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ON SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH "COMMON ZERO" AT INFINITY

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Dedicated to the memory of Professor Otakar Borůvka

Abstract. The zeros $c_k(\nu)$ of the solution $z(t,\nu)$ of the differential equation $z''+q(t,\nu)\,z=0$ are investigated when $\lim\limits_{t\to\infty} q(t,\nu)=1$, $\int^\infty |q(t,\nu)-1|\,dt<\infty$ and $q(t,\nu)$ has some monotonicity properties as $t\to\infty$. The notion $c_\kappa(\nu)$ is introduced also for κ real, too. We are particularly interested in solutions $z(t,\nu)$ which are "close" to the functions $\sin t$, $\cos t$ when t is large.

We derive a formula for $dc_\kappa(\nu)/d\nu$ and apply the result to Bessel differential equation, where we introduce new pair of linearly independent solutions replacing the usual pair $J_\nu(t), Y_\nu(t)$. We show the concavity of $c_\kappa(\nu)$ for $|\nu|\ge\frac12$ and also for $|\nu|<\frac12$ under the restriction $c_\kappa(\nu)\ge\pi\nu^2(1-2\nu)$.

1. Introduction.

Almost 50 years ago O. Borůvka introduced the function $\varphi_1(t)$, known as the first dispersion which can be defined as the first right zero of a solution of the second order differential equation

$$z''+q(t)\,z=0 \tag{1.1}$$

vanishing at t. This function is studied in [1] and is connected to the transformation theory of (1.1). The method, using similar transformation as in [1] is used e.g. in [6] to study distribution of zeros and certain quantities, connected to zeros.

In case when the coefficient $q(t)$ in (1.1) involves some parameter ν the solutions depend on the parameter and the zeros can be considered also as functions of this parameter (see e.g. [5]). Up to now the ususal approach was to fix one finite zero of a solution for all $\nu\in J$. Our aim here is to "send" this fixed zero to infinity.

This paper is concerned with the differential equation

$$z''+q(t,\nu)\,z=0,\qquad t\in I=(0,\infty),\quad \nu\in J,\quad {}'=\frac{d}{dt} \tag{1.2}$$

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which is oscillatory at infinity but not oscillatory at $t = 0$. There is defined also the zero $c_\kappa(\nu)$ for any real κ as a function of a parameter ν and derived some its general properties.

Then application is made to the Bessel differential equations, namely the concavity of $c_\kappa(\nu)$ is proved almost for all ν and $t > 0$ and certain new properties of Bessel function are derived.

The function $f(t)$ is said to be of class $M_n(0,\infty)$, briefly M_n or monotonic of order n on $(0,\infty)$, if it possess n ($n \geq 0$) continuous derivatives satisfying

$$(-1)^i f^{(i)}(t) \geq 0 \qquad \text{for} \quad t > 0 \quad \text{and} \quad i = 0, 1, \dots, n. \tag{1.3}$$

2. Preliminary results.

Consider the family of differential equations (1.2) where we assume

$$\lim_{t\to\infty} q(t,\nu) = 1, \qquad \int^\infty |q(t,\nu) - 1|\, dt < \infty \qquad \text{for } \nu \in J. \tag{2.1}$$

Let

$$q' \in M_2 \quad (\text{or } q \in M_3) \quad \text{and} \quad q(t,\nu_1) - q(t,\nu_2) \in M_2 \quad \text{for} \quad \nu_1 > \nu_2. \tag{2.2}$$

Let $x = x(t,\nu)$, $y = y(t,\nu)$ denote a pair of solutions of (1.2) such that their Wronskian

$$w(x,y) := xy' - x'y = 1 \quad \text{for all} \quad \nu \in J\,.$$

It is known (see e.g. [5], [6]) that the function $v(t) := x^2 + y^2$ complies with the so called Mammana identity

$$\mathcal{M}(v) := v''v - \frac{1}{2}v'^2 + 2qv^2 = 2 \tag{2.3}$$

which is the first integral of the Appel equation

$$\mathcal{A}(v) := v''' + 4qv' + 2q'v = 0. \tag{2.4}$$

From [6], [7], [2] follows that the assumptions (2.1) and (2.2) imply the existence of certain exceptional unique solution (principal solution) $v = v(t,\nu)$ of (2.4) such that $v \in M_1$, (or $v' \in M_0$), $[v(t,\nu_1) - v(t,\nu_2)] \in M_1$ for $\nu_1 > \nu_2$ and

$$\lim_{t\to\infty} v(t,\nu) = 1, \qquad \int^\infty |1 - \frac{1}{v(t,\nu)}|\, dt < \infty. \tag{2.5}$$

