 \to X^4$ be an isometric immersion of a surface into a 4-dimensional Riemannian manifold X^4. Then the indicatrix of the normal curvature is one of the following curves:*

(1) *$I(m) = \{0\}$*
(2) *$I(m)$ is a stretch symmetric with respect to the origin $0 \in S_m$*
(3) *$I(m)$ is the intersection of a cylinder over an ellipse and a two-plane in S_m.*

The indicatrix $I(m)$ is a set of operators acting on the tangent space $T_m(M^2)$. We can evaluate this family of operators on a fixed tangent vector $\vec{t} \in T_m(M^2)$ and then we obtain a closed curve $I(m;\vec{t})$ in the tangent space $T_m(M^2)$,

$$I(m;\vec{t}) = \{\mathrm{II}(\vec{n})(\vec{t}) : \vec{n} \in N_m(M^2), ||\vec{n}|| = 1\}.$$

Fix an orthonormal basis $e_3, e_4 \in N_m(M^2)$ in such a way that the mean curvature $H(e_4)$ vanishes. We choose the tangent vectors $e_1, e_2 \in T_m(M^2)$ to be eigenvectors of the second fundamental form $\mathrm{II}(e_4)$. Then we obtain the matrix representations

$$\mathrm{II}(e_3) = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{pmatrix} \qquad \mathrm{II}(e_4) = \begin{pmatrix} \mu_1 & 0 \\ 0 & -\mu_1 \end{pmatrix},$$

and the indicatrix $I(m)$ is given by the formula

$$I(m) = \left\{ \begin{pmatrix} \lambda_1 \cos\varphi + \mu_1 \sin\varphi & \lambda_2 \cos\varphi \\ \lambda_2 \cos\varphi & \lambda_3 \cos\varphi - \mu_1 \sin\varphi \end{pmatrix} : 0 \le \varphi \le 2\pi \right\}.$$

We introduce the isometry $S_m \simeq R^3$ given by the formula

$$\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \simeq \left(\frac{a_1 + a_3}{\sqrt{2}}, \sqrt{2} a_2, \frac{a_1 - a_3}{\sqrt{2}} \right).$$

Then $I(m)$ as a curve in R^3 has the parametrization

$$I(m) = \left\{ \left(\frac{1}{\sqrt{2}} (\lambda_1 + \lambda_3) \cos\varphi, \sqrt{2}\lambda_2 \cos\varphi, \sqrt{2}\mu_1 \sin\varphi \right) : 0 \le \varphi \le 2\pi \right\}.$$

Suppose now that $f : M^2 \to X^4$ is a minimal immersion, i.e., the mean curvature vanishes for all normal vectors. Then $\lambda_1 = -\lambda_3$ and we obtain

Proposition 2 (Kommerell 1905, Eisenhart 1912). *The indicatrix of a minimal surface at each point is an ellipse, a circle, or a stretch.*

Consider a surface $f : M^2 \to X^4$ such that for any tangent vector $\vec{t} \in T_m(M^2)$ the curve $I(m;\vec{t}) \subset T_m(M^2)$ is a circle with center 0. An easy calculation yields the conditions $\lambda_1 = \lambda_3 = 0 \quad , \quad \lambda_2 = \pm\mu_1$. In particular, in this case the indicatrix $I(m)$ is the circle $\{(0, \pm\sqrt{2}\mu_1 \cos\varphi, \sqrt{2}\mu_1 \sin\varphi) : 0 \le \varphi \le 2\pi\}$.

Definition. *A surface $f : M^2 \rightarrow X^4$ is called superminimal if any curve $I(m, \vec{t})$ is a circle with center 0 ($\lambda_2 = \mu_1$).*

Proposition 3. *Any superminimal surface is a minimal surface. Its indicatrix at each point is a circle with center 0.*

Example (R-surfaces in R^4; Kommerell 1905). Let $U \subset \mathbb{C} \approx R^2$ be an open subset of the complex plain and let $f(z)$ be a holomorphic function. The graph of the function f

$$M^2 = \{(z, f(z)) : z \in U\}$$

is a superminimal surface of the Euclidean space R^4.

In case of a superminimal surface, the length $||\mathrm{II}(\vec{n})|| = \sqrt{2}\mu_1 = \sqrt{2}\lambda_2$ does not depend on the normal vector $\vec{n} \in N_m(M^2)$ and equals the radius of the indicatrix $I(m)$ at the point $m \in M^2$.

2. Superminimal Surfaces from the Point of View of Twistor Theory.

Let (X^4, g) be an oriented, 4-dimensional Riemannian manifold. Consider a point $x \in X^4$ and let Z_x be the set of all linear maps $J : T_x(X^4) \rightarrow T_x(X^4)$ satisfying the following conditions:

(1) $J^2 = -Id$
(2) J is compatible with the metric and preserves the orientation.
(3) If $\Omega(t_1, t_2) = g(Jt_1, t_2)$, then $-\Omega \wedge \Omega$ defines the orientation of X^4.

