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ON THE ITERATED ABSOLUTE DIFFERENTIATION ON SOME FUNCTIONAL BUNDLES

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Dedicated to the memory of Professor Otakar Borůvka

Abstract. We deduce further properties of connections on the functional bundle of all smooth maps between the fibers over the same base point of two fibered manifolds over the same base, which we introduced in [2]. In particular, we define the vertical prolongation of such a connection, discuss the iterated absolute differentiation by means of an auxiliary linear connection on the base manifold and prove the general Ricci identity.

Let $p_1 : Y_1 \to M$, $p_2 : Y_2 \to M$ be two classical fibered manifold over the same base. Consider the set of all fiber maps

$$\mathcal{F}(Y_1, Y_2) = \bigcup_{x \in M} C^\infty(Y_{1x}, Y_{2x}) \tag{1}$$

and denote by $p : \mathcal{F}(Y_1, Y_2) \to M$ the canonical projection. The set $\mathcal{F}(Y_1, Y_2)$ is a smooth space in the sense of Frölicher, [5]. In [2] we introduced the first jet prolongation $J^1\mathcal{F}(Y_1, Y_2)$ of $\mathcal{F}(Y_1, Y_2)$ and defined a connection on $\mathcal{F}(Y_1, Y_2)$ as a smooth section $\Gamma : \mathcal{F}(Y_1, Y_2) \to J^1\mathcal{F}(Y_1, Y_2)$. Since such a connection is a kind of differential operator, we have a well defined concept of finite order connection. In [2] we also introduced the curvature of Γ and the absolute differential $\nabla_\Gamma s$ of a smooth section $s : M \to \mathcal{F}(Y_1, Y_2)$ with respect to Γ and we deduced their basic properties.

The main aim of the present paper is to study the iterated absolute differentiation on $\mathcal{F}(Y_1, Y_2)$. Analogously to the case of an arbitrary fibered manifold $Y \to M$, we use an auxiliary linear connection Λ on M. We first construct the vertical prolongation $\mathcal{V}\Gamma : V\mathcal{F}(Y_1, Y_2) \to J^1V\mathcal{F}(Y_1, Y_2)$ of a differentiable connection

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Γ on $\mathcal{F}(Y_1, Y_2)$, where $V\mathcal{F}(Y_1, Y_2)$ means the vertical tangent bundle of $\mathcal{F}(Y_1, Y_2)$. The absolute differential of $\nabla_\Gamma s$ with respect to the tensor product $\mathcal{V}\Gamma \otimes \Lambda^*$ is called the iterated absolute differential $\nabla^2_{\Gamma,\Lambda} s$ of s. In Section 4 we assume that Y_2 is a vector bundle and Γ is a finite order connection (non-linear in general) and deduce that the alternation of $\nabla^2_{\Gamma,\Lambda} s$ satisfies a direct modification of the Ricci identity for non-linear connections on a vector bundle.

However, the first author has recently clarified that the general Ricci identity holds even for a general connection on an arbitrary fibered manifold $Y \to M$, provided one replaces the tensor alternation by a more sophisticated operation of the difference tensor of some distinguished iterated 2-jets, [1]. In the second half of the present paper we show that the same is true for the case of $\mathcal{F}(Y_1, Y_2)$. At this occasion we define the general concept of space $J^r(N, \mathcal{F}(Y_1, Y_2))$ of all r-jets of a manifold N into $\mathcal{F}(Y_1, Y_2)$. In the second order we introduce the corresponding space of iterated jets, characterize among them the semiholonomic and nonholonomic 2-jets and describe their basic properties. The last section is devoted to the proof of the general Ricci identity for $\mathcal{F}(Y_1, Y_2)$.

If we deal with two finite dimensional manifolds and a map between them, we always assume they are of class C^∞, i.e. smooth in the classical sense. On the other hand, the idea of smoothness in the infinite dimension is taken from the theory of smooth structures by Frölicher, [5], see also [3].

1. Connections on $\mathcal{F}(Y_1, Y_2)$.

The definition of the r-th jet prolongation $J^r\mathcal{F}(Y_1, Y_2)$ of $\mathcal{F}(Y_1, Y_2)$ is based on the idea of fiber jet, [8], [10], p. 395. In general, let $\pi : Y \to M$ be a fibered manifold, let $Y_x = \pi^{-1}(x)$, $x \in M$, let N be a manifold and $f, g : Y \to N$ be two C^∞-maps. We say that f and g determine the same fiber r-jet $j^r_x f = j^r_x g$ at $x \in M$, if

$$j^r_y f = j^r_y g \qquad \text{for all} \quad y \in Y_x\,. \tag{2}$$

For every section $s : M \to \mathcal{F}(Y_1, Y_2)$, we can construct the associated map $\widetilde{s} : Y_1 \to Y_2$, $\widetilde{s}(y) = s(p_1 y)(y)$. We say that s is smooth (in the sense of Frölicher, [5]), if $\widetilde{s}$ is a C^∞-map. Two smooth sections $s_1, s_2 : M \to \mathcal{F}(Y_1, Y_2)$ determine the same r-jet $j^r_x s_1 = j^r_x s_2$ at $x \in M$, if $j^r_x \widetilde{s}_1 = j^r_x \widetilde{s}_2$ in the sense of (2).