Consider now the set of solutions $z(t,\nu)$ of (1.2) having common zero at $t = c_0$ for all $\nu \in J$. Such solution can be expressed (see e.g. in [6], [7]) as

$$z(t,\nu) = \text{const}\,\sqrt{v(t,\nu)}\sin\Big(\int_{c_0}^t \frac{1}{v(s,\nu)}\, ds\Big) \tag{2.6}$$

with some const$\neq 0$. From this it is clear that the zero of this solution $z(t,\nu)$ next to c_0 occurs where the relation

$$\int_{c_0}^{t} \frac{1}{v(s,\nu)}\,ds = \pi$$

holds. This will be the first zero $c_1(\nu)$. The notion of the second zero $c_2(\nu)$, the third zero $c_3(\nu)$ and so on, as a function of ν is natural. We can extend this notion of $c_k(\nu)$ with $k = 0, 1, \dots$ to $c_\kappa(\nu)$ (see [8]) for $\kappa \in \mathbb{R}$ by the relation

$$\int_{c_0}^{c_\kappa(\nu)} \frac{ds}{v(s,\nu)} = \kappa\pi. \tag{2.7}$$

The notion of noninteger κ as index was introduced for the Bessel functions in [3], [4].

Differentiating (2.7) with respect to ν, we get

$$c'_\kappa(\nu) = \frac{d}{d\nu}\,c_\kappa(\nu) = -Q(c_\kappa(\nu),\nu), \tag{2.8}$$

where

$$Q(t,\nu) = -v(t,\nu)\int_{c_0}^{t} \frac{\partial v(s,\nu)/\partial\nu}{v^2(s,\nu)}\,ds. \tag{2.9}$$

3. Differential equation for the function $Q(t,\nu)$.

Lemma 3.1. *Let $v(t,\nu)$ complies with the Mammana identity (2.3) and let the function $q(t,\nu)$ be continuously differentiable with respect to ν. Then the function $Q(t,\nu)$ defined in (2.9) is a solution of the inhomogeneous third order differential equation*

$$\mathcal{A}(Q) = 2\frac{\partial}{\partial\nu}q(t,\nu) \tag{3.1}$$

where the operator $\mathcal{A}$ is defined in (2.4).

Proof. In the sequel we make use of the abbreviation $v_\nu = v_\nu(t,\nu) = \partial v(t,\nu)/\partial\nu$. Differentiating $Q(t,\nu)$ in (2.9) with respect to t, we obtain

$$\begin{aligned}
Q' &= -v'\int_{c_0}^{t} \frac{v_\nu}{v^2}\,ds - \frac{v_\nu}{v}\\
Q'' &= -v''\int_{c_0}^{t} \frac{v_\nu}{v^2}\,ds - \frac{v'_\nu}{v}\\
Q''' &= -v'''\int_{c_0}^{t} \frac{v_\nu}{v^2}\,ds - \frac{v''v_\nu + v''_\nu v - v'_\nu v'}{v^2}
\end{aligned}$$

hence

$$\mathcal{A}(Q) = -\mathcal{A}(v)\int_{c_0}^{t} \frac{v_\nu}{v^2}\,ds \;-\frac{1}{v^2}(v''v_\nu + vv_\nu'' - v_\nu'v' + 4qv_\nu v). \tag{3.2}$$

By our assumption $\mathcal{M}(v) \equiv 2$ which implies $\mathcal{A}(v) = 0$. Differentiating (2.3) with respect to ν, we have

$$\frac{\partial}{\partial\nu}\mathcal{M}(v) = v_\nu''v + v''v_\nu - v'v_\nu' + 2q_\nu v^2 + 4qv\,v_\nu = 0,$$

hence the value of $\mathcal{A}(Q)$ in (3.2) is reduced to (3.1), which proves Lemma 3.1. □

Consider the following pair x, y of solutions of (1.2):

$$x = \sqrt{v}\cos\Big(\int_{c_0}^{t}\frac{ds}{v(s)}\Big), \qquad y = \sqrt{v}\sin\Big(\int_{c_0}^{t}\frac{ds}{v(s)}\Big). \tag{3.3}$$

