ARCHIVUM MATHEMATICUM (BRNO)
Tomus 32 (1996), 355 – 372

CALCULUS OF FLOWS ON CONVENIENT MANIFOLDS

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To Ivan Kolář, on the occasion of his 60th birthday

Abstract. The study of diffeomorphism group actions requires methods of infinite dimensional analysis. Really convenient tools can be found in the Frölicher - Kriegl - Michor differentiation theory and its geometrical aspects. In terms of it we develop the calculus of various types of one parameter diffeomorphism groups in infinite dimensional spaces with smooth structure. Some spectral properties of the derivative of exponential mapping for manifolds are given.

1. Introduction

The foundations of differential geometry stream visibly towards the geometry in infinite dimensional spaces and to methods of infinite analysis. Adaptations to Hilbert and Banach manifolds were done by P.Libermann and P. de la Harpe in the seventies. Then R.S.Hamilton gave beautiful examples of making geometrical use from the inverse function theorem of Nash and Moser in tame Fréchet spaces. The theory of infinite Lie groups and Lie algebras was intensively developed, to mention H. Omori, J. Milnor, J. Grabowski and others. Recently A. Kriegl and P. Michor introduced, studied and effectively applied the concept of regular Lie groups, which is more general and simpler than the one originating from Omori. They start from the Milnor's idea that smooth curves in the Lie algebra should integrate to smooth curves in the group (an evolution operator exists). This allowed them to advance immensely the foundations of geometry of principal bundles of infinite dimension, e.g.: the theory of connections, invariant calculus and Lie theory of regular Lie groups. It is worth to underline their result that *Lie algebra homomorphisms integrate to Lie group homomorphisms, if the source group is simply connected and the image group is regular.*

Classical differentiation in linear spaces of arbitrary dimension uses Banach spaces; but most function spaces are not Banach spaces, in particular those occurring on smooth manifolds. Another deficiency is that the space $C^\infty(E,F)$ of

1991 *Mathematics Subject Classification*: 58B25, 22E65.
Key words and phrases: flow, diffeomorphism group, regular Lie group action, Frölicher-Kriegl differential calculus, 1-parameter group of bounded operators.

smooth maps is no longer a Banach space. The space of vector fields on a smooth manifold M is naturally a modelling vector space for the diffeomorphism group $Diff(M)$, but it is in a canonical way a non-normable (nuclear) Fréchet space. Other difficulties in applying the Banach space theory to diffeomorphism groups are indicated by Hamilton in [4].

Among many theories of differentiation in non-normable spaces, seemingly the most appropriate and conceptually simpliest is that created by Frölicher and Kriegl [1]. It is based on the Boman's idea of testing smoothness along smooth curves.

A mapping $f : E \to F$ between locally convex vector spaces is called *smooth* or C^∞, if it maps smooth curves into smooth curves, i.e., if

$$f \circ c \in C^\infty(R, F) \quad \text{for all} \quad c \in C^\infty(R, E).$$

Definition. A locally convex vector space E is called *convenient* (we shall call it briefly a *Con-space*) if for a curve $c : R \to E$, $f \circ c$ smooth for all continuous functionals f on E implies c is smooth.

An equivalent defining property is that for every smooth curve in E Riemann integrals exist over compact intervals.

Remark. For finite dimensional smooth manifolds the locally convex topology of $C^\infty(M, N)$ is the classical C^∞ compact-open topology of uniform convergence on compact sets of all derivatives (cf. M.Hirsch, Differential Topology, 1976).

We shall be in need to apply some fundamental properties of the category of convenient spaces which we quote below after [1] and [6].

1. The space $C^\infty(E, F)$ of smooth maps is canonically a Con-space. The subspace $L(E, F)$ of all bounded linear maps is closed in $C^\infty(E, F)$. A linear (or multilinear) map is smooth if and only if it is bounded.

2. The category Con is cartesian closed, i.e.,

$$C^\infty(E \times F, G) \cong C^\infty(E, C^\infty(F, G))$$

is a linear diffeomorphism of Con-spaces.