The set $Z = \bigcup_{x \in X^4} Z_x$ is a $P^1(\mathbb{C})$-fibre bundle over X^4 that is associated to the frame bundle of the oriented Riemannian manifold. Denote by π the projection into the bundle and consider the decomposition induced by the Levi-Civita connection of the tangent bundle of Z

$$T(Z) = T^v(Z) \oplus T^h(Z)$$

into the vertical and horizontal subspaces. There exists an almost complex structure $\mathcal{J}$ on Z preserving this decomposition and coinciding with the canonical complex structure on the fibres $SO(4)/U(2) = P^1(\mathbb{C})$. On the horizontal space $T^h_J(Z)$ at the $J \in Z, \mathcal{J}$ is defined by $\mathcal{J} = \pi_*^{-1} J \pi_*$. It is well-known that $(Z, \mathcal{J})$ is a complex manifold if and only if X^4 is self-dual, i.e., if one part of the Weyl tensor vanishes. The almost complex manifold $(Z, \mathcal{J})$ is called the twistor space of X^4.

Now consider an oriented, 2-dimensional manifold M^2 and an immersion $f : M^2 \rightarrow X^4$. Using the orientation of M^2 and X^4 we see that the spaces $T_m(M^2)$ and $N_m(M^4)$ are oriented, 2-dimensional Euclidean vector spaces. We define

$F(m) : T_{f(m)}(X^4) = T_m(M^2) \oplus N_m(M^2) \to T_m(M^2) \oplus N_m(M^2) = T_{f(m)}(X^4)$ by

$$F(m) = \text{rotation around the angle } \tfrac{\pi}{2} \text{ in the positive (negative) direction on } T_m(M^2) \text{ (on } N_m(M^2)).$$

Then $F : M^2 \to Z$ is a lift of the immersion $f : M^2 \to X^4$ into the twistor space,

$$\begin{tikzcd} & Z \arrow[d, "\pi"] \\ M^2 \arrow[ur, "F"] \arrow[r, "f"'] & X^4 \end{tikzcd}$$

Proposition 4 (see [F], §1). *An immersion $f : M^2 \to X^4$ is superminimal if and only if the lift $F : M^2 \to Z$ is horizontal, i.e.,*

$$dF(T(M^2)) \subset T^h(Z).$$

In this case the lift $F : M^2 \to Z$ is a holomorphic map. Conversely, let $F : M^2 \to Z$ be a holomorphic and horizontal immersion. Then $f := \pi \circ F : M^2 \to X^4$ is a superminimal immersion.

Now we give a further geometric characterization of superminimal immersions. This description is well-known in case of the Euclidean space (Kwietniewski 1902) and has been generalized in the paper [F]. First let us recall some linear-algebraic facts. Let V be a four-dimensional Euclidean vector space and consider two planes E and F in V. Then E is called *isoclinic* to F if the angle between $e \in E$ and its projection $pr_F(e)$ into F does not depend on $e \in E$. The relation has the following properties:

I.1.) If E is isoclinic to F, then F is isoclinic to E.

I.2.) E is isoclinic to F if and only if the projection $pr_F : E \to F$ is a conformal map.

I.3.) If E is isoclinic to F, then E is isoclinic to the orthogonal complement $F^\perp$.

Suppose now that V has a fixed orientation. If E is an oriented plane, we denote by $E^\perp$ the orthogonal complement with the orientation given by the condition $E \oplus E^\perp = V$. Two oriented planes E, F are called *oriented-isoclinic* if either $E = F^\perp$ (as oriented planes) or the projection $pr_F : E \to F$ is a non-trivial, conformal map preserving the orientations. Then we have the properties:

I.1.*) If E is oriented-isoclinic to F, then F is oriented-isoclinic to E.

I.2.*) If E is oriented-isoclinic to F, then $E^\perp$ is oriented-isoclinic to $F^\perp$.

I.3.*) If E is an oriented plane and F is a non-oriented plane such that E and F are isoclinic, then F admits exactly one orientation with respect to this E, and F are oriented-isoclinic.

In general it is not true that the condition "E is oriented isolclinic to F" implies "E is oriented-isoclinic to $+F^{\perp}$ or $-F^{\perp}$". Therefore, we define that an oriented plane E is *negatively oriented-isoclinic* to F if E is oriented-isoclinic to F and to $(-F^{\perp})$. The link between this relation and complex structures is given by the following

Lemma (see [F]). *Let E, F be two oriented planes in V and denote by $J^E : V \to V$ the map acting as the rotation around $\pi/2$ in the positive (negative) direction on E (on $E^{\perp}$). E is negatively oriented-isoclinic to F if and only if $J^E = J^F$.*

Consider an oriented surface $f : M^2 \to X^4$ in a 4-dimensional, oriented Riemannian manifold X^4. If γ is a curve in X^4, we denote by τ_γ the parallel displacement along γ in the tangent bundle $T(X^4)$. We say that M^2 is a *negatively oriented-isoclinic surface* if, for every curve γ in M^2 from x to y, the planes $\tau_{f\circ\gamma}(T_{f(x)}M^2)$ and $T_{f(y)}(M^2)$ are negatively oriented isoclinic planes in $T_{f(y)}(X^4)$. The mentioned geometric characterization of superminimal surfaces can be formulated now.