In the rest of this section we discuss the case $r = 1$. Let $X = j^1_x s \in J^1_x\mathcal{F}(Y_1, Y_2)_\psi$, where x is the source αX of X and $\psi = s(x) \in C^\infty(Y_{1x}, Y_{2x})$ is the target βX of X. Then X determines a map $\widetilde{X} : J^1_x Y_1 \to J^1_x Y_2$,

$$\widetilde{X}(j^1_x \sigma) = j^1_x(s \circ \sigma) \qquad \text{for all} \quad j^1_x \sigma \in J^1_x Y_1\,. \tag{3}$$

Let (x^i, y^p) or (x^i, z^a) be some local fiber coordinates on Y_1 or Y_2, let $z^a = \varphi^a(x^i, y^p)$ be the coordinate expression of $\widetilde{s}$ and let $x = x_0$. Write

$$\psi^a(y) = \varphi^a(x_0, y), \quad \psi^a_i(y) = \frac{\partial \varphi^a(x_0, y)}{\partial x^i}\,. \tag{4}$$

Then the coordinate form of $\widetilde{X}$ is

$$z^a = \psi^a(y), \quad z_i^a = \frac{\partial \psi^a(y)}{\partial y^p} y_i^p + \psi_i^a(y)\,, \tag{5}$$

where y_i^p or z_i^a are the induced coordinates on Y_1 or Y_2. It is well known that $J_x^1 Y_i \to Y_{ix}$, $i = 1, 2$, is an affine bundle over Y_{ix} with derived vector bundle $V_x Y_i \otimes T_x^* M$, provided VY_i denotes the vertical tangent bundle of Y_i. Hence (5) yields that $\widetilde{X}$ is an affine bundle morphism over $\psi : Y_{1x} \to Y_{2x}$, whose derived linear morphism is $T\psi \otimes \mathrm{id}_{T_x^* M} : V_x Y_1 \otimes T_x^* M \to V_x Y_2 \otimes T_x^* M$. Conversely, Lemma 2 from [2] reads that for every affine bundle morphism $\Psi : J_x^1 Y_1 \to J_x^1 Y_2$ over $\psi : Y_{1x} \to Y_{2x}$ with derived linear morphism $T\psi \otimes \mathrm{id}_{T_x^* M}$ there exists a unique element $X \in J_x^1 \mathcal{F}(Y_1, Y_2)_\psi$ such that $\Psi = \widetilde{X}$. Thus, on one hand, the associated map $\widetilde{X}$ characterizes an element $X \in J^1 \mathcal{F}(Y_1, Y_2)$ geometrically. On the other hand, the numbers x_0^i and the functions $\psi^a(y)$, $\psi_i^a(y)$ from (4) are a kind of the coordinate expression of X.

A connection on $\mathcal{F}(Y_1, Y_2)$ is defined as a section $\Gamma : \mathcal{F}(Y_1, Y_2) \to J^1 \mathcal{F}(Y_1, Y_2)$, which is smooth in the Frölicher sense, [2]. Since Γ is a kind of differential operator, one can characterize an r-th order connection, $r \geqslant 1$, as follows. We say that Γ is of order r, if the condition $j_y^r \varphi = j_y^r \psi$, $\varphi, \psi \in C^\infty(Y_{1x}, Y_{2x})$, $y \in Y_{1x}$ implies

$$\widetilde{\Gamma(\varphi)} | (J^1 Y_1)_y = \widetilde{\Gamma(\psi)} | (J^1 Y_1)_y\,, \tag{6}$$

i.e. the restrictions of the associated maps $\widetilde{\Gamma(\varphi)}$, $\widetilde{\Gamma(\psi)} : J_x^1 Y_1 \to J_x^1 Y_2$ to the fiber $(J^1 Y_1)_y$ over y coincide, [2].

Write $\mathcal{F}J^r(Y_1, Y_2) = \bigcup\limits_{x \in M} J^r(Y_{1x}, Y_{2x})$, which is a finite dimensional manifold. The jet coordinates on $\mathcal{FJ}^r(Y_1, Y_2)$ are x^i, y^p, z_α^a, where α is a multiindex of the range equal to the range of y^p with $0 \le |\alpha| \le r$. Let $S(J^1 Y_1, J^1 Y_2)$ be the space of all affine maps $(J^1 Y_1)_y \to (J^1 Y_2)_z$ with the derived linear map of the form $B \otimes \mathrm{id}_{T_y^* M}$, $B \in V_z Y_2 \otimes V_y^* Y_1$. This is a fibered manifold over $Y_1 \times_M Y_2$ with the fiber coordinates b_p^a, c_i^a induced by

$$z_i^a = b_p^a\, y_i^p + c_i^a \tag{7}$$

An r-th order connection Γ determines the associated map $\mathcal{G} : \mathcal{FJ}^r(Y_1, Y_2) \to S(J^1 Y_1, J^1 Y_2)$ by

$$\mathcal{G}(j_y^r \psi) = \widetilde{\Gamma(\psi)} | (J^1 Y_1)_y\,. \tag{8}$$

By [2], $\mathcal{G}$ is a C^∞-map. The coordinate form of $\mathcal{G}$ corresponding to (5) is

$$z_i^a = z_p^a\, y_i^p + \Phi_i^a(x^i, y^p, z_\alpha^a)\,, \qquad 0 \le |\alpha| \le r\,. \tag{9}$$

We say that Φ_i^a is the coordinate expression of Γ. Conversely, given any C^∞-map $\mathcal{G} : \mathcal{F}\mathcal{J}^r(Y_1, Y_2) \to S(J^1Y_1, J^1Y_2)$ of the form (9), it determines an r-th order connection Γ on $\mathcal{F}(Y_1, Y_2)$ by

$$\widetilde{\Gamma(\psi)} = \bigcup_{y \in Y_{1x}} \mathcal{G}(j_y^r \psi)\,. \tag{10}$$

If x^i, $\psi^a(y)$ is the coordinate expression of ψ, then $\widetilde{\Gamma(\psi)}$ is given by

$$z_i^a = \frac{\partial \psi^a}{\partial y^p} y_i^p + \Phi_i^p(x^i, y^p, \partial_\alpha \psi^a(y))\,. \tag{11}$$

In other words, $\Phi_i^p(x^i, y^p, \partial_\alpha \psi^a(y))$ is the coordinate expression of $\Gamma(\psi)$.