Direct calculation shows that their Wronskian

$$w(x,y) = x(t,\nu)y'(t,\nu) - x'(t,\nu)y(t,\nu) = 1\,.$$

Differentiation with respect to ν gives

$$\begin{aligned} x_\nu &= \frac{v_\nu}{2\sqrt{v}}\cos\Big(\int_{c_0}^{t}\frac{ds}{v}\Big) + \sqrt{v}\int_{c_0}^{t}\frac{v_\nu}{v^2}\,ds\,\sin\Big(\int_{c_0}^{t}\frac{ds}{v}\Big),\\ y_\nu &= \frac{v_\nu}{2\sqrt{v}}\sin\Big(\int_{c_0}^{t}\frac{ds}{v}\Big) - \sqrt{v}\int_{c_0}^{t}\frac{v_\nu}{v^2}\,ds\cos\Big(\int_{c_0}^{t}\frac{ds}{v}\Big)\,, \end{aligned} \tag{3.4}$$

hence by (2.9)

$$\begin{vmatrix} x & y \\ x_\nu & y_\nu \end{vmatrix} = -v(t,\nu)\int_{c_0}^{t}\frac{v_\nu(s,\nu)}{v^2(s,\nu)}\,ds = Q(t,\nu), \tag{3.5}$$

which is in accordance with [5], where the function $c(\nu)$ is defined as a zero of the linear combination $\cos\alpha\,x(t,\nu) + \sin\alpha\,y(t,\nu)$ with fixed α.

Let us check what happens if in the representation (3.3) c_0 varies. We observe that c_0 may tend to 0 if the integral $\int_{+0} ds/v$ is convergent. But this is equivalent to the fact that the solutions of (1.2) are not oscillatory at $t = 0$, and indeed, we have this assumption. In case $c_0 = 0$ the pair of solutions in (3.3) becomes

$$x_0(t,\nu) = \sqrt{v(t,\nu)}\cos\Big(\int_0^t\frac{ds}{v(s,\nu)}\Big),\; y_0(t,\nu) = \sqrt{v(t,\nu)}\sin\Big(\int_0^t\frac{ds}{v(s,\nu)}\Big), \tag{3.6}$$

and correspondingly

$$Q_0(t,\nu) = -v(t,\nu)\int_0^t\frac{v_\nu(s,\nu)}{v^2(s,\nu)}ds. \tag{3.7}$$

However, c_0 can not be replaced by ∞ in (3.3) because the integral $\int^\infty ds/v$ is divergent. Owing to our choice of $v(t,\nu)$ as principal one, by (2.5) we can use the fact that the integral $\int^\infty (1-1/v)ds$ is convergent (see [7]). Let $N(\nu)$ and $\varphi(t,\nu)$ be defined by

$$N(\nu) = \int_0^\infty \left(1 - \frac{1}{v(t,\nu)}\right) dt, \quad \varphi(t,\nu) = t + \int_t^\infty \left(1 - \frac{1}{v(s,\nu)}\right) ds, \tag{3.8}$$

then

$$\int_0^t \frac{ds}{v(s,\nu)} = \varphi(t,\nu) - N(\nu). \tag{3.9}$$

Let us introduce the new pair of solutions of (1.2)

$$C(t,\nu) = \sqrt{v}\cos\varphi(t,\nu), \qquad S(t,\nu) = \sqrt{v}\sin\varphi(t,\nu). \tag{3.10}$$

From here it is clear that the zeros of $C(t,\nu)$ and the ones of $\cos t$ will be asymptotically equal when $t \to \infty$. The same observation is true for the zeros of $S(t,\nu)$ and the function $\sin t$, too. Owing to (3.8) we have $\frac{\partial\varphi}{\partial\nu} = \int_t^\infty (v_\nu/v^2)\, ds$, hence

$$Q_1(t,\nu) = \begin{vmatrix} C & S \\ \frac{\partial}{\partial\nu}C & \frac{\partial}{\partial\nu}S \end{vmatrix} = v(t,\nu)\int_t^\infty \frac{v_\nu(s,\nu)}{v^2(s,\nu)} ds. \tag{3.11}$$

Let us mention that the convergence of the integral $\int_t^\infty v_\nu(s,\nu)/v^2(s,\nu)ds$ follows from [7].

