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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THIRD ORDER DELAY DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Professor Otakar Borůvka

Abstract. We give an equivalence criterion on property A and property B for delay third order linear differential equations. We also give comparison results on properties A and B between linear and nonlinear equations, whereby we only suppose that nonlinearity has superlinear growth near infinity.

Introduction

Consider the third–order nonlinear equations with deviating argument of the form

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}\,x'(t)\right)'\right)' + q(t)f(x(h(t))) = 0 \tag{N,h}$$

and

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}\,z'(t)\right)'\right)' - q(t)f(z(h(t))) = 0, \tag{N$^{\mathcal{A}}$,h}$$

where

$$\begin{gathered} r,p,q,h \in C^0([a,\infty),\mathbb{R}), \quad r(t)>0,\ p(t)>0,\ q(t)>0 \text{ on } [a,\infty), \\ \int_a^\infty r(t)\,dt = \int_a^\infty p(t)\,dt = \infty, \\ h(t)\le t,\ \lim_{t\to\infty} h(t)=\infty, \\ f\in C^0(\mathbb{R},\mathbb{R}), \quad f(u)u>0 \ \text{ for } u\ne 0. \end{gathered} \tag{H}$$

If x is a solution of (N,h) then the functions

$$x^{[0]}=x, \quad x^{[1]}=\frac{1}{r}x', \quad x^{[2]}=\frac{1}{p}(\frac{1}{r}x')'=\frac{1}{p}(x^{[1]})'$$

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are called quasiderivatives of x. Similarly we can proceed for $(N^{\mathcal{A}},h)$. Equation $(N^{\mathcal{A}},h)$ is obtained from (N,h) by interchanging coefficients p,r and by replacing q with $-q$. The notation $(N^{\mathcal{A}},h)$ is suggested by the fact that for the linear equation without deviating argument, i.e. for equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)'+q(t)x(t)=0, \tag{L}$$

the *adjoint equation* is

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'(t)\right)'\right)'-q(t)z(t)=0. \tag{L$^{\mathcal{A}}$}$$

For simplicity, when $h(t)\equiv t$, we will denote (N,h) and $(N^{\mathcal{A}},h)$ with (N) and $(N^{\mathcal{A}})$, respectively.

As usual, a solution x of (N,h) is said to be *proper* if it is defined for all large t and $\sup\{|x(t)|, t\geq T\}>0$ for every large T. A proper solution x is said to be *oscillatory* or *nonoscillatory* according to whether does or does not have arbitrarily large zeros. Similar definitions hold for $(N^{\mathcal{A}},h)$. In addition, (L) $[(L^{\mathcal{A}})]$ is called *oscillatory* if it has at least one nontrivial oscillatory solution, and *nonoscillatory* if its all nontrivial solutions are nonoscillatory.

The study of asymptotic behavior of solutions, both in the ordinary case and in the case with deviating argument, is often accomplished by introducing the concepts of *equation with property A* and *equation with property B*. More precisely, (N,h) is said to have *property A* if any proper solution x of (N,h) is either oscillatory or satisfies

$$|x^{[i]}(t)|\downarrow 0 \quad \text{as } t\to\infty,\ i=0,1,2,$$

and $(N^{\mathcal{A}},h)$ is said to have *property B* if any proper solution z of $(N^{\mathcal{A}},h)$ is either oscillatory or satisfies

$$|z^{[i]}(t)|\uparrow\infty \quad \text{as } t\to\infty,\ i=0,1,2.$$

Here the notation $y(t)\downarrow 0$ $[y(t)\uparrow\infty]$ means that y monotonically decreases to zero as $t\to\infty$ [monotonically increases to infinity as $t\to\infty$]. Among the numerous results dealing with this topic, we refer the reader in particular to [6,7,9–12] and to the references contained therein.