In [2] we defined a differentiable connection Γ by using the idea of jet prolongation of Γ. However, now we find it more convenient to express such an idea in terms of the tangent prolongation. We recall only briefly that the tangent bundle $Tp : T\mathcal{F}(Y_1, Y_2) \to TM$ is the space of all tangent vectors $X = \frac{\partial}{\partial t}\big|_0 \gamma(t)$ to the smooth curves $\gamma : \mathbb{R} \to \mathcal{F}(Y_1, Y_2)$, [2]. The vertical tangent bundle $V\mathcal{F}(Y_1, Y_2) \subset T\mathcal{F}(Y_1, Y_2)$ consists of all X satisfying $Tp(X) = 0$.

Definition 1. A connection $\Gamma : \mathcal{F}(Y_1, Y_2) \to \mathcal{F}J^1(Y_1, Y_2)$ is called differentiable, if the formula

$$T\Gamma\left(\frac{\partial}{\partial t}\Big|_0 \gamma(t)\right) = \frac{\partial}{\partial t}\Big|_0 \Gamma(\gamma(t)) \tag{12}$$

defines a smooth map $T\Gamma : T\mathcal{F}(Y_1, Y_2) \to TJ^1(Y_1, Y_2)$.

One verifies easily that a connection differentiable in this sense is also differentiable in the sense of [2]. Clearly, every finite order connection is differentiable.

In [2], the curvature $C\Gamma$ of a differentiable connection Γ has been defined as a map $C\Gamma : \mathcal{F}(Y_1, Y_2) \to \mathcal{F}(Y_1, VY_2 \otimes \Lambda^2 T^*M)$. If Γ is an r-th order connection with the coordinate expression $\Phi_i^a(x^i, y^p, z_\alpha^a)$, then $C\Gamma$ is an operator of the order $2r$, whose coordinate form is the antisymmetrization (in i and j) of

$$\frac{\partial \Phi_i^a}{\partial x^j} + \frac{\partial \Phi_i^a}{\partial z^b}\Phi_j^b + \frac{\partial \Phi_i^a}{\partial z_p^b} D_p \Phi_j^b + \dots + \frac{\partial \Phi_i^a}{\partial z_\alpha^b} D_\alpha \Phi_j^b \tag{13}$$

where D_p or D_α denotes the formal derivative with respect to y^p or with respect to multiindex α, [2].

Given a section $s : M \to \mathcal{F}(Y_1, Y_2)$, its absolute differential $\nabla_\Gamma s$ is a section $M \to \mathcal{F}(Y_1, VY_2 \otimes T^*M)$ defined by the difference $\nabla_\Gamma s(x) = j_x^1 s - \Gamma(s(x))$, [2]. If Γ is an r-th order connection (11) and $z^a = f^a(x, y)$ is the coordinate form of s, then the coordinate expression of $\nabla_\Gamma s$ is

$$\frac{\partial f^a(x, y)}{\partial x^i} - \Phi_i^a(x^i, y^p, \partial_\alpha f^a(x, y)) \tag{14}$$

Example 1. We present the simpliest example of a connection on $\mathcal{F}(Y_1, Y_2)$. A pair of connections Γ on Y_1 and Δ on Y_2 defines a connection (Γ, Δ) on $\mathcal{F}(Y_1, Y_2)$ as follows. We have to prescribe $(\Gamma, \Delta)(\psi)$, $\psi \in C^\infty(Y_{1x}, Y_{2x})$ as an affine morphism $J^1_x Y_1 \to J^1_x Y_2$ with the derived linear morphism $T\psi \otimes \mathrm{id}_{T^*_x M}$. Such a morphism is uniquely determined by requiring that $\Gamma(y)$ should be transformed into $\Delta(\psi(y))$, $y \in Y_{1x}$. Thus, if

$$dy^p = \Gamma^p_i(x, y)\, dx^i \quad \text{or} \quad dz^a = \Delta^a_i(x, z)\, dx^i \tag{15}$$

are the equations of Γ or Δ, respectively, then the equations of $(\Gamma, \Delta)(\psi)$ are

$$z^a_i = \frac{\partial \psi^a}{\partial y^p}(y^p_i - \Gamma^p_i(x, y)) + \Delta^a_i(x, \psi(y))\,. \tag{16}$$

Comparing with (11) we find that (Γ, Δ) is a first order connection with the associated map

$$\Delta^a_i(x, z) - z^a_p \Gamma^p_i(x, y)\,. \tag{17}$$

The second author, [7], or L. Mangiarotti and M. Modugno, [11], introduced the absolute differential $\nabla_{\Gamma,\Delta} f : Y_1 \to VY_2 \otimes T^*M$ of a base preserving morphism $f : Y_1 \to Y_2$ with respect to the pair Γ, Δ. Its coordinate form is

$$\frac{\partial f^a}{\partial x^i} + \frac{\partial f^a}{\partial y^p}\Gamma^p_i(x, y) - \Delta^a_i(x, f) \tag{18}$$

provided f is given by $z^a = f^a(x, y)$. Comparing with (14) and (17), we find that $\nabla_{\Gamma,\Delta} f$ coincides with the absolute differential $\nabla_{(\Gamma,\Delta)} f$ in the functional sense.

2. Projectable connections.

Consider a fibered manifold $\pi : Y_3 \to Y_2$, so that we have the total projection $\pi \circ p_2 : Y_3 \to M$. Denote by $\bar{\pi} : \mathcal{F}(Y_1, Y_3) \to \mathcal{F}(Y_1, Y_2)$ or $J^1\bar{\pi} : J^1\mathcal{F}(Y_1, Y_3) \to J^1\mathcal{F}(Y_1, Y_2)$ the induced projections. Analogously to the finite dimensional case, [7], we introduce the following concept.