Proposition 5 (Kwietniewski 1902; [F]). *An immersion $f : M^2 \to X^4$ is superminimal if and only if it is negatively oriented-isoclinic.*

3. Superminimal Surfaces in Spaces of Constant Curvature.

Let X^4 be the Euclidean space R^4 (or, more generally, a space form). Denote by H^* the standard positive line bundle on $P^1(\mathbb{C})$. The twistor space Z of R^4 is isomorphic to $H^* \oplus H^*$. Therefore, we have a projection $p : Z \to P^1(\mathbb{C})$, the projection in the vector bundle $H^* \oplus H^*$.

$$\begin{array}{ccc} H^* \oplus H^* = & Z & \xrightarrow{\pi} R^4 \\ & \downarrow p & \\ & P^1(\mathbb{C}) & \end{array}$$

The map $p : Z = H^* \oplus H^* \to P^1(\mathbb{C})$ can also be described in the following way: Consider the twistor space $\pi : Z \to R^4$. Since R^4 is flat and simply connected, the parallel transport defines a fibration $p : \mathcal{Z} \approx \mathcal{Z}_x \approx P^1(\mathbb{C})$ of the twistor space over one of its fibres.

If $F : M^2 \to Z = H^* \oplus H^*$ is a holomorphic map, then $p \circ F : M^2 \to P^1(\mathbb{C})$ is a meromorphic function on M^2. Therefore, the holomorphic maps $F : M^2 \to \mathcal{Z} = H^* \oplus H^*$ correspond to the sets (g, s_1, s_2) such that

a.) $g : M^2 \to P^1(\mathbb{C})$ is a meromorphic function on M^2.

b.) s_1, s_2 are holomorphic sections of the bundle $g^*(H^*)$ over M^2.

$F = (g, s_1, s_2)$ is horizontal if and only if $dg = 0$, i.e., g is constant. Consequently, the superminimal immersions $f : M^2 \to R^4$ correspond to pairs (h_1, h_2) of holomorphic functions such that $|dh_1| + |dh_2| > 0$. A similar argument for spaces of constant curvature yields the following

Theorem 1 (Borůvka 1928). *Let $X^4(c)$ be a space of constant curvature. The family of superminimal immersions $f : M^2 \to X^4(c)$ depends (locally) on two holomorphic functions.*

In particular, the isoclinic surfaces $M^2 \hookrightarrow R^4$ are locally R-surfaces, i.e., graphs of holomorphic functions h (Eisenhart 1912).

1982 R. Bryant proved the following global existence results for superminimal surfaces in a space of positive constant curvature:

Theorem 2 (Bryant 1982). *Every compact Riemann surface M^2 admits a conformal, superminimal immersion into the sphere S^4.*

We sketch the idea of the proof. The twistor space of the sphere S^4 is the projective space $P^3(\mathbb{C})$. On the subset

$$\{[z_1 : z_2 : z_3 : z_4] \in P^3(\mathbb{C}) : z_1 \neq 0\} = \{[1 : z_2 : z_3 : z_4] \in P^3(\mathbb{C})\}$$

the horizontal distribution $T^h(P^3(\mathbb{C}))$ of the twistor fibration is defined by the equation

$$dz_2 - z_4 dz_3 + z_3 dz_4 = 0.$$

A general holomorphic and horizontal map $F : M^2 \to P^3(\mathbb{C})$ depends on two meromorphic functions $A, B : M^2 \to P^1(\mathbb{C})$:

$$F = \left[1 : A - \frac{1}{2} B \frac{dA}{dB} : B : \frac{1}{2} \frac{dA}{dB}\right].$$

Example. Consider a torus $T = \mathbb{C}/\Gamma$ and the Weierstrass function

$$p(z) = \frac{1}{z^2} + \sum_{\lambda \in \Gamma, \lambda \neq 0} \left[\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}\right].$$

With $A = p(z), B = p'(z)$ we obtain a holomorphic, horizontal immersion $F : T \to P^3(\mathbb{C})$ that defines a superminimal immersion $f : T \to S^4$. The Euler number of this immersion equals $e = -12$.

We study now the radius R of the indicatrix of a superminimal immersion $f : M^2 \to X^4(c)$. With respect to a local frame e_1, e_2, e_3, e_4 on the surface we have

$$\mathrm{II}(e_3) = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix} \qquad \mathrm{II}(e_4) = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix},$$

and this radius equals $R = \sqrt{2}\mu$.

Theorem 3 (Borůvka 1928). *Let $X^4(c)$ be a space of constant non-positive curvature. Then there is no superminimal immersion with constant radius $R = const > 0$.*

Proof. We sketch the proof using the Cartan method of moving frames. Denote by $\sigma_1, ...\sigma_4$ the dual frame to $e_1, ..., e_4$ and let $\omega_{ij} = < \nabla e_i, e_j >$ be the connection forms. The special form of the second fundamental form yields

$$\omega_{31} = \mu\sigma_2 \qquad \omega_{32} = \mu\sigma_1$$

$$\omega_{41} = \mu\sigma_1 \qquad \omega_{42} = -\mu\sigma_2.$$

From the structure equation of $X^4(c)$ restricted to M^2 we obtain

$$d\omega_{13} = \omega_{12} \wedge \omega_{23} + \omega_{14} \wedge \omega_{43} = -\mu\omega_{12} \wedge \sigma_1 - \mu\sigma_1 \wedge \omega_{43}.$$