In this paper we will study relationships between property A for (N,h) and property B for $(N^{\mathcal{A}},h)$. In particular in section 2 the linear case is considered. Here we give an equivalence criterion on property A for (L,h) and property B for $(L^{\mathcal{A}},h)$. Such a result enables us to obtain criteria on property A for (L,h) from criteria on property B for $(L^{\mathcal{A}},h)$ and vice versa. As far as we know, such approach seems new, since usually properties A and B are studied independently each other. As consequence relationships between the case with or without deviating argument are obtained.
The nonlinear case is considered in section 3. Here comparison results on properties A and B between linear and nonlinear case, which extend a previous one from [10],

are given. The obtained results also generalize recent criteria which have been given by the authors in [2,3] for the case without deviating argument.
Our approach is based on a study of asymptotic behavior of nonoscillatory solutions of (N, h) and $(N^{\mathcal{A}}, h)$ as well as on a comparison result [10] between equations with different deviating argument. Such a comparison criterion, in the form here used, is quoted in section 1. Finally we remark that our results are closely related to those in [11] in which oscillation of a delay differential equation is deduced from that of the corresponding differential equation without delay.

1. Preliminary results

We introduce the following notation:

$$I(u_i) = \int_a^\infty u_i(t)\,dt, \quad I(u_i, u_j) = \int_a^\infty u_i(t) \int_a^t u_j(s)\,ds\,dt, \quad i, j = 1, 2$$
$$I(u_i, u_j, u_k) = \int_a^\infty u_i(t) \int_a^t u_j(s) \int_a^s u_k(\tau)\,d\tau\,ds\,dt, \quad i, j, k = 1, 2, 3,$$

where u_i, $i = 1, 2, 3$, are continuous positive functions on $[a, \infty)$.

In the recent papers [3,4] we have studied relationships among oscillation and properties A and B for linear and nonlinear equations without delay. The main results for linear equations without delay are summarised in the following:

Theorem A. ([3]) The following assertions are equivalent:

(i) (L) has property A.
(i') $(L^{\mathcal{A}})$ has property B.
(ii) (L) is oscillatory and it holds $I(q, p, r) = \infty$.
(ii') $(L^{\mathcal{A}})$ is oscillatory and it holds $I(q, p, r) = \infty$.

For nonlinear equations without delay the following holds:

Theorem B. ([4]) *Assume*

$$\limsup_{u\to 0} \frac{f(u)}{u} < \infty, \qquad \textit{and } \liminf_{|u|\to\infty} \frac{f(u)}{u} > 0$$

and let the linear equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + kq(t)x(t) = 0$$

be oscillatory for all $k > 0$. If (N) has property A, then $(N^{\mathcal{A}})$ has property B.

Remark. Theorem A holds even if $I(r) < \infty$ and/or $I(p) < \infty$. In this case condition $I(q, p, r) = \infty$ becomes $I(q, p, r) = I(r, q, p) = I(p, r, q) = \infty$.

Similarly, Theorem B holds even if $I(r) < \infty$, $I(p) = I(r, p) = \infty$.

To extend these results to delay differential equations we will use the following linear comparison criterion. It is a particular case of a more general theorem which is stated in [10] for functional differential equations of higher order.

Theorem C. ([10]) *Consider the differential equations (i=1,2)*

$$(\mathrm{L},\mathrm{h}_i)_i \qquad \left(\frac{1}{p(t)}\left(\frac{1}{r(t)}\,x'(t)\right)'\right)' + q_i(t)x(h_i(t)) = 0$$

$$(\mathrm{L}^{\mathcal{A}},\mathrm{h}_i)_i \qquad \left(\frac{1}{r(t)}\left(\frac{1}{p(t)}\,z'(t)\right)'\right)' - q_i(t)z(h_i(t)) = 0$$

where $q_i, h_i \in C^0([a,\infty),\mathbb{R})$, $q_i(t) > 0$, $\lim_{t\to\infty} h_i(t) = \infty$ *and*

$$h_1(t) \le h_2(t), \quad q_1(t) \le q_2(t) \quad \text{for } t > t_0 \ge a.$$

If $(L,h_1)_1$ *has property A then* $(L,h_2)_2$ *has property A.*
If $(L^{\mathcal{A}},h_1)_1$ *has property B then* $(L^{\mathcal{A}},h_2)_2$ *has property B.*