Definition 2. A connections $\Gamma_1 : \mathcal{F}(Y_1, Y_3) \to J^1\mathcal{F}(Y_1, Y_3)$ is said to be projectable over a connection $\Gamma : \mathcal{F}(Y_1, Y_2) \to J^1\mathcal{F}(Y_1, Y_2)$, if the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(Y_1, Y_3) & \xrightarrow{\Gamma_1} & J^1\mathcal{F}(Y_1, Y_3) \\ {\scriptstyle \bar{\pi}}\downarrow & & \downarrow{\scriptstyle J^1\bar{\pi}} \\ \mathcal{F}(Y_1, Y_2) & \xrightarrow{\Gamma} & J^1\mathcal{F}(Y_1, Y_2) \end{array} \tag{19}$$

Let w^s be some additional fiber coordinates on $Y_3 \to Y_2$. Then the coordinate expression of the associated map $\mathcal{G}_1$ of an r-th order projectable connection Γ_1 over Γ is of the form

$$\Phi^a_i(x^i, y^p, z^a_\alpha)\,, \ \Psi^s_i(x^i, y^p, z^a_\alpha, w^s_\alpha)\,, \quad 0 \le |\alpha| \le r\,. \tag{20}$$

If $Y_3 \to Y_2$ is a vector bundle, then $\mathcal{F}(Y_1, Y_3) \to \mathcal{F}(Y_1, Y_2)$ and $J^1\mathcal{F}(Y_1, Y_3) \to J^1\mathcal{F}(Y_1, Y_2)$ are vector bundles, too. If Γ_1 is a linear morphism over Γ, then we say (analogously to [7]) that Γ_1 is a semilinear connection. The coordinate characterization of such a case is that the functions Ψ_i^s from [20] are linear in w_α^s, $0 \le |\alpha| \le r$.

Consider now the case $Y_3 = VY_2$, so that $\mathcal{F}(Y_1, VY_2) \approx V\mathcal{F}(Y_1, Y_2)$, [2]. In [9] the second author has constructed a canonical identification $i : J^1V\mathcal{F}(Y_1, Y_2) \to VJ^1\mathcal{F}(Y_1, Y_2)$. If Γ is a differentiable connection on $\mathcal{F}(Y_1, Y_2)$, the restriction on $T\Gamma$ to $V\mathcal{F}(Y_1, Y_2) \subset T\mathcal{F}(Y_1, Y_2)$ is a smooth map $V\Gamma : V\mathcal{F}(Y_1, Y_2) \to VJ^1\mathcal{F}(Y_1, Y_2)$. The composition

$$\mathcal{V}\Gamma = i^{-1} \circ V\Gamma : \mathcal{F}(Y_1, VY_2) \to J^1\mathcal{F}(Y_1, VY_2) \tag{21}$$

is a connection on $\mathcal{F}(Y_1, VY_2)$, which will be called the vertical prolongation of Γ. If Γ is an r-th order connection with the associated map

$$\Phi_i^a(x^i, y^p, z_\alpha^a) \tag{22}$$

then $\mathcal{V}\Gamma$ is also an r-th order connection with the associated map (22) and

$$\frac{\partial \Phi_i^a}{\partial z^b} Z^b + \cdots + \frac{\partial \Phi_i^a}{\partial z_\alpha^b} Z_\alpha^b \tag{23}$$

provided Z^a are the induced coordinates on $VY_2 \to Y_2$ and Z_α^a are the induced jet coordinates on $\mathcal{F}J^r(Y_1, VY_2)$. Thus, for every finite order connection Γ, $\mathcal{V}\Gamma$ is a semilinear connection over Γ.

Example 2. Consider the connection (Γ, Δ) from Example 1. Then $\mathcal{V}(\Gamma, \Delta)$ is a connection on $\mathcal{F}(Y_1, VY_2)$. On the other hand, if we construct the classical vertical prolongation $\mathcal{V}\Delta : VY_2 \to J^1VY_2$, then $(\Gamma, \mathcal{V}\Delta)$ is another connection on $\mathcal{F}(Y_1, VY_2)$. The comparison of both approaches is given by the following assertion.

Proposition 1. *We have* $\mathcal{V}(\Gamma, \Delta) = (\Gamma, \mathcal{V}\Delta)$.

Proof. Applying (23) to the associated map (17) of (Γ, Δ), we find

$$\frac{\partial \Delta_i^a}{\partial z^b} Z^b - Z_p^a \Gamma_i^p\,. \tag{24}$$

On the other hand, the additional equations of $\mathcal{V}\Delta$ are

$$dZ^a = \frac{\partial \Delta_i^a}{\partial z^b} Z^b \, dx^i \tag{25}$$

see [7]. Hence the associated map of $(\Gamma, \mathcal{V}\Delta)$ coincides with (17) and (24), which is the associated map of $\mathcal{V}(\Gamma, \Delta)$. $\square$

3. Tensor products.

We first recall a suitable approach to the tensor product of linear connections in finite dimension, [11]. Let $E_i \to M$, $i = 1, 2$, be two vector bundles and $\Lambda_i : E_i \to J^1E_i$ be linear connections. Consider the tensor map $\otimes : E_1 \times_M E_2 \to E_1 \otimes E_2$, so that $J^1\otimes : J^1E_1 \times_M J^1E_2 \to J^1(E_1 \otimes E_2)$. Then there exists a unique linear connection $\Lambda_1 \otimes \Lambda_2 : E_1 \otimes E_2 \to J^1(E_1 \otimes E_2)$ such that following diagram commutes

$$\begin{tikzcd} J^1E_1 \times_M J^1E_2 \arrow[r, "J^1\otimes"] & J^1(E_1 \otimes E_2) \\ E_1 \times_M E_2 \arrow[u, "\Lambda_1 \times_M \Lambda_2"] \arrow[r, "\otimes"] & E_1 \otimes E_2 \arrow[u, "\Lambda_1 \otimes \Lambda_2"'] \end{tikzcd} \tag{26}$$

where $\Lambda_1 \times_M \Lambda_2$ means the fiber product of maps over id_M.