However, $d\omega_{13} = -d(\mu\sigma_2) = -\mu d\sigma_2 = -\mu\omega_{21} \wedge \sigma_1$, and finally we conclude

$$2\mu\omega_{12} \wedge \sigma_1 = \mu\omega_{43} \wedge \sigma_1.$$

Using the form ω_{23} a similar calculation provides the equation

$$2\mu\omega_{12} \wedge \sigma_2 = \mu\omega_{43} \wedge \sigma_2.$$

This implies $2\omega_{12} = \omega_{43}$. On the other hand, we have

$$d\omega_{43} = \omega_{41} \wedge \omega_{13} + \omega_{42} \wedge \omega_{23} = -2\mu^2\ \sigma_1 \wedge \sigma_2$$

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} + \omega_{14} \wedge \omega_{42} - c\sigma_1 \wedge \sigma_2 = +2\mu^2\ \sigma_1 \wedge \sigma_2 - c\sigma_1 \wedge \sigma_2.$$

The equation $2\omega_{12} = \omega_{43}$ yields now $3\mu^2 = c$. □

Remark. The Gaussian curvature K of the surface is related to the radius $R^2 = 2\mu^2$ of the indicatrix by the formula $K = c - R^2$.

Remark. In case μ is not constant, we obtain the differential equations

$$*d\mu = -\mu(2\omega_{12} + \omega_{34})$$

and

$$-|d(\mu^2)|^2 + 4\mu^4(3\mu^2 - c) = \mu^2 \triangle (\mu^2)$$

for the radius $R^2 = 2\mu^2$ of the indicatrix of a superminimal surface $f : M^2 \hookrightarrow X^4(c)$.

The superminimal surfaces in S^4 with a constant radius $R > 0$ of the indicatrix were described by Borůvka:

Theorem 4 (Borůvka 1928). *A superminimal surface $f : M^2 \to S^4$ with a constant radius $R > 0$ of the indicatrix is a Veronese surface.*

We consider now a compact superminimal surface $f : M^2 \to X^4$. Then there is a link between the Euler number e of the normal bundle and the volume of the surface in case X^4 is a self-dual Einstein manifold.

Theorem 5 (see [F]). *Let X^4 be a self-dual Einstein space with scalar curvature τ and consider a superminimal immersion $f : M^2 \to X^4$ of a compact surface. Then*

$$e = \chi(M^2) - \frac{\tau vol(M^2)}{24\pi}$$

holds.

The Killing-Lipschitz curvature $G : N^1(M^2) \to R^4$ on the set $N^1(M^2)$ of all unit normal vectors of a superminimal immersion does not depend on the normal vector and is given by

$$G(\vec{n}) = -\mu^2 = -\frac{R^2}{2}.$$

Therefore, the total absolute curvature of the surface coincides with the mean value of R^2,

$$\int_{N^1} |G| = \pi \int_{M^2} R^2.$$

In case of a space X^4 of constant curvature, the mean value of R^2 is a topological invariant:

Theorem 6 (see [F]). *If $f : M^2 \to X^4(c)$ is a superminimal immersion of a compact surface M^2 into a space $X^4(c)$ of constant curvature, then*

$$e = \chi - \frac{c vol(M^2)}{2\pi} = -\frac{1}{2\pi}\int_{M^2} R^2.$$

4. Complete Superminimal Surfaces in the Hyperbolic Space H^4.

We identify the four-dimensional sphere S^4 with $\mathbb{C}^2 \cup \{\infty\}$ and use the coordinates (w_1, w_2). The twistor space of S^4 is the complex projective space $P^3(\mathbb{C})$. Let $[z_1 : z_2 : z_3 : z_4]$ be its homogeneous coordinates. The projection $\pi : P^3(\mathbb{C}) \longrightarrow S^4$ in the twistor bundle is given by

$$w_1 = \frac{-\bar{z}_2 z_3 + \bar{z}_4 z_1}{|z_3|^2 + |z_4|^2} \quad , \quad w_2 = \frac{\bar{z}_2 z_4 + \bar{z}_3 z_1}{|z_3|^2 + |z_4|^2}.$$

Note that the following formulas hold:

$$|w_1|^2 + |w_2| = \frac{|z_1|^2 + |z_2|^2}{|z_3|^2 + |z_4|^2},$$

$$z_1 = w_2 z_3 + w_1 z_4 \quad , \quad z_2 = -\bar{w}_1 z_3 + \bar{w}_2 z_4.$$

Therefore, the equations

$$dz_1 = w_2 dz_3 + w_1 dz_4 \quad , \quad dz_2 = -\bar{w}_1 dz_3 + \bar{w}_2 dz_4$$

describe the vertical bundle T^v of the twistor fibration. We consider the space of constant curvature $(c = 4, 0, -4)$

$$H^4(c) = \{(w_1, w_2) \in S^4 : 1 + \frac{c}{4}(|w_1|^2 + |w_2|^2) > 0\}$$

with the Riemannian metric

$$ds_c^2 = \frac{|dw_1|^2 + |dw_2|^2}{(1 + \frac{c}{4}(|w_1|^2 + |w_2|^2))^2}.$$

The analytic structure of the twistor space depends only on the conformal structure of the underlying 4-dimensional Riemannian manifold. Consequently, the twistor space $Z(c)$ of $H^4(c)$ coincides with the preimage $\pi^{-1}(H^4(c))$:

$$Z(c) = \{[z_1 : z_2 : z_3 : z_4] \in P^3(\mathbb{C}) : c(|z_1|^2 + |z_2|^2) + 4(|z_3|^2 + |z_4|^2) > 0\}.$$