3. If $f : U \subset E \to F$ is smooth then the derivative

$$Df(x)v := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

exists and is linear, and smooth as a map $Df : U \times E \to F$ as well as a map $Df : U \to L(E, F)$ where $L(E, F)$ has the topology of uniform convergence on bounded subsets. The chain rule holds.

4. (Taylor expansion) Let $U \subset E$ be an open subset in the final topology with respect to all smooth curves on E (i.e., the "$c^\infty - topology$)". Let $f : U \to F$ be a smooth map, $x \in U$ and $x + [0, 1]v \in U$. Then the mean value theorem

$$f(x + v) = f(x) + \int_0^1 (Df)(x + tv).vdt$$

and the usual Taylor's formula hold.

5. (The smooth uniform boundedness theorem) A linear map $f : E \to C^\infty(F, G)$ is smooth if and only if the evaluation map $ev_y f : E \to G$ is smooth for each $y \in F$.

6. All Fréchet spaces are convenient; also the Schwartz space of test functions with compact support in R^n is convenient.

7. The notions of a smooth manifold, tangent bundle, Lie group, smooth Lie group action, etc., are defined in a classical way. In particular, for finite dimensional smooth manifolds M, N, where M is supposed to be compact, the space $C^\infty(M, N)$ has a natural structure of a smooth manifold in the Con category (de facto it is a Fréchet manifold, with the collection of all C^r norms)

The group $Diff(M)$ of smooth diffeomorphisms of M is open in $C^\infty(M, M)$ and is a smooth convenient Lie group. The tangent space at the identity is naturally identified with its model space, and equally with the linear space $\Gamma(TM)$ of smooth vector fields on M.

It should be noted that in the noncompact case, there have to be assumed certain behaviors at infinity; for instance, some or all C^r-norms bounded, compact supports, etc. Accordingly, we get a Banach, Fréchet or convenient vector space structure.

For any vector bundle E over M, we let $\Gamma(E)$ denote the real vector space of continuous sections of E. In turn, for $f : N \to M$, $\Gamma(f^*TM)$ is the space of *vector fields along* f (= maps $N \to TM$ covering f).

We shall make the identifications for the tangent spaces to the diffeomorphism group at the identity and at an arbitrary diffeomorphism f:

$$T_{id}Diff(M) = \Gamma(TM) \quad \text{and} \quad T_f Diff(M) = \Gamma(f^*TM).$$

We let f_* and f^* be the smooth linear maps in $\Gamma(TM)$ given by

$$f_*X = (Tf.X) \circ f^{-1}, \qquad f^*X = (Tf)^{-1}.(X \circ f),$$

which represent the adjoint actions of f on $\Gamma(TM)$. In the next section we give particulars concerning the X-derivative of the exponential map and its Taylor expansions. The formula for the derivative was computed first in another way by J. Grabowski [3]. For basic concepts in modern foundations of differential geometry the reader can refer to Kolář-Michor-Slovak in [5].

2. The map $X \to \exp X$

2.1. Higher order derivatives. Let M be a finite dimensional smooth and compact manifold. Then $\mathcal{D}(M) = Diff(M)$ has the structure of a Fréchet Lie group with the strong C^∞ topology; it is also a convenient regular Lie group (in the sense of Kriegl-Michor) and its c^∞- topology coincides with the above mentioned Whitney topology. The Lie algebra of $\mathcal{D}(M)$ is $\Gamma(TM)$, the space of vector fields on M with the negative of the usual Lie bracket.

Each vector field X generates a global one-parameter group $t \to \exp tX$ for $t \in R$. Thus we have a map

$$\phi : \Gamma(TM) \times M \times R \to M$$

$$(X, x, t) \to (\exp tX)(x).$$

X being smooth, the map ϕ is also smooth in x, t. We want to look closer at the smoothness in X. We denote by D and T denote the derivative in X and the usual tangent map operator, respectively. The X-derivative at tX in direction Y is then

$$D\exp(tX)Y = \frac{d}{ds}\Big|_{s=0} \exp t(X + sY).$$

To compute the derivative we start with computing the (left) logarithmic derivative of exp.