We need a modification of this idea to the functional case. First of all, we study the "pure" case $E \to M$ is a vector bundle, Γ is an r-th order linear connection on $\mathcal{F}(Y_1, E)$ and Λ is a classical linear connection on another vector bundle $E_2 \to M$. Let x^i, w^s be some linear fiber coordinates on E, so that the coordinate expression of the associated map $\mathcal{G} : \mathcal{F}J^r(Y_1, E) \to S(J^1Y_1, J^1E)$ is of the form

$$\Psi^s_{it}(x, y)w^t + \cdots + \Psi^{s\alpha}_{it}(x, y)w^t_\alpha \, . \tag{27}$$

Let x^i, u^h be some linear fiber coordinates on $E_2 \to M$ and let the equations of Λ be

$$du^h = \Lambda^h_{ki}(x)u^h \, dx^i \, . \tag{28}$$

We have the tensor map $\otimes : \mathcal{F}J^r(Y_1, E) \times_M E_2 \to \mathcal{F}J^r(Y_1, E \otimes E_2)$, $\otimes(j^r_y\psi, u) = j^r_y(\psi\otimes u)$, $\psi : Y_{1x} \to E_x$, $u \in E_{2x}$. On the other hand, $\otimes : E \times_M E_2 \to E \otimes E_2$ defines $J^1\otimes : J^1E \times_M J^1E_2 \to J^1(E \otimes E_2)$ and this induces a map $\tau : S(J^1Y, J^1E) \times_M J^1E_2 \to S(J^1Y, J^1(E \otimes E_2))$. The coordinate form of τ is

$$\tau(x^i, y^p, w^s, b^s_p y^p_i + c^s_i, u^h, u^h_i) = (x^i, y^p, w^s u^h, u^h(b^s_p y^p_i + c^s_i) + w^s u^h_i) \tag{29}$$

Then one verifies easily that there is a unique linear r-th order connection $\Gamma \otimes \Lambda$ on $\mathcal{F}(Y_1, E \otimes E_2)$ such that its associated map $\mathscr{H}$ makes the following diagram commutative

$$\begin{tikzcd} S(J^1Y_1, J^1E) \times_M J^1E_2 \arrow[r, "\tau"] & S(J^1Y_1, J^1(E \otimes E_2)) \\ \mathcal{F}J^r(Y_1, E) \times_M E_2 \arrow[u, "\mathcal{G} \times_M \Lambda"] \arrow[r, "\otimes"] & \mathcal{F}J^r(Y_1, E \otimes E_2) \arrow[u, "\mathscr{H}"'] \end{tikzcd} \tag{30}$$

The coordinate expression of $\mathscr{H}$ is

$$\Psi^s_{it}(x, y)v^{th} + \cdots + \Psi^{s\alpha}_{it}(x, y)v^{th}_\alpha + \Lambda^h_{ki}(x)v^{sk} \tag{31}$$

provided v^{sh} are the induced fiber coordinates on $E \otimes E_2$.

However, we need a more general situation. Let $E_1 \to Y_2$ be a vector bundle, Γ_1 be an r-th order semilinear connection on $\mathcal{F}(Y_1, E_1)$ over a connection Γ on $\mathcal{F}(Y_1, Y_2)$ and Λ be a classical linear connection on a vector bundle $E_2 \to M$. Then a direct modification of the previous construction leads to an r-th order semilinear connection $\Gamma_1 \otimes \Lambda$ on $\mathcal{F}(Y_1, E_1 \otimes E_2)$ over Γ, which is called the tensor product of Γ_1 and Λ. If (22) and (27) with the Ψ's being functions of x^i, y^p and z^a_α is the coordinate expression of Γ_1 and (28) are the equations of Λ, then the coordinate expression of $\Gamma_1 \otimes \Lambda$ is (22) and

$$\Psi^s_{it}(x, y, z^a_\alpha)v^{th} + \cdots + \Psi^{s\alpha}_{it}(x, y, z^a_\alpha)v^{th}_\alpha + \Lambda^h_{ki}(x)v^{sk}\,. \tag{32}$$

Example 3. In the situation of Example 2, we find easily $\mathcal{V}(\Gamma, \Delta) \otimes \Lambda = (\Gamma, \mathcal{V}\Delta \otimes \Lambda)$, where $\mathcal{V}\Delta \otimes \Lambda$ is a finite dimensional concept defined e.g. in [10], p. 365.

4. The iterated absolute differentiation.

In this section we assume Γ is a finite order connection on $\mathcal{F}(Y_1, Y_2)$. Let us consider a section $s : M \to \mathcal{F}(Y_1, Y_2)$ in the form of the associated morphism $\widetilde{s} = f : Y_1 \to Y_2$. Then $\nabla_\Gamma f : M \to \mathcal{F}(Y_1, VY_2 \otimes T^*M)$. Construct $\mathcal{V}\Gamma$ and take an auxiliary linear connection Λ on M. Then Λ^* is a linear connection on T^*M and we can construct $\mathcal{V}\Gamma \otimes \Lambda^*$.

Definition 3. $\nabla^2_{\Gamma,\Lambda} f = \nabla_{\mathcal{V}\Gamma\otimes\Lambda^*}(\nabla_\Gamma f)$ is called the iterated absolute differential of f.