The relation between the pairs of the solutions $x_0(t,\nu)$, $y_0(t,\nu)$ and $C(t,\nu)$, $S(t,\nu)$ can be established by making use of (3.9):

$$\begin{aligned} x_0(t,\nu) &= \cos(N(\nu))\, C(t,\nu) + \sin(N(\nu))\, S(t,\nu), \\ y_0(t,\nu) &= \cos(N(\nu))\, S(t,\nu) - \sin(N(\nu))\, C(t,\nu), \end{aligned} \tag{3.12}$$

or equivalently

$$\begin{aligned} C(t,\nu) &= \cos(N(\nu))\, x_0(t,\nu) - \sin(N(\nu))\, y_0(t,\nu), \\ S(t,\nu) &= \sin(N(\nu))\, x_0(t,\nu) + \cos(N(\nu))\, y_0(t,\nu). \end{aligned} \tag{3.13}$$

In the same way we find from (3.6), (3.11)

$$Q_1(t,\nu) - Q_0(t,\nu) = v(t,\nu)\frac{d\,N(\nu)}{d\nu}. \tag{3.14}$$

Since by (3.11) we have $\lim_{t\to\infty} Q_1(t,\nu) = 0$, we obtain from (2.5), (3.14)

$$\frac{d\,N(\nu)}{d\nu} = -\lim_{t\to\infty} Q_0(t,\nu). \tag{3.15}$$

4. Application to the Bessel differential equation.

The transformed Bessel differential equation which is relevant to (1.2) has the form

$$Z'' + \left(1 - \frac{\nu^2 - 1/4}{t^2}\right) Z = 0 \qquad t > 0 \tag{4.1}$$

and a principal pair of its solutions is $\sqrt{\frac{\pi t}{2}}\, J_\nu(t)$, $\sqrt{\frac{\pi t}{2}}\, Y_\nu(t)$, where $J_\nu(t)$, $Y_\nu(t)$ are the standard Bessel functions of order ν (see [9]). The function $v(t, \nu)$ given by

$$v(t, \nu) = \frac{\pi t}{2}[J_\nu^2(t) + Y_\nu^2(t)] \tag{4.2}$$

is the principal solution of the corresponding Appel equation (see [6]) and the Nicholson formula ([9], p. 444)

$$v(t, \nu) = \frac{4t}{\pi}\int_0^\infty K_0(2t\mathrm{sinh}s)\mathrm{cosh}(2\nu s)\, ds \tag{4.3}$$

provides an efficient tool for the investigations.

In particular, for $\nu = 1/2$ we have the pair of solutions $\sin t$ and $-\cos t$, accordingly and $v(t, 1/2) \equiv 1$.

The differential equation (4.1) is singular but not oscillatory at $t = 0$ — except $\nu = \pm 1/2$ when there is no singularity at all — and $\lim_{t\to 0+} J_\nu(t)/Y_\nu(t) = 0$, $\lim_{t\to 0+} Y_\nu(t) = -\infty$, which imply the representation in the spirit of (3.6)

$$x_0(t, \nu) = -\sqrt{\frac{\pi t}{2}}\, Y_\nu(t), \quad y_0(t, \nu) = \sqrt{\frac{\pi t}{2}}\, J_\nu(t). \tag{4.4}$$

By (3.6) the function $Q_0(t, \nu)$ in (3.7) is

$$Q_0(t, \nu) = \frac{t\pi}{2}\left(J_\nu(t)\tfrac{\partial}{\partial\nu}Y_\nu(t) - Y_\nu(t)\tfrac{\partial}{\partial\nu}J_\nu(t)\right)$$

hence by the Watson formula [9, p. 445]

$$Q_0(t, \nu) = -2t\int_0^\infty K_0(2t\sinh s)\, e^{-2\nu s}\, ds. \tag{4.5}$$

Making use of the integral ([9, p. 388])

$$\int_0^\infty K_0(u)u^{\mu-1}du = 2^{\mu-2}\Gamma^2(\frac{\mu}{2}), \tag{4.6}$$

we obtain by (4.5)

$$\lim_{t\to\infty} Q_0(t, \nu) = -\int_0^\infty K_0(u)\, du = -\frac{\pi}{2}, \quad \nu \in \mathbb{R} \tag{4.7}$$

which will be the main ingradient in proving the next theorem.