In case $c \neq 0$, the twistor space $Z(c)$ admits a natural metric g_c such that $(Z(c), g_c) \xrightarrow{\pi} (H^4(c), ds_c^2)$ is a Riemannian submersion. The metric g_c is given by the formula

$$g_c = \frac{1}{b_c^2(\mathfrak{z}, \mathfrak{z})}\{b_c(\mathfrak{z}, \mathfrak{z}) b_c(d\mathfrak{z}, d\mathfrak{z}) - |b_c(d\mathfrak{z}, \mathfrak{z})|^2\},$$

where b_c denotes the Hermitian form in $\mathbb{C}^4$:

$$b_c(\mathfrak{z}, \mathfrak{z}^*) = c(z_1 \bar{z}_1^* + z_2 \bar{z}_2^*) + 4(z_3 \bar{z}_3^* + z_4 \bar{z}_4^*).$$

If $c = 4$, the metric g_c is the Fubini-Study metric of the projective space $P^3(\mathbb{C})$. In case $c = -4$, g_c is a pseudo-Riemannian metric of signature $(2, 4)$, and $(Z(-4), g_{-4})$ is an Einstein space as well as a (pseudo-) Kähler manifold. It is a matter of fact that the horizontal bundle T^h of the twistor space $Z(c)$ with respect to the metric ds_c^2 coincides with the g_c-orthogonal complement of T^v:

$$T^h(Z(c)) = \{\vec{t} \in T(Z(c)) : g_c(\vec{t}, \vec{t_1}) = 0 \qquad \text{for all} \quad \vec{t_1} \in T^v\}.$$

A direct calculation yields now the following result:

Proposition 6. *The horizontal distribution $T^h(Z(c))$ on the subset $z_1 \equiv 1$ is given by the equation*

$$cdz_2 + 4(-z_4 dz_3 + z_3 dz_4) = 0.$$

In particular, we consider the case of $c \leq 0$. The twistor space $Z(0)$ is a rank-two vector bundle over $P^1(\mathbb{C})$. Indeed, let

$$P^1(\mathbb{C}) = \{[z_1 : z_2 : z_3 : z_4] \in P^3(\mathbb{C}) : z_1 = z_2 = 0\}$$

and denote by $p : Z(0) \to P^1(\mathbb{C})$ the map $p[z_1 : z_2 : z_3 : z_4] = [0 : 0 : z_3 : z_4]$. A point in the dual Hopf bundle H^* is a pair $([z_3 : z_4], \xi)$, where $[z_3 : z_4] \in P^1(\mathbb{C})$ is a line in $\mathbb{C}^2$ and ξ is a linear map on this line. We identify $Z(0)$ with $H^* \oplus H^*$ via the map $\Psi : H^* \oplus H^* \longleftrightarrow Z(0)$,

$$\Psi([z_3 : z_4], \xi_1, \xi_2) = [\xi_1[z_3 : z_4] : \xi_2[z_3 : z_4] : z_3 : z_4].$$

Then, the diagram

$$\begin{tikzcd} H^* \oplus H^* \arrow[rr, "\Psi"] \arrow[dr] & & Z(0) \arrow[dl, "p"] \\ & P^1(\mathbb{C}) & \end{tikzcd}$$

commutes and the twistor space $Z(c) \quad (c < 0)$ corresponds to

$$Z(c) = \{(\xi_1, \xi_2) \in H^* \oplus H^* : |\xi_1|^2 + |\xi_2|^2 < -\frac{4}{c}\}.$$

Consequently, a holomorphic map $F : M^2 \to Z(c)$ is given by a meromorphic function $\Phi = p \circ F : M^2 \to P^1(\mathbb{C})$ and two holomorphic sections s_1, s_2 in the induced bundle $\Phi^*(H^*)$ such that

$$|s_1(m)|^2 + |s_2(m)|^2 < -\frac{4}{c}.$$

We fix the holomorphic section α in H^* given by $\alpha[z_3 : z_4] = z_4$.
The sections $s_1, s_2 \in \Gamma(\Phi^*(H^*))$ are multiples of $\Phi^*(\alpha)$,

$$s_1 = A\Phi^*(\alpha) \quad s_2 = B\Phi^*(\alpha).$$

$A, B : M^2 \to P^1(\mathbb{C})$ are meromorphic functions on the Riemann surface M^2, and the holomorphic map $F : M^2 \to H^* \oplus H^* = Z(0)$ can be written in the form

$$F = [A\Phi^4 : B\Phi^4 : \Phi^3 : \Phi^4] = [1 : \frac{B}{A} : \frac{1}{A}\Phi : \frac{1}{A}].$$

F is horizontal if and only if

$$cd\left(\frac{B}{A}\right) + 4\left(-\frac{1}{A}d\left(\frac{\Phi}{A}\right) + \frac{\Phi}{A}d\left(\frac{1}{A}\right)\right) = 0.$$

The equation is equivalent to

$$\frac{c}{4}A^2 d\left(\frac{B}{A}\right) = d\Phi$$

and, finally, we obtain

Theorem 7. *A conformal superminimal immersion $f : M^2 \to H^4(c) \quad (c \leq 0)$ is given by three meromorphic functions $A, B, \Phi : M^2 \to P^1(\mathbb{C})$ such that*

a.) $\frac{c}{4}A^2 d\left(\frac{B}{A}\right) = d\Phi$

b.) $|A|^2 + |B|^2 < -\frac{4}{c}\left(1 + |\Phi|^2\right)$.