If $Y_2 = E$ is a vector bundle, we can identify $\nabla_\Gamma f$ with a morphism $Y_1 \to E \otimes T^*M$ and $\nabla^2_{\Gamma,\Lambda} f$ with a morphism $Y_1 \to E \otimes \bigotimes^2 T^*M$. Hence we can construct the alternation $\operatorname{Alt}(\nabla^2_{\Gamma,\Lambda} f) : Y_1 \to E \otimes \Lambda^2 T^*M$. In the vector bundle case, the curvature $C\Gamma$ can be interpreted as a map $\mathcal{F}(Y_1, E) \to \mathcal{F}(Y_1, E \otimes \Lambda^2 T^*M)$. Let S_Λ be the torsion of Λ, so that the contraction $\langle S_\Lambda, \nabla_\Gamma f\rangle$ is a map $Y_1 \to E \otimes \Lambda^2 T^*M$. The following assertion generalizes the Ricci identity for non-linear connections, [1], to the functional case.

Proposition 2. *We have*

$$\operatorname{Alt}(\nabla^2_{\Gamma,\Lambda} f) = -(C\Gamma)(f) + \langle S_\Lambda, \nabla_\Gamma f\rangle \tag{33}$$

Proof. Let Γ be given by (22) and $\Lambda^k_{ij}(x)$ be the Christoffels of Λ. By (23) and (32), the equations of $\mathcal{V}\Gamma \otimes \Lambda^*$ are (22) and

$$\frac{\partial \Phi^a_j}{\partial z^b} v^b_i + \cdots + \frac{\partial \Phi^a_j}{\partial z^b_\alpha} v^b_{i\alpha} - \Lambda^k_{ij} v^a_k \tag{34}$$

By (14), the coordinate form of $\nabla_\Gamma f$ is

$$f^a_i = \frac{\partial f^a}{\partial x^i} - \Phi^a_i(x^i, y^p, \partial_\alpha f^a) \tag{35}$$

Hence the coordinate form of $\nabla_{\mathcal{V}\Gamma\otimes\Lambda^*}(f_i^a)$ is

$$(36)\qquad \frac{\partial}{\partial x^j}(f_i^a) - \frac{\partial \Phi_j^a}{\partial z^b} f_i^b - \frac{\partial \Phi_j^a}{\partial z_p^b}\partial_y(f_i^b) - \cdots - \frac{\partial \Phi_j^a}{\partial z_\alpha^b}\partial_\alpha(f_i^b) + \Lambda_{ij}^k f_k^a$$

If we evaluate the partial derivatives, we first obtain an expression

$$\frac{\partial^2 f^a}{\partial x^i \partial x^j} - \frac{\partial \Phi_j^a}{\partial z^b}\frac{\partial f^p}{\partial x^i} - \frac{\partial \Phi_i^a}{\partial z^b}\frac{\partial f^p}{\partial x^j} - \cdots - \frac{\partial \Phi_j^a}{\partial z_\alpha^b}\frac{\partial}{\partial x^i}\partial_\alpha f^p - \frac{\partial \Phi_i^a}{\partial z_\alpha^b}\frac{\partial}{\partial x^j}\partial_\alpha f^b$$

which is symmetric in i and j. Using (13) we find that the alternation of the remaining terms is equal to the right hand side of (33). □

In the finite order case, the first author deduced the Ricci identity on an arbitrary fibered manifold $Y \to M$, provided she replaced the tensor alternation by a more sophisticated operation on some special iterated 2-jets. We are going to develop such an operation in the functional case as well.

5. The jet space $J^r(N, \mathcal{F}(Y_1, Y_2))$.

Given a manifold N, a map $f : N \to \mathcal{F}(Y_1, Y_2)$ is called smooth, if
(i) $p \circ f : N \to M$ is a C^∞-map,
(ii) the induced map $\widetilde{f} : (p \circ f)^* Y_1 \to Y_2$,

$$\widetilde{f}(a, y) = f(a)(y), \qquad (a, y) \in (p \circ f)^* Y_1\,,$$

is also C^∞, provided $(p \circ f)^* Y_1 \to N$ denotes the induced bundle, [2].

Consider the map $J_N^r p_i : J^r(N, Y_i) \to J^r(N, M)$, $j_a^r h \mapsto j_a^r(p_i \circ h)$, $h : N \to Y_i$, $i = 1, 2$. Write $J_X^r(N, Y_i) = (J_N^r p_i)^{-1}\,(X) \subset J^r(N, Y_i)$, $i = 1, 2$, $X = j_a^r(p \circ f)$, $a \in N$. The smooth map f induces a map

$$J_a^r\, f : J_X^r(N, Y_1) \to J_X^r(N, Y_2),\ \ J_a^r\, f(j_a^r h) = j_a^r f(u)(h(u))\,,$$

where $h : N \to Y_1$ satisfies $p \circ f = p_1 \circ h$, $u \in N$. Let $g : N \to \mathcal{F}(Y_1, Y_2)$ be another map satisfying $X = j_a^r(p \circ g)$.

Definition 4. We say that f and g determine the same r-jet $j_a^r f = j_a^r g$ at $a \in N$, if

$$J_a^r f = J_a^r g : J_X^r(N, Y_1) \to J_X^r(N, Y_2)$$

The set of all such r-jets is denoted by $J^r(N, \mathcal{F}(Y_1, Y_2))$. This is a smooth space in the sense of Frölicher, [5]. In the same way we proceed if we have a subbundle $\mathcal{E} \subset \mathcal{F}(Y_1, Y_2)$.