Concerning the nonlinear case, by Theorem 1 in [10], we obtain the following:

Theorem D. ([10])
Let $f(u)$ *be nondecreasing in* $\mathbb{R}$ *and assume* $f(u)\,\mathrm{sgn}\,u \ge u\,\mathrm{sgn}\,u$ *for all* $u \in \mathbb{R}$. *Then:*
if (N,h) *has property A,* (N) *has property A;*
if $(N^{\mathcal{A}},h)$ *has property B,* $(N^{\mathcal{A}})$ *has property B.*
In particular, the statements holds for linear equations.

From a slight modification of the well–known lemma of Kiguradze (see, e.g., [8]) it follows that nonoscillatory solutions x of (L,h) and (N,h) can be divided into the following two classes:

$$\mathcal{N}_0 = \{x \text{ solution}, \exists\, T_x : x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) > 0 \text{ for } t \ge T_x\}$$
$$\mathcal{N}_2 = \{x \text{ solution}, \exists\, T_x : x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) > 0 \text{ for } t \ge T_x\}.$$

Similarly nonoscillatory solutions z of $(L^{\mathcal{A}},h)$ and $(N^{\mathcal{A}},h)$ can be divided into the following two classes:

$$\mathcal{M}_1 = \{z \text{ solution}, \exists\, T_z : z(t)z^{[1]}(t) > 0,\ z(t)z^{[2]}(t) < 0 \text{ for } t \ge T_z\}$$
$$\mathcal{M}_3 = \{z \text{ solution}, \exists\, T_z : z(t)z^{[1]}(t) > 0,\ z(t)z^{[2]}(t) > 0 \text{ for } t \ge T_z\}.$$

It is clear that (L,h) $[(N,h)]$ has property A if and only if all nonoscillatory solutions x of (L,h) $[(N,h)]$ belong to the class $\mathcal{N}_0$ and $\lim_{t\to\infty} x^{[i]}(t) = 0$, $i = 0,1,2$. Similarly $(L^{\mathcal{A}},h)$ $[(N^{\mathcal{A}},h)]$ has property B if and only if all nonoscillatory solutions z of $(L^{\mathcal{A}},h)$ $[(N^{\mathcal{A}},h)]$ belong to the class $\mathcal{M}_3$ and $\lim_{t\to\infty} |z^{[i]}(t)| = \infty$, $i = 0,1,2$. Independently on properties A and B, it is easy to show the following.

Lemma 1.1. *It holds:*

(i) *any solution* x *of* (L,h) $[(N,h)]$ *from* $\mathcal{N}_0$ *satisfies* $\lim_{t\to\infty} x^{[i]}(t) = 0$, $i = 1,2$;
(ii) *any solution* z *of* $(L^{\mathcal{A}},h)$ $[(N^{\mathcal{A}},h)]$ *from* $\mathcal{M}_3$ *satisfies* $\lim_{t\to\infty} |z^{[i]}(t)| = \infty$, $i = 0,1$.

When $I(r) = I(p) = \infty$, then property B may be interpreted in a different way, as the following lemma states.

Lemma 1.2. *Let z be a nonoscillatory solution of $(L^{\mathcal{A}}, h)$ $[(N^{\mathcal{A}}, h)]$. Then the following assertions are equivalent:*

(i) $z \in \mathcal{M}_3$, $\lim_{t\to\infty} |z^{[2]}(t)| = \infty$;

(ii) $z \in \mathcal{M}_3$, $\lim_{t\to\infty} |z^{[i]}(t)| = \infty$, $i = 0, 1, 2$;

(iii)

$$\lim_{t\to\infty} \left|\frac{z(t)}{\int_a^t r(s) \int_a^s p(u) du\, ds}\right| = \infty.$$

Proof. (i)⇒(ii). It follows by Lemma 1.1.
(ii)⇒(iii). It follows by l'Hôpital rule.
(iii)⇒(i). Because z is nonoscillatory, we get $(z^{[2]}(t))' > 0$ for all large t. Integrating this inequality we obtain $z \in \mathcal{M}_3$.