Theorem 4.1. *In the case of Bessel differential equation* (4.1) *the function* $N(\nu)$ *defined by* (3.8) *has the form*

$$N(\nu) = \frac{\pi}{2}(|\nu| - \frac{1}{2}). \tag{4.8}$$

Proof. By the definition of $N(\nu)$ in (3.8) and by (4.3) we find $N(-\nu) = N(\nu)$, i.e. $N(\nu)$ is even function of ν. Hence it will be sufficient to consider the case $\nu > 0$.

From (3.15), (4.7) we have $N'(\nu) = \pi/2$, hence $N(\nu) = \pi\nu/2 + \text{const}$. Particularly, for $\nu = 1/2$ it is $v(t, 1/2) \equiv 1$, hence in (3.8) we find $N(1/2) = 0$ which supplies the value of constant for $N(\nu)$. □

In possesion of the formula (4.8), the formulas (3.13), (4.4) suggest the following pair of solutions of the Bessel differential equation:

$$\begin{aligned} S_\nu(t) &= -\sin(\frac{\pi}{2}(\nu - \frac{1}{2}))Y_\nu(t) + \cos(\frac{\pi}{2}(\nu - \frac{1}{2}))J_\nu(t) \\ C_\nu(t) &= -\cos(\frac{\pi}{2}(\nu - \frac{1}{2}))Y_\nu(t) - \sin(\frac{\pi}{2}(\nu - \frac{1}{2}))J_\nu(t). \end{aligned} \tag{4.9}$$

For noninteger ν we have also

$$\begin{aligned} S_\nu(t) &= \frac{\sin\frac{\nu+1/2}{2}\pi J_\nu(t) + \sin\frac{\nu-1/2}{2}\pi J_{-\nu}(t)}{\sin\nu\pi}, \\ C_\nu(t) &= -\frac{\cos\frac{\nu+1/2}{2}\pi J_\nu(t) - \cos\frac{\nu-1/2}{2}\pi J_{-\nu}(t)}{\sin\nu\pi}. \end{aligned}$$

It is not difficult to check that $S_{-\nu}(t) = S_\nu(t)$, $C_{-\nu}(t) = C_\nu(t)$. This symmetry with respect to ν is reflected a little also by (4.3) where clearly the relation $v(t, -\nu) = v(t, \nu)$ holds.

Now we are going to investigate the zero $c_\kappa(\nu)$ of the linear combination $\cos\alpha\, S_\nu(t) + \sin\alpha\, C_\nu(t)$. By the above mentioned symmetry we have that the function $c_\kappa(\nu)$ is even function and it exists for $-\nu_0(\kappa) < \nu < \nu_0(\kappa)$ with some $\nu_0(\kappa)$. By (3.8) we have the impicite equation for $c_\kappa(\nu)$:

$$c_\kappa(\nu) + \int_{c_\kappa(\nu)}^{\infty} (1 - \frac{1}{v(s,\nu)})ds = \kappa\pi.$$

Here the left hand side expression is a stricly increasing function of $c = c_\kappa(\nu) > 0$ for fixed ν, therefore by (3.8), (4.8)

$$\kappa\pi > \int_0^{\infty} (1 - \frac{1}{v(s,\nu)})ds = N(\nu) = \frac{\pi}{2}(|\nu| - \frac{1}{2})$$

which implies that $\nu_0(\kappa) = 2\kappa + \frac{1}{2}$ and $c_\kappa(\nu)$ is defined on $-(2\kappa + \frac{1}{2}) < \nu < 2\kappa + \frac{1}{2}$, moreover $c_\kappa(\nu)$ exists only for $\kappa > -\frac{1}{4}$.

Another observation can be made also. In case $\kappa > 0$ $c_\kappa(\nu)$ is defined also at $\nu = \frac{1}{2}$ and recalling the fact that $v(t, \frac{1}{2}) \equiv 1$ we obtain the relation $c_\kappa(\frac{1}{2}) = \kappa\pi$, too.