$$\begin{aligned}
T(\exp(-X)) \cdot D\exp(X)Y &= T(\exp(-X))\ \frac{d}{ds}\Big|_{s=0} \exp(X + sY) \\
&= \frac{d}{ds}\Big|_{s=0} \int_0^1 \frac{d}{dt} \exp(-tX) \circ \exp t(X + sY) dt \\
&= \int_0^1 \frac{d}{ds}\Big|_{s=0} \{-T(\exp(-tX)) \cdot X \circ \exp t(X + sY) \\
&\quad + T(\exp(-tX)) \cdot (X + sY) \circ \exp t(X + sY)\} dt \\
&= \int_0^1 \frac{d}{ds}\Big|_{s=0} sT(\exp(-tX)) \cdot Y \circ \exp t(X + sY) dt \\
&= \int_0^1 T(\exp(-tX)) \cdot Y \circ \exp tX dt = \int_0^1 (\exp X)^* Y\, dt.
\end{aligned}$$

Hence we obtain

$$D\exp(X)Y = T(\exp X) \cdot \int_0^1 (\exp tX)^* Y\, dt = \int_0^1 (\exp tX)_* Y\, dt \circ \exp X. \tag{2.1}$$

From this we get for $\phi_t = \exp tX$ the formula

$$D\phi_t Y = T\phi_t \cdot \int_0^t \phi_s^* Y\, ds = \int_0^t (\phi_s)_* Y\, ds \circ \phi_t. \tag{2.2}$$

Clearly $D\exp(X)Y$ is a smooth vector field on M and the map $(X, Y) \to D\exp tX \cdot Y$ is continuous in the C^∞-topology.

Remark. In the noncompact case we may also consider the derivative $D\exp(tX)$ defined by formulae (2.1). If necessary one may change the vector field by a scalar factor to obtain one which defines a global flow. If we deal with local problems in which the exponential mapping is involved, we can take the derivative at tX for t small enough so that the formula

$$D\exp(tX)Y = T(\exp tX) \cdot \int_0^t (\exp sX)^* Y\, ds$$

is well defined in an open subset of $M \times R$.

In this way one can apply the calculus developed on compact manifolds to study groups of compactly supported diffeomorphisms, or groups of germs of diffeomorphisms at a fixed point and their smooth actions, also in the noncompact case (one of typical subjects in the theory of natural bundles). Contrary to the global compact approach, there seems to be no literature on whether the behavior of the exponential map is also so bad in the local version, at least that I know of.

For a fixed X we introduce the *mean adjoint operators* A and B on $\Gamma(TM)$ defined by

$$AY = \int_0^1 (\phi_s)_* Y\, ds \quad BY = \int_0^1 \phi_s^* Y\, ds. \tag{2.3}$$

Now, (2.1) can be written

$$D\phi Y = AY \circ \phi = T\phi \cdot BY,$$

or briefly

$$D\phi = A \circ \phi = T\phi \cdot B \tag{2.4}$$

where we set $\phi = \exp X$.

In order to compute higher order X-derivatives $D^k\phi$, it is enough to use repeatedly (2.2) and the following formula for the derivative of the adjoint map $X \to (\exp tX)_* Y = (\phi_t)_* Y$

$$D((\exp tX)_* Y) \cdot Z = \big[(\exp tX)_* Z, \int_0^t (\exp sX)_* Y\, ds\big]$$

where $[\,,]$ denotes the Lie bracket of vector fields. This formula will be derived in Section 2. Clearly all the higher order derivatives are smooth families (with X as parameter) of multilinear differential operators in $\Gamma(TM)$. Therefore the map $X \to \exp X$ is smooth.