Without loss of generality we may assume that $z^{[2]}$ is eventually positive. Now assume there exists a positive constant L such that $z^{[2]}(\infty) < L$, i.e., $z^{[2]}(t) < L$ for $a \le t < \infty$. Integrating twice the last inequality we obtain

$$z(t) < z(a) + z^{[1]}(a) \int_a^t p(s)\, ds + L \int_a^t p(s) \int_a^s r(u)\, du\, ds,$$

which implies that $z(t)/ \int_a^t p(s) \int_a^s r(u)\, du\, ds$ is bounded from above, that is a contradiction. □

2. Linear case

A first answer to the problem of equivalency between property A and property B is given by the following:

Theorem 2.1. *Let $h(t) \le t$, $g(t) \ge t$ where $g \in C^0([a,\infty), \mathbb{R})$, $\lim_{t\to\infty} g(t) = \infty$.*
a) If (L, h) has property A, then $(L^{\mathcal{A}}, g)$ has property B.
b) If $(L^{\mathcal{A}}, h)$ has property B, then (L, g) has property A.

Proof. Claim a). By Theorem C ($h_2(t) = t$), (L) has property A and so, by Theorem A, $(L^{\mathcal{A}})$ has property B. The assertion follows again using Theorem C with $h_1(t) = t$ and $h_2(t) = g(t)$.

Claim b). The argument is similar to this given in the claim a). □

For delay equations, the following holds:

Lemma 2.1. *If (L) has property A then every solution x of (L, h) which is from the class $\mathcal{N}_0$ satisfies $\lim_{t\to\infty} x^{[i]}(t) = 0$, $i = 0, 1, 2$.*

Proof. By Lemma 1.1, it is sufficient to prove that any nonoscillatory solution x of (L, h), $x \in \mathcal{N}_0$, satisfies $\lim_{t\to\infty} x(t) = 0$. Assume by contradiction that $x(\infty) \ne 0$.

Without loss of generality we may suppose that there exists $T \ge a$ such that $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) > 0$ for all $t \ge T$. Integrating (L, h) three times in (t, ∞), with $t \ge T$ and taking into account that $x^{[1]}(\infty) = x^{[2]}(\infty) = 0$, we obtain

$$x(t) = x(\infty) + \int_t^\infty r(s) \int_s^\infty p(u) \int_u^\infty q(\sigma)x(h(\sigma))\, d\sigma\, du\, ds.$$

Because x is eventually decreasing, we get

$$x(t) > x(\infty) \int_t^\infty r(s) \int_s^\infty p(u) \int_u^\infty q(\sigma)\, d\sigma\, du\, ds$$

and then, by interchanging order of integration, we get $I(q,p,r) < \infty$. Since (L) has property A, it holds by Theorem A that $I(q,p,r) = \infty$, which is a contradiction.□

Theorem 2.2. *a) Assume*

$$\int_a^\infty q(t) \int_a^{h(t)} r(s)\, ds\, dt = \infty. \tag{2.1}$$

If $(L^{\mathcal{A}}, h)$ has property B, then (L, h) has property A.

b) Assume $I(q,r) = \infty$ and

$$\int_a^\infty q(t) \int_{h(t)}^t p(s) \int_a^s r(u)\, du\, ds\, dt < \infty. \tag{2.2}$$

If (L, h) has property A, then $(L^{\mathcal{A}}, h)$ has property B.