On the other hand, by making use of the asymptotic expansions of the functions $J_\nu(t)$ and $Y_\nu(t)$ for large values of t from [9, p.199], we may obtain

$$c_\kappa(\nu) = \kappa\pi + \frac{1-4\nu^2}{8\kappa\pi} + \mathcal{O}(\frac{1}{\kappa^3}) \quad \text{as } \kappa \to \infty.$$

Due to the symmetry $c_\kappa(\nu) = c_\kappa(-\nu)$, we can restrict our investigations to the interval $[0, \nu_0(\kappa))$.

By (2.8), (3.11) we have $dc/d\nu = -Q_1(c, \nu)$, and by (3.14), (4.8) $Q_1(t, \nu) = Q_0(t, \nu) + \frac{\pi}{2} v(t, \nu)$, and making use of the Watson formula (4.5) and the Nicholson formula (4.3) we get

$$Q_1(t, \nu) = 2t \int_0^\infty K_0(2t\sinh s)\sinh(2\nu s)\, ds$$

which gives the (nonlinear) differential equation for the zero $c = c_\kappa(\nu)$

$$c' = \frac{dc}{d\nu} = -2c \int_0^\infty K_0(2c\sinh t)\sinh 2\nu t\, dt. \tag{4.10}$$

On the behaviour of the function $c_\kappa(\nu)$ we have got the following results.

Theorem 4.2. *The zero function $c_\kappa(\nu)$ is defined on $(-\frac{1}{2}-2\kappa, 2\kappa+\frac{1}{2})$ for $\kappa > -\frac{1}{4}$, it is symmetric with respect to $\nu = 0$, i.e. $c_\kappa(-\nu) = c_\kappa(\nu)$, moreover it is concave if $|\nu| \ge \frac{1}{2}$ and also in the case when $c_\kappa(\nu) > \pi\nu^2(1-2|\nu|)$ for $-\frac{1}{2} \le \nu \le \frac{1}{2}$.*

Proof. First we calculate the function $\dfrac{d^2c}{d\nu^2} = c''$:

$$\begin{aligned} c'' = &-2c' \int_0^\infty K_0(2c\sinh)\sinh 2\nu t\, dt - 2c \int_0^\infty K_0'(2c\sinh t) 2c' \sinh t \cdot \sinh 2\nu t - \\ &- 2c \int_0^\infty K_0(2c\sinh t)\cosh 2\nu t \cdot 2t\, dt. \end{aligned}$$

Integrating by parts in the second term on the right hand side, we obtain

$$\begin{aligned} 2c' \int_0^\infty K_0'(2c\sinh t) 2c\cosh t \,\frac{\sinh t}{\cosh t} \sinh 2\nu t\, dt &= \\ = [2c' K_0(2c\sinh t)\tanh t \cdot \sinh 2\nu t]_0^\infty &- \\ -2c' \int_0^\infty K_0(2c\sinh t) \left[\frac{1}{\cosh^2 t}\sinh 2\nu t + 2\nu \tanh t \cdot \cosh 2\nu t\right] dt&, \end{aligned}$$

hence

$$c'' = -2\int_0^\infty K_0(2c\sinh t)\, t\cosh 2\nu t \left[c' \frac{\tanh^2 t \cdot \tanh 2\nu t - 2\nu \tanh t}{t} + 2c\right] dt\,. \tag{4.11}$$

Since $K_0(u) > 0$ for $u > 0$, we have $c' < 0$ for $\nu > 0$, moreover for $\nu \geq 1/2$

$$\tanh^2 t \cdot \tanh 2\nu t - 2\nu \tanh t = \tanh t \, [\tanh t \cdot \tanh 2\nu t - 2\nu] < 0,$$

hence $c''(\nu) < 0$ for $\nu \geq 1/2$. An inspection at (4.11) shows that $c'' < 0$ if $\nu = 0$, too. On the interval $(0, 1/2)$ we need more sophisticated investigation. First we derive a lower bound for $c'(\nu)$. Due to the convexity of the function $\sinh t$ for $t > 0$ we have

$$\frac{\sinh 2\nu t}{2\nu} < \frac{\sinh t}{1} < \cosh t \,,$$

hence by (4.10) and [9, p. 388] for $0 < \nu < \frac{1}{2}$

$$c'(\nu) > -2\nu \int_0^\infty K_0(2c \sinh t) 2c \cosh t \, dt = -2\nu \int_0^\infty K_0(u) \, du = -\pi\nu \,. \tag{4.12}$$