It is easy to derive the formula for the immersion $f : M^2 \to H^4(c)$ depending on A, B, Φ:

$$f = \frac{1}{1 + |\Phi|^2}\left(-\bar{B}\Phi + A, \bar{B} + A\bar{\Phi}\right).$$

Example. Denote by $\alpha^2 \approx 0,630415$ the unique root of the polynomial $x^3 - 9x^2 - 9x + 9$ in the interval $[0,1]$. On the unit disk $M^2 = \{z \in \mathbb{C} : |z| < 1\}$ we consider the functions

$$A(z) = \alpha^2 z^2 \quad , \quad B(z) = \alpha z \quad , \quad \Phi(z) = \frac{1}{3}\alpha^3 z^3.$$

Then $-A^2 d\left(\frac{B}{A}\right) = d\Phi$ as well as $|A|^2 + |B|^2 < 1 + |\Phi(z)|^2$ hold for all $|z| < 1$. The map $F : M^2 \to P^1(\mathbb{C})$ is given by $F(z) = [\alpha^2 z : \alpha z : \frac{1}{3}\alpha^3 z^3 : 1]$. The metric $f^*(ds^2_{-4})$ induced by the corresponding immersion $f : M^2 \to H^4(-4)$ coincides with

$$F^*(g_{-4}) = \frac{1}{b^2_{-4}(F,F)}\{b_{-4}(F,F)b_{-4}(dF,dF) - |b_{-4}(dF,F)|^2\}.$$

A calculation of $F^*(g_{-4})$ yields the following result: $f^*(ds^2_{-4}) = G(|z|^2)\frac{|dz|^2}{(1-|z|^2)^2}$, where $\lim\limits_{|z| \to 1} G(|z|^2) = const \neq 0$. Since the hyperbolic metric $\frac{|dz|^2}{(1-|z|^2)^2}$ is a complete Riemannian metric on $M^2 = \{z \in \mathbb{C} : |z| < 1\}$, the metric $f^*(ds^2_{-4})$ is complete, too.

The formula for the superminimal immersion

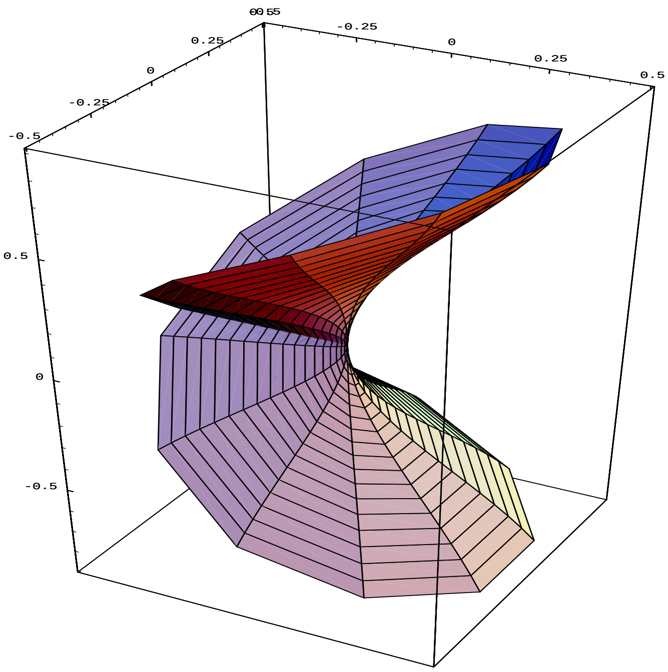
$$f : \{z \in \mathbb{C} : |z| < 1\} \longrightarrow H^4(-4) \subset \mathbb{C}^2$$

is

$$f(z) = \frac{1}{\alpha^6 |z|^6 + 9} \left(3\alpha^2 z^2 (3 - \alpha^2 |z|^2), 3\alpha \bar{z}(3 + \alpha^4 |z|^4)\right)$$

and f is a complete and superminimal embedding of the unit disk into $H^4(-4)$. We project $f(M^2) \subset H^4(-4)$ onto the 3-dimensional Euclidean space $R^3 = \mathbb{C} \oplus R \subset \mathbb{C}^2$. Then we obtain the following picture of this projected surface in the unit ball of R^3:

```
ParametricPlot3D[{3(0.793987)^2r^2Cos[2t]
(3-(0.793987)^2r^2)/((0.793987)^6r^6+9), 3(0.793987)^2r^2Sin[2t]
(3-(0.793987)^2r^2)/((0.793987)^6r^6+9),
3(0.793987)r Cos[t](3+(0.793987)^4r^4)/
((0.793987)^6r^6+9)},{r,0,1},{t,0,2Pi}]
```



In particular, we obtain

Theorem 8. *There are embedded, complete, simply-connected and superminimal surfaces $M^2 \subset H^4$ that are not totally geodesic.* □

Remark. The constructed surface is holomorphic to the unit disk. A surface of this type that is holomorphic to $\mathbb{C}$ (or to S^2) cannot exist since the curvature K of a superminimal surface is $K = -4 - 2\mu^2 \leq -4$. On the manifold $\mathbb{C} \approx R^2$, there are no complete Riemannian metrics such that $K \leq -4$ (Sattinger 1972).