Proof. *Claim a).* Assume that (L,h) does not have property A. Let x be a nonoscillatory solution of (L,h).

By Theorem D, equation $(L^{\mathcal{A}})$ has property B and so, by Theorem A, equation (L) has property A. Hence, by Lemma 2.1 any solution $x \in \mathcal{N}_0$ satisfies $x(\infty) = 0$.

Let $x \in \mathcal{N}_2$. Without loss of generality we may suppose that there exists $T \geq a$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \geq T$. Because for $t \geq T$ $(x^{[2]}(t))' = -q(t)x(h(t)) < 0$, we have $x^{[2]}(\infty) < \infty$. Let $T_1 > T$ be such that $h(t) > T$ for all $t \geq T_1$. Integrating (L,h) in (T_1, ∞) we obtain

$$\int_{T_1}^\infty q(t) x(h(t))\, dt < \infty\,. \tag{2.3}$$

Because $x^{[1]}$ is an eventually positive increasing function, we have $x^{[1]}(t) > x^{[1]}(T)$, i.e.,

$$x(t) > x^{[1]}(T) \int_T^t r(s)\, ds \qquad (t \geq T).$$

Hence

$$x(h(t)) > x^{[1]}(T) \int_T^{h(t)} r(s)\, ds > x^{[1]}(T) \int_{T_1}^{h(t)} r(s)\, ds \qquad (t \geq T_1).$$

Substituting into (2.3) we get

$$\int_{T_1}^\infty q(t) \int_{T_1}^{h(t)} r(s)\, ds\, dt < \infty,$$

which contradicts (2.1). Claim a) is now proved.

Claim b). Assume that $(L^{\mathcal{A}}, h)$ does not have property B. Let z be a proper nonoscillatory solution of $(L^{\mathcal{A}}, h)$. By Lemmas 1.1 and 1.2 there are two possibilities: I) $z \in \mathcal{M}_3$ such that $|z^{[2]}(\infty)| < \infty$; II) $z \in \mathcal{M}_1$.
Case I). Without loss of generality we may suppose that there exists $T \geq a$ such that $z(t) > 0$, $z^{[1]}(t) > 0$, $z^{[2]}(t) > 0$ for all $t \geq T$. Let $T_1 > T$ be such that $h(t) > T$ for all $t \geq T_1$. Because $z^{[2]}(\infty) < \infty$, by integrating $(L^{\mathcal{A}}, h)$ in (t, ∞), we obtain

$$z^{[2]}(\infty) - z^{[2]}(t) = \int_t^\infty q(s) z(h(s))\, ds$$

and so

$$\int_{T_1}^\infty q(t) z(h(t))\, dt < \infty\,. \tag{2.4}$$

Because $z^{[2]}$ is increasing for $t \geq T$, we have $z^{[2]}(t) > z^{[2]}(T) > 0$. Integrating twice in (T, t) we get

$$z(t) > z^{[2]}(T) \int_T^t p(s) \int_T^s r(u)\, du\, ds \qquad (t \geq T)$$

or

$$z(h(t)) > z^{[2]}(T) \int_T^{h(t)} p(s) \int_T^s r(u)\, du\, ds > z^{[2]}(T) \int_{T_1}^{h(t)} p(s) \int_{T_1}^s r(u)\, du\, ds$$

for $t \geq T_1$. From (2.4) we obtain

$$\int_{T_1}^\infty q(t) \int_{T_1}^{h(t)} p(s) \int_{T_1}^s r(u)\, du\, ds\, dt < \infty$$

which implies

$$\int_a^\infty q(t) \int_a^{h(t)} p(s) \int_a^s r(u)\, du\, ds\, dt < \infty.$$

From (2.2) it follows $I(q, p, r) < \infty$. By Theorem D, equation (L) has property A and so, by Theorem A, $I(q, p, r) = \infty$, which is a contradiction.
Case II). Without loss of generality we can suppose that z is eventually positive. Hence $0 \leq -z^{[2]}(\infty) < \infty$. Integrating $(L^{\mathcal{A}}, h)$ in (t, ∞), with t sufficiently large, we obtain

$$-z^{[2]}(t) = -z^{[2]}(\infty) + \int_t^\infty q(s) z(h(s))\, ds > \int_t^\infty q(s) z(h(s))\, ds.$$