Then we are going to show the relation

$$\frac{\tanh^2 t \cdot \tanh 2\nu t - 2\nu \tanh t}{2\nu\,(1-2\nu)\,t} < 1 \qquad \text{for} \quad t > 0, \quad 0 < \nu < \frac{1}{2},$$

or equivalently — using the notation $\gamma = 2\nu$ —

$$\phi(t,\gamma) = \gamma(1-\gamma)t - \tanh^2 t \cdot \tanh \gamma t + \gamma \tanh t > 0 \quad \text{for } t > 0, \quad 0 < \gamma < 1\,. \tag{4.13}$$

Fix the value t and calculate $\frac{\partial \phi}{\partial \gamma}, \frac{\partial^2 \phi}{\partial \gamma^2}$, we obtain

$$\begin{aligned} \frac{\partial \phi}{\partial \gamma} &= (1-2\gamma)t - \tanh^2 t \frac{t}{\cosh^2 \gamma t} + \tanh t, \\ \frac{\partial^2 \phi}{\partial \gamma^2} &= -2t \left[1 - t \tanh^2 t \cdot \frac{\sinh \gamma t}{\cosh^3 \gamma t} \right] . \end{aligned} \tag{4.14}$$

Observing that $\max\limits_{u \geq 0} \frac{\sinh u}{\cosh^3 u} = \max\limits_{u \geq 0} \{\tanh u - \tanh^3 u\} = \max\limits_{0 \leq x \leq 1} \{x - x^3\} = \frac{2}{3\sqrt{3}}$, we define the value t_0 by the relation

$$t_0 \tanh^2 t_0 = \frac{3\sqrt{3}}{2}.$$

We find $t_0 = 2.650426\ldots$ and from (4.14) $\frac{\partial^2 \phi}{\partial \gamma^2} \leq 0$ for $0 \leq t \leq t_0$, $0 \leq \gamma \leq 1$. On the other hand $\phi(t, 0) = 0$, $\phi(t, 1) = \tanh t - \tanh^3 t > 0$, hence the concavity of $\phi(t,\gamma)$ with respect to γ proves the relation (4.13) for $0 \leq t \leq t_0$, $0 \leq \gamma \leq 1$.

Our another observation is concerned with the function $\tanh u/u$. Since

$$\frac{d^2}{du^2}\left(\frac{\tanh u}{u}\right) = \frac{S(u)}{\cosh^2 u}$$

where

$$S(u) = \frac{\sinh 2u - 2u - 2u^2 \tanh u}{u^3} = \sum_{i=1}^{\infty} \frac{(2u)^{2i+1}}{(2i+1)!\, u^3} - 2\frac{\tanh u}{u} \tag{4.15}$$

and $S(u)$ is increasing function of u for $u > 0$, $S(0) < 0$, $\lim_{u\to\infty} S(u) = \infty$, there exists unique $u^* \in (0,\infty)$ where $S(u^*) = 0$. Numerically we get $u^* = .919937668\ldots$. For our purpose it will be also important the value $u_1 = 1.6140830\ldots$, where the tangent of the curve $\tanh u/u$ goes through $(0;1)$. Let the function $\tau(u)$ be defined by the relation

$$\tau(u) = \begin{cases} \dfrac{\tanh u}{u} & u \ge u_1 \\ 1 + \frac{u}{u_1}(\frac{\tanh u_1}{u_1} - 1) & 0 \le u \le u_1 . \end{cases}$$

Then $\tau(u)$ is convex function and $\tanh u/u \ge \tau(u)$ for $u \ge 0$. Since $\tau(0) = 1$, the convexity of $\tau(u)$ implies the relation

$$1 - \gamma + \gamma\tau(t) \ge \tau(\gamma t) \quad \text{for } t > 0,\ 0 \le \gamma \le 1. \tag{4.16}$$

Let $\psi(t,\gamma)$ be defined as

$$\psi(t,\gamma) = \frac{\phi(t,\gamma)}{\gamma t} = 1 - \gamma - \tanh^2 t \frac{\tanh \gamma t}{\gamma t} + \frac{\tanh t}{t} \qquad t \ge t_0 .$$