To find a suitable description of the space $J^r(N, \mathcal{F}(Y_1, Y_2))$, we first discuss the case of one-point base, so that the bundles are identified with the standard fibers $Y_1 = Q_1$, $Y_2 = Q_2$. Then $\mathcal{F}(Y_1, Y_2) = C^\infty(Q_1, Q_2)$. A smooth map $f : N \to C^\infty(Q_1, Q_2)$ defines a map

$$\widetilde{J}_a^r f : Q_1 \to J_a^r(N, Q_2), \quad q \mapsto j_a^r f(u)(q)\,, \quad u \in N\,.$$

Consider another smooth map $g : N \to C^\infty(Q_1, Q_2)$. The following simple assertion is equivalent to some results from [9] and [14].

Proposition 3. *$j^r_a f = j^r_a g$ if and only if $\widetilde{J}^r_a f = \widetilde{J}^r_a g$. Conversely, for every C^∞-map $h : Q_1 \to J^r_a(N, Q_2)$, there exists a smooth map $f : N \to C^\infty(Q_1, Q_2)$ such that $h = \widetilde{J}^r_a f$.* □

Thus, $J^r_a(N, C^\infty(Q_1, Q_2))$ is identified with $C^\infty(Q_1, J^r_a(N, Q_2))$. Write

$$C^\infty_\alpha(Q_1, J^r(N, Q_2)) = \bigcup_{a\in N} C^\infty(Q_1, J^r_a(N, Q_2))$$

(The subscript α indicates we consider the maps into the fibers of the projection $\alpha : J^r(N, Q_2) \to N$.) Then we have

$$J^r(N, C^\infty(Q_1, Q_2)) = C^\infty_\alpha(Q_1, J^r(N, Q_2)) \tag{37}$$

Consider now the case of trivial bundles $Y_1 = M \times Q_1$, $Y_2 = M \times Q_2$, so that $\mathcal{F}(Y_1, Y_2) = M \times C^\infty(Q_1, Q_2)$. Hence a smooth map $f : N \to \mathcal{F}(Y_1, Y_2)$ is a pair of smooth maps $f_0 : N \to M$ and $f_1 : N \to C^\infty(Q_1, Q_2)$. Given another smooth map $g = (g_0, g_1)$ of N into $\mathcal{F}(Y_1, Y_2)$, one finds easily

$$j^r_a f = j^r_a g \quad \text{iff} \quad j^r_a f_0 = j^r_a g_0 \quad \text{and} \quad j^r_a f_1 = j^r_a g_1 \,. \tag{38}$$

By Proposition 3, we obtain

$$J^r(N, \mathcal{F}(Y_1, Y_2)) = J^r(N, M) \times_N C^\infty_\alpha(Q_1, J^r(N, Q_2)) \tag{39}$$

In other words, an element $X \in J^r_a(N, M \times C^\infty(Q_1, Q_2))$ is a pair (X_0, X_1), where $X_0 \in J^r_a(N, M)$ and X_1 is a map $X_1 : Q_1 \to J^r_a(N, Q_2)$. If we use some local coordinates on M, Q_1, Q_2 and N, (39) gives a coordinate description of $J^r(N, \mathcal{F}(Y_1, Y_2))$.

It is worthwhile to show a simple application of (39). Let $X = j^r_a f \in J^r_a(N, \mathcal{F}(Y_1, Y_2))$ and $A = j^r_b g \in J^r_b(P, N)_a$ be a classical jet of a manifold P into N. Then we can define the jet composition

$$X \circ A = j^r_b(f \circ g) \in J^r_b(P, \mathcal{F}(Y_1, Y_2)) \,. \tag{40}$$

To show correctness of this definition, we can write X in the above form $X = (X_0, X_1)$. Then $X \circ A = (X_0 \circ A, X_1 \circ A)$, where $X_1 \circ A : Q_1 \to J^r_b(P, Q_2)$ is the map $q \mapsto X_1(q) \circ A$.

6. Iterated jets.

The classical space of iterated 2-jets $\check{J}^2(N, M)$ of N into M is defined by the iteration $\check{J}^2(N, M) = J^1(N, J^1(N, M))$, [15]. Beside the source and target projections $\alpha_1 : \check{J}^2(N, M) \to N$ and $\beta_1 : \check{J}^2(N, M) \to J^1(N, M)$ we have the induced maps $J^1(1_N, \alpha) : \check{J}^2(N, M) \to J^1(N, N)$ and $J^1(1_N, \beta) : \check{J}^2(N, M) \to J^1(N, M)$, where 1_N means the identity of N, [1]. The set $\widetilde{J}^2(N, M)$ of nonholonomic 2-jets by Ehresmann, [4], is the subset of all $A \in \check{J}^2(N, M)$ satisfying $J^1(1_N, \alpha)(A) =$

$j^1_a 1_N$, $a = \alpha_1 A = \alpha(\beta_1 A)$. The semiholonomic 2-jets are further characterized by $\beta_1(A) = J^1(1_N, \beta)(A)$. If u^s or x^i are some local coordinates on N or M, then the induced coordinates on $\check{J}^2(N, M)$ are, [1],

$$(x^i, x^i_s, u^s, x^i_{os}, x^i_{st}, u^s_t, v^s)\,. \tag{41}$$

The first author introduced the concept of distinguished iterated 2-jet, [1]. Let $[a]$ denotes the constant map of N into $a \in N$. An element $A \in \widehat{J}^2(N, M)$ is called distinguished, if $J^1(1_N, \alpha)(A) = j^1_a[a]$ for some $a \in N$ and $\beta_1(A) = J^1(1_N, \beta)(A)$. The coordinate form of a distinguished iterated 2-jet is

$$(x^i, x^i_s, u^s, x^i_s, x^i_{st}, 0, u^s)\,. \tag{42}$$

The set of all distinguished iterated 2-jets is denoted by $\widehat{J}^2(N, M)$.