Because $0 \leq z^{[1]}(\infty) < \infty$, by integrating again in (t, ∞) we get

$$z^{[1]}(t) > z^{[1]}(\infty) + \int_t^\infty r(s) \int_s^\infty q(u) z(h(u)) du\, ds >$$
$$z(h(t)) \int_t^\infty r(s) \int_s^\infty q(u)\, du\, ds = z(h(t)) \int_t^\infty q(u) \int_t^u r(s)\, ds\, du,$$

which gives a contradiction with $I(q, r) = \infty$. The proof is now complete. □

Under additional assumptions, we can state a comparison theorem, which gives an opposite result with respect to Theorem D:

Theorem 2.3.

a) Assume (2.1). If (L) has property A then (L,h) has property A.

b) Assume $I(q,r)=\infty$ and (2.2). If $(L^{\mathcal{A}})$ has property B then $(L^{\mathcal{A}},h)$ has property B.

Proof. The assertion follows by using a similar argument to this given in the proof of Theorem 2.2. The details are omitted. □

Denote by (L,τ) $[(L^{\mathcal{A}},\tau)]$ equation (L,h) $[(L^{\mathcal{A}})]$ with the delay $h(t)=t-\tau(t)$, where τ is a bounded function. Kusano and Naito [10] proved the equivalency of property A between (L) and (L,τ) and of property B between $(L^{\mathcal{A}})$ and $(L^{\mathcal{A}},\tau)$. From this and Theorem A we get the following result:

Corollary 2.1. *Let $I(r)=I(p)=\infty$, r,p be nonincreasing for $t\in[0,\infty)$. Then, (L,τ) has property A if and only if $(L^{\mathcal{A}},\tau)$ has property B.*

3. Nonlinear case

Here we state comparison theorem between linear and nonlinear delay equations.

Theorem 3.1. *Let*

$$\lim_{u\to\infty}\frac{f(u)}{u}=\infty. \tag{3.1}$$

a) If (L,h) has property A, then (N,h) has property A.

b) If $(L^{\mathcal{A}},h)$ has property B, then $(N^{\mathcal{A}},h)$ has property B.

Proof. *Claim a).* Because (L,h) has property A, by Theorem D, (L) has property A and, by Theorem A, $I(q,p,r)=\infty$. Assume that (N,h) does not have property A. Let x be a proper nonoscillatory solution of (N,h). By Lemma 1.1 there are two possibilities: I) $x\in\mathcal{N}_0$ such that $x(\infty)\neq 0$; II) $x\in N_2$.
Case I). Without loss of generality we may suppose that there exists $T\geq 0$ such that $x(t)>0$ for all $t\geq T$. Because $\lim_{t\to\infty}h(t)=\infty$, there exists $t_1\geq T$ such that $h(t_1)=T$, $h(t)>T$ for $t>t_1$. Let H be the function

$$H(t)=\begin{cases} t_1 & \text{for } t\in[T,t_1]\\ h(t) & \text{for } t\in(t_1,\infty)\end{cases}$$

and consider, for $t\geq T$, the function F given by $F(t)=f(x(H(t)))/x(t)$. Then the nonlinear ordinary differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}\,w'(t)\right)'\right)'+q(t)F(t)w(t)=0 \qquad t\in[t_1,\infty)$$

has a nonoscillatory solution (the function x) such that $x\in\mathcal{N}_0$, $x(\infty)\neq 0$. Hence by Lemma 4–(i) in [3], we get $I(qF,p,r)<\infty$. Because x does not approach zero as $t\to\infty$, there exists a positive constant k such that $F(t)>k$ for all $t\geq t_1$, which implies $I(q,p,r)<\infty$, that is a contradiction.