As example we give a formula for the second order derivative, for brevity we use the operator $A_t = \int_0^t (\exp sX)_* Y\, ds$,

$$D^2(\exp tX)_*(Y, Z) = \{A_t[Y, A_t Z] + T(A_t Y) \cdot A_t Z\} \circ \exp X.$$

The map $X \to A(X)$ is exactly the (right) *logarithmic derivative* $\delta \exp$ of the exponential map. (For the details on logarithmic derivatives of smooth maps from manifolds into convenient Lie groups cf. [6])

2.2. Taylor expansion of $\exp X$**.** Let $M = E$ be a Con-space. We can identify $TM = E \times E$ and $\Gamma(TM) = C^\infty(E, E)$. Let $X \in C^\infty(E, E)$ be a vector field on E which admits a global flow $\exp tX$. Then

$$D_t \exp tX = X \circ \exp tX, \qquad \exp 0 = id_E. \tag{2.5}$$

X can also be considered as a differential operator on $C^\infty(E, E)$ by $Y \to XY = D_t(Y \circ \exp tX)(t = 0)$.

Since $D^k(\exp X)(X=0) = D_t^k(\exp tX)(t=0)$, the Taylor expansion of $\exp X$ at $X=0$ coincides with the one for $t \to \exp tX$ at $t=0$.

From (2.5) we obtain easily

$$D_t^k(Y \circ \exp tX) = (X^{(k)}Y) \circ \exp tX \tag{2.6}$$

Thus

$$Y \circ \exp tX = Y + tXY + \cdots + \frac{t^k}{k!}X^{(k)}Y + R_k \tag{2.7}$$

$$R_k = \int_0^t \frac{(t-s)^k}{k!}X^{(k+1)}Y \circ \exp sX\,ds$$

The formal Taylor series will be then $Y \circ \exp tX \simeq e^{tX}Y$.

In view of (2.5) by integration we get

$$\exp tX \simeq id + tX + \frac{t^2}{2!}XX + \cdots + \frac{t^k}{k!}X^{(k-1)}X + \cdots$$

or finite expansion at $X=0$

$$\exp tX = id + tX + \cdots + \frac{t^k}{k!}X^{(k-1)}X + \int_0^t \frac{(t-s)^k}{k!}X^{(k)}X \circ \exp sX\,ds \tag{2.8}$$

We can show that if $E = R^n$ and X is real analytic then the Taylor series converges uniformly for small t to yield

$$\exp tX = id + \sum_{k=1}^{\infty} \frac{t^k}{k!}X^{(k-1)}X.$$

2.3. Invertibility of the X-derivative. There has been written a lot on the bad behavior of the exponential map and its derivative $D\exp(X)$ in respect of their invertibility on compact manifolds (N. Kopell, J. Palis, R. S. Hamilton, J. Grabowski). We would like to say something positive. To gain this we ask first for the invertibility of the derivative at a point. More on spectral properties of maps connected with the exponential will appear in a separate paper.

We start with the modified formula (2.4)

$$D\phi \circ \phi^{-1} = A. \tag{2.9}$$

We observe that the injectivity or surjectivity of the derivative $D\phi$, where $\phi = \exp X$, coincides with that for the mean adjoint operator A. Therefore we study our problem via A.

It should be noted that the differential operator $ad(X) : Y \to [X,Y]$ is the infinitesimal generator of the group $\{\phi_t^*\}$, $t \in R$, of continuous linear operators in $\Gamma(TM)$. In turn, $ad(-X)$ is the infinitesimal generator of the group $\{(\phi_t)_*\}$.

These one-parameter groups are *strongly continuous* in the sense that for all Y, $\lim_{t\to 0}\phi_t^* Y = Y$. Consequently, we have

$$ad(X)\int_0^t (\phi_s)_* Y\,ds = \int_0^t (\phi_s)_* ad(X) Y\,ds = Y - (\phi_t)_* Y.$$

For $t = 1$ it can be written

$$ad(X)\circ A = A\circ ad(X) = I - \phi_*. \tag{2.10}$$

Lemma 1. *The following facts are true.*

(1) *If a closed subspace of* $\Gamma(TM)$ *is invariant under the group* $(\phi_t)_*$, *then it is also invariant under* $ad(X)$ *and* A.

(2) *If* $I-\phi_*$ *is injective, surjective or invertible on a closed* $(\phi_t)_*$*-invariant subspace of* $\Gamma(TM)$, *then so are respectively both* $ad(X)$ *and* A.

(3) $\ker(I-\phi_*) = \{Y$; *such that* $Y_