The example explained above is a special case of a more general family of complete superminimal immersions. Let $Q(z)$ be a holomorphic function. We put

$$B(z) = z, \quad A(z) = 2zQ(z) + z^2Q'(z), \quad \Phi(z) = z^3Q'(z).$$

Then we have a solution of the differential equation $-A^2 d\left(\frac{B}{A}\right) = d\Phi$. Suppose now that the connected component Ω_Q of the domain defined by

$$\psi(z) := 1 + |z|^6|Q'(z)|^2 - |z|^2 - |z|^2|2Q(z) + zQ'(z)|^2 > 0$$

is a bounded domain with a smooth boundary and denote by $K_Q(z, z)$ its Bergman kernel. Then $K_Q(z, z)$ has the form (see [BFG])

$$K(z, z) = \frac{\varphi(z)}{\psi^2(z)} + \tilde{\varphi}(z) \log \psi(z),$$

where φ and $\tilde{\varphi}$ are smooth functions on $\bar{\Omega}_Q$, and $\varphi(z) \neq 0$ on $\partial\Omega_Q$. The immersion $f : \Omega_Q \to H^4$ is given by its lift $F : \Omega_Q \to Z(-4) \subset P^3(\mathbb{C})$,

$$F(z) = [2zQ(z) + z^2Q'(z) : z : z^3Q'(z) : 1]$$

and we obtain the following formula for the induced metric : $f^*(ds^2_{-4}) = -G(z)|dz|^2$, where

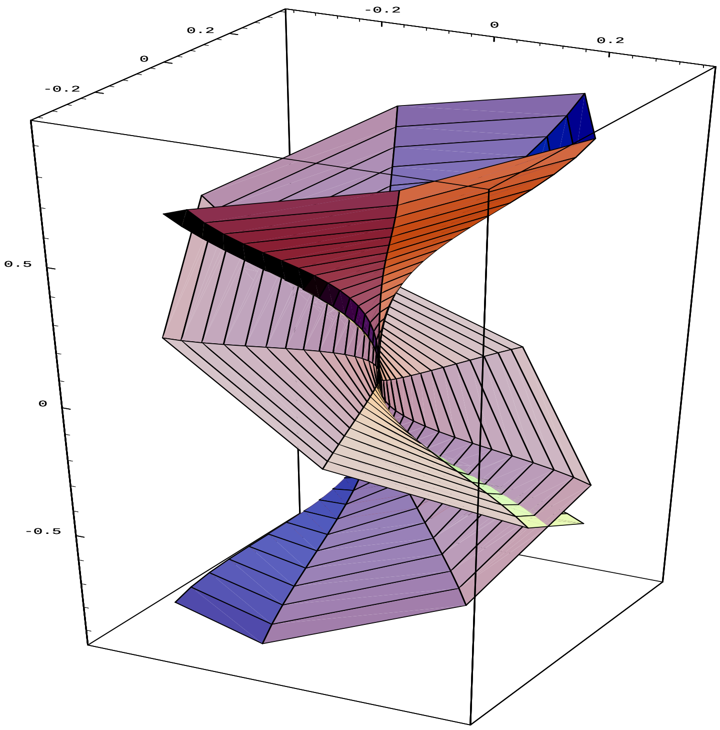
$$G(z) = \frac{1}{\psi^2(z)}\{\psi(z)b_{-4}(dF, dF) - |b_{-4}(dF, F)|^2\}.$$

If $z \to z_0 \in \partial\Omega_Q$, we have

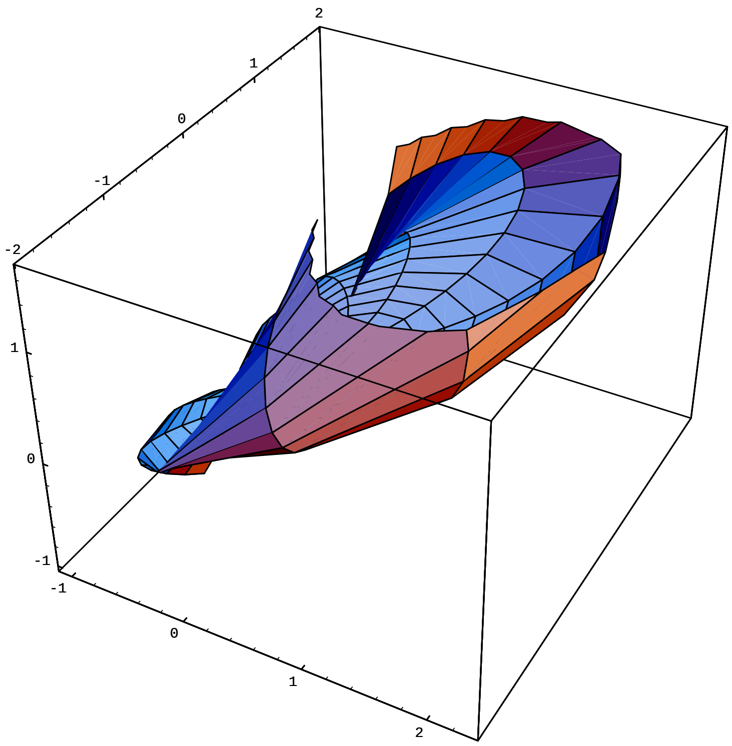
$$\lim_{z\to z_0} \frac{G(z)}{K(z, z)} = \lim_{z\to z_0} \frac{\psi(z)b_{-4}(dF, dF)(z) - |b_{-4}(dF, F)|^2(z)}{\varphi(z) + \psi^2(z)\tilde{\varphi}(z)\log\psi(z)} = -\frac{|b_{-4}(dF, F)|^2(z_0)}{\varphi(z_0)}.$$