Case II). Without loss of generality we may suppose that x is eventually positive. Now the linear equation with delay (t large)

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}\,z'(t)\right)'\right)' + q(t)F_1(t)w(h(t)) = 0$$

where $F_1(t) = f(x(h(t)))/x(h(t))$, does not have property A, because x is an its nonoscillatory solution, $x \in \mathcal{N}_2$. In view of the facts $x(\infty) = \infty$ and (3.1), there exists $T \geq 0$ such that $F_1(t) > 1$ for $t \geq T$. Hence by Theorem C ($h_1(t) = h_2(t) = h(t)$), (L, h) does not have property A, which is a contradiction.

Claim b). Assume that $(N^{\mathcal{A}}, h)$ does not have property B. Let z be a proper nonoscillatory solution of $(N^{\mathcal{A}}, h)$. Without loss of generality we may suppose that z is eventually positive. By Lemma 1.1 and Lemma 1.2 there are two possibilities: I) $z \in \mathcal{M}_3$ such that $z^{[2]}(\infty) \neq \infty$; II) $z \in M_1$.

Case I). Consider, for all large t, the linear equation with delay

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}\,w'(t)\right)'\right)' - q(t)F_2(t)w(h(t)) = 0 \tag{3.2}$$

where $F_2(t) = f(z(h(t)))/z(h(t))$. Since z is an its nonoscillatory solution, (3.2) does not have property B. In view of the facts $z(\infty) = \infty$ and (3.1), there exists a large $T \geq 0$ such that $F_2(t) > 1$ for $t \geq T$. Hence by Theorem D, $(L^{\mathcal{A}}, h)$ does not have property B, which is a contradiction.

Case II). If $z(\infty) = \infty$, then, by reasoning as above, we get a contradiction. Assume $z(\infty) < \infty$. Because $z^{[1]}$ is eventually positive decreasing, we have for all large t that $z^{[1]}(t) > z^{[1]}(\infty)$ or (T large, $t > T$)

$$z(t) > z(T) + z^{[1]}(\infty)\int_T^t p(s)\,ds.$$

Because z is bounded as $t \to \infty$ and $I(p) = \infty$, we get $z^{[1]}(\infty) = 0$. By using a similar argument we obtain $z^{[2]}(\infty) = 0$. Integrating $(N^{\mathcal{A}}, h)$ three times in (t, ∞), with $t \geq T$, we obtain

$$\begin{aligned} z(\infty) - z(t) = \int_t^\infty p(s)\int_s^\infty r(u)\int_u^\infty q(v)f(z(h(v)))\,dv\,du\,ds = \\ \int_t^\infty q(v)f(z(h(v)))\int_t^v r(u)\int_t^u p(s)\,ds\,du\,dv. \end{aligned} \tag{3.3}$$

Taking into account that z is positive increasing, there exists a positive constant k such that $f(z(h(t))) > k$ for all large t. Hence (3.3) implies that $I(q, r, p) < \infty$. By a result in [1, Theorem 5], (L) is nonoscillatory. On the other hand, by Theorem D, $(L^{\mathcal{A}})$ has property B and so, by Theorem A, (L) is oscillatory, which is a contradiction. $\square$

Remark. Theorem 3.1 does not require that the nonlinearity $|f(u)|$ dominates the linear term $|u|$ in the whole $\mathbb{R}^+$. In addition monotonicity assumptions of f are unnecessary. Then Theorem 3.1 extends, as far as third order delay equations, the quoted criterion in [10] (Theorem 1).

Concluding remarks. When $p = r$, the linear/nonlinear equations without as well as with delay argument present special properties. This, together with application of Theorems 3.2, 3.3, and 4.1 to obtain integral criteria under which the linear/nonlinear delay equations have property A, resp. B, will be given elsewhere.

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