In the functional case, we define analogously

$$\check{J}^2(N, \mathcal{F}(Y_1, Y_2)) = J^1(N, J^1(N, \mathcal{F}(Y_1, Y_2)))\,. \tag{43}$$

Even here we have the projections

$$\alpha_1 : \check{J}^2(N, \mathcal{F}(Y_1, Y_2)) \to N, \quad \beta_1 : \check{J}^2(N, \mathcal{F}(Y_1, Y_2)) \to J^1(N, \mathcal{F}(Y_1, Y_2))$$

and the induced maps $J^1(1_N, \alpha) : \check{J}^2(N, \mathcal{F}(Y_1, Y_2)) \to J^1(N, N)$ and $J^1(1_N, \beta) : \check{J}^2(N, \mathcal{F}(Y_1, Y_2)) \to J^1(N, \mathcal{F}(Y_1, Y_2))$. To find a description of $\check{J}^2(N, \mathcal{F}(Y_1, Y_2))$ analogous to (39), we first remark, that if $Y \to M$, $Z \to M$ are two fibered manifolds, then $J^1(N, Y \times_M Z) = J^1(N, Y) \times_{J^1(N,M)} J^1(N, Z)$. In particular, if P is another manifold, then $\check{J}^2(N, M \times P) = J^1(N, J^1(N, M) \times_N J^1(N, P)) = \check{J}^2(N, M) \times_{J^1(N,N)} \check{J}^2(N, P)$. To simplify the notation, we write $C^\infty_1(Q_1, \check{J}^2(N, Q_2))$ for the space of all C^∞-maps from Q_1 into the fibers of $J^1(1_N, \alpha) : \check{J}^2(N, Q_2) \to J^1(N, N)$. Then we deduce analogously to Section 5

$$\check{J}^2(N, M \times C^\infty(Q_1, Q_2)) = \check{J}^2(N, M) \times_{J^1(N,N)} C^\infty_1(Q_1, \check{J}^2(N, Q_2))\,. \tag{44}$$

The idea of distinguished iterated 2-jet takes place in the functional case as well.

Definition 5. An element $A \in \check{J}^2(N, \mathcal{F}(Y_1, Y_2))$ will be called distinguished, if $J^1(1_N, \alpha)(A) = j^1_a[a]$ for some $a \in N$ and $\beta_1(A) = J^1(1_N, \beta)(A)$.

The set of all distinguished iterated 2-jets from N into $\mathcal{F}(Y_1, Y_2)$ will be denoted by $\widehat{J}^2(N, \mathcal{F}(Y_1, Y_2))$.

One verifies easily, that if A is expressed in the form (44) as (A_0, A_1), $A_0 \in \check{J}^2(N, M)$, $A_1 : Q_1 \to \check{J}^2(N, Q_2)$, then $A \in \widehat{J}^2(N, \mathcal{F}(Y_1, Y_2))$ iff $A_0 \in \widehat{J}^2(N, M)$ and the values of A_1 lie in $\widehat{J}^2(N, Q_2)$.

7. The difference tensor.

Using an idea by Pradines, [13], the first author clarified that every distinguished iterated 2-jet $A \in \widehat{J}^2_a(N,M)_x$ determines a tensor $\delta(A) \in T_xM \otimes \Lambda^2 T^*_a N$, which is called the difference tensor of A. If (42) is the coordinate form of A, then the coordinate expression of $\delta(A)$ is

$$x^i_{st} - x^i_{ts}\,. \tag{45}$$

In the functional case, $T_\psi \mathcal{F}(Y_1,Y_2)\otimes\Lambda^2T^*_aN$ will mean the space of all bilinear antisymmetric maps from T_aN into $T_\psi(Y_1,Y_2)$. For every $A \in \widehat{J}^2(N,\mathcal{F}(Y_1,Y_2))_\psi$, we construct in the same way as in [1] an element

$$\delta(A) \in T_\psi\mathcal{F}(Y_1,Y_2)\otimes\Lambda^2T^*_aN$$

which is also called the difference tensor of A. Our construction implies directly the following assertion.

Proposition 4. *If $A \in \widehat{J}^2(N, M\times C^\infty(Q_1,Q_2))$ is of the form (A_0,A_1), $A_0 \in \widehat{J}^2(N,M)$, $A_1 : Q_1 \to \widehat{J}^2(N,Q_2)$, then $\delta(A) = (\delta(A_0),\delta(A_1))$.*

8. The general Ricci identity.

Consider a finite order connection Γ on an arbitrary bundle $\mathcal{F}(Y_1,Y_2)$ and a linear connection Λ on M. Analogously to the finite dimensional case, [1], we find that for every base preserving morphism $f : Y_1 \to Y_2$, the values of $\nabla^2_{\Gamma,\Lambda}f$ lie in $\widehat{J}^2(M,\mathcal{F}(Y_1,Y_2))$ and the values of $\delta(\nabla^2_{\Gamma,\Lambda}f)$ lie in $V\mathcal{F}(Y_1,Y_2)\otimes\Lambda^2T^*M \subset T\mathcal{F}(Y_1,Y_2)\otimes\Lambda^2T^*M$. Hence $\delta(\nabla^2_{\Gamma,\Lambda}f)$ can be interpreted as a map $Y_1 \to VY_2\otimes\Lambda^2T^*M$. The following assertion is the general Ricci identity for finite order connections on $\mathcal{F}(Y_1,Y_2)$.

Proposition 5. *We have*

$$\delta(\nabla^2_{\Gamma,\Lambda}f) = -(C\Gamma)(f) + \langle S_\Lambda, \nabla_\Gamma f\rangle$$

Proof. By (45) and Proposition 4, the coordinate form of (46) is the same as in Proposition 2. □

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