The Bergman metric $K(z, z)|dz|^2$ is a complete metric on Ω_Q. Consequently, in case $|b_{-4}(dF, F)|^2 \neq 0$ on $\partial\Omega_Q$, the metric $f^*(ds^2_{-4})$ is a complete metric, too. This construction provides a whole family of complete superminimal immersions of the unit disk into the hyperbolic four-space H^4. The example explained above corresponds to the case $Q(z) = \frac{1}{3}$. In case we consider $Q(z) = z$ or $Q(z) = e^z$ for example, we obtain a superminimal surface in H^4 whose projection onto the 3-dimensional Euclidean space looks as follows:

```
ParametricPlot3D[{(-2r^5+4r^3)Cos[3t]/(r^8+4),
(-2r^5+4r^3)Sin[3t]/(r^8+4),(4r+2r^7)Cos[t]/(r^8+4)},
{r,0,0.841406},{t,0,2Pi}]
```




```
ParametricPlot3D[{((1-r^2)r^2Exp[r Cos[t]](Cos[r Sin[t]]Cos[2t]
-Sin[r Sin[t]]Sin[2t])+2r Exp[r Cos[t]](Cos[r Sin[t]]Cos[t]
-Sin[r Sin[t]]Sin[t]))/(1+r^6Exp[2r Cos[t]]),
((1-r^2)r^2Exp[r Cos[t]](Sin[r Sin[t]]Cos[2t]
+Cos[r Sin[t]]Sin[2t])+2r Exp[r Cos[t]](Sin[r Sin[t]]Cos[t]
+Cos[r Sin[t]]Sin[t]))/(1+r^6Exp[2r Cos[t]]),
(r Cos[t]+Exp[2r Cos[t]]r^4((2+Cos[t])Cos[2t]+r Sin[t]Sin[2t]))
/(1+r^6Exp[2r Cos[t]])},{t,0,2Pi},{r,0,1}]
```



The hyperbolic space H^4 does not contain compact minimal surfaces. On the other hand, the Riemann surface $M^2 = H^2/\Gamma$ is a totally geodesic (compact) submanifold in $X^4(c) = H^4/\Gamma$. It seems to be an open question whether or not there exist conformal, superminimal and non-totally geodesic immersions $f : M^2 \to X^4(c)$ of a compact Riemann surface into a (non-simply connected) space form $X^4(c)$. Compact surfaces of this type do not exist, we will return to this problem occasionally.

References

[B1] Borůvka, O., *Sur une classe de surfaces minima plone'es dans un espace á quatre dimensions a courbure constante*, C.R. Acad. Sci 187 (1928), 334-336.

[B2] Borůvka, O., *Sur une classe de surfaces minima plouge'es un espace á cinq dimension á courbure constante*, C.R. Acad. Sci. 187 (1928), 1271-1273.

[BFG] Beals, M., Fefferman, C., and Grossman, R., *Strictly pseudoconvex domains in* $\mathbb{C}^n$, Bull. Amer. Math. Soc., vol. 8 (1983), 125-326.

[Br] Bryant, R. L., *Conformal and minimal immersions of compact surfaces into the 4-sphere*, Journ. Diff. Geom. 17 (1982), 455-473.

[E] Eisenhart, L. P., *Minimal surfaces in Euclidean four-spaces*, Amer. Journ. of Math. 34 (1912), 215-236.

[F] Friedrich, Th., *On Surfaces in four-spaces*, Ann. Glob. Anal. and Geom. No. 3 (1984), 257-287.

[K.1] Kommerell, K., *Die Krümmung der zweidimensionalen Gebilde im ebenen Raum von vier Dimensionen*, Dissertation Tübingen 1897.

[K.2] Kommerell, K., *Riemannsche Flächen im ebenen Raum von vier Dimensionen*, Math. Ann. 60 (1905), 548-596.

[Kw] Kwietniewski, St., *Über Flächen des vierdimensionalen Raumes, deren sämtliche Tangentialebenen untereinander gleichwinklig sind, und ihre Beziehung zu ebenen Kurven*, Dissertation Zürich 1902.

[L] Lumiste, Ü., *On the theory of two-dimensional minimal surfaces I-IV*, Tartu Riikl. Ül. Tomestised vol. 102 (1961), 3-15 and 16-28, vol. 129 (1962), 74-89 and 90-102.

[S] Schiemangk, Chr. and Sulanke, R., *Submanifolds of the Möbius space*, Math. Nachr. 96 (1980), 165-183.

Institut für Reine Mathematik
Humboldt-Universität zu Berlin
Sitz: Ziegelstr. 13a
10099 Berlin, GERMANY
E-mail: friedric@mathematik.hu-berlin.de