Particularly we have $\psi(t,0) = 1 - \tanh^2 t + \dfrac{\tanh t}{t} > 0$, $\psi(t,1) = \dfrac{\tanh t}{t}(1 - \tanh^2 t) > 0$. We find that $\psi(t,\gamma) \ge (1-\gamma)\psi(t,0) + \gamma\, \psi(t,1)$ if and only if the inequality $1 - \gamma + \gamma\dfrac{\tanh t}{t} \ge \dfrac{\tanh \gamma\, t}{\gamma\, t}$ holds. By definition of $\tau(u)$ and (4.16) this inequality holds if $\gamma\, t \ge u_1$, and we have got

$$\psi(t,\gamma) > 0 \quad \text{if } \gamma t \ge u_1 . \tag{4.17}$$

Let $u = \gamma\, t$, hence $\gamma = u/t$, and we have $\psi(t,\gamma) = \Psi(t,u)$:

$$\Psi(t,u) = 1 - \frac{u}{t} - \tanh^2 t \cdot \frac{\tanh u}{u} + \frac{\tanh t}{t}\,, \qquad t \ge t_0, \quad u \le t.$$

Let $u_0 = \tanh t_0 = .990074\ldots$. Then we have $u_0 > u^*$ and

$$\Psi(t,u) = 1 - \tanh^2 t \frac{\tanh u}{u} + \frac{\tanh t - u}{t} > 0 \qquad \text{for} \quad u \le \tanh t_0 = u_0 . \tag{4.18}$$

Finally, it remains the case $u_0 < u < u_1$, $t \geq t_0$. By calculation we find

$$\frac{\partial^2 \Psi(t,u)}{\partial u^2} = -\frac{S(u)}{\cosh^2 u} \tanh^2 t$$

where $S(u)$ is the same as in (4.15). Since u^* is the unique zero of $S(u)$ and $u_0 > u^*$, we realize that $\Psi(t,u)$ is concave function on $u_0 \leq u \leq u_1$ for any fixed t and the inequalities $\Psi(t,u_0) > 0$, $\Psi(t,u_1) > 0$ have been established in (4.17), (4.18), consequently $\Psi(t,u) > 0$ also on $[u_0, u_1]$, which completes the proof of (4.13).

Summing up our results, we get $c'' < 0$ if $2c + 2\nu(1-2\nu)c' > 0$ according to (4.11) and this inequality holds if we assume $c > \pi\nu^2(1-2\nu)$ when we make use of the inequality (4.12).

Remark. For any fixed $\kappa > -\frac{1}{4}$ $c = c_\kappa(\nu)$ represents certain curve in the (ν, c) plane ($c_\kappa(\nu)$ is here the zero of linear combination of (4.9) so the common zero at infinity is considered).

Let us calculate $\lim\limits_{c \to 0_+} \frac{d}{d\nu} c_\kappa(\nu)$. We have

$$\frac{d}{dc} c_\kappa(\nu) = -Q_1(c_\kappa, \nu), \text{ where } \; Q_1(t,\nu)$$

is in general case given in (3.11).

For the Bessel functions we have for $v(t,\nu)$ deefined by (4.2) and any $\nu \geq 0$

$$\lim_{t \to 0_+} \int_t^\infty \frac{v_\nu(s,\nu)}{v^2(s,\nu)}\, ds = \frac{d}{d\nu} N(\nu) = \frac{\pi}{2}$$

Frobenius method shows (it follows also from the well known properties of the Bessel functions)

$$\lim_{t \to 0_+} v(t,\nu) = \begin{cases} \infty & \text{for} \quad |\nu| > \frac{1}{2} \\ 1 & \text{for} \quad |\nu| = \frac{1}{2} \\ 0 & \text{for} \quad |\nu| < \frac{1}{2} \end{cases}$$

so

$$\lim_{t \to 0_+} Q_1(t,\nu) = \begin{cases} \infty & \text{for} \quad |\nu| > \frac{1}{2} \\ \pi/2 & \text{for} \quad |\nu| = \frac{1}{2} \\ 0 & \text{for} \quad |\nu| < \frac{1}{2} \end{cases}$$

and

$$\lim_{c_\kappa \to 0_+} \frac{d}{d\nu} c_\kappa(\nu) = \begin{cases} +\infty & \text{for} \quad \nu < -\frac{1}{2} \\ -\infty & \text{for} \quad \nu > \frac{1}{2} \\ \mp\pi/2 & \text{for} \quad \nu = \pm\frac{1}{2} \\ 0 & \text{for} \quad |\nu| < \frac{1}{2} \end{cases}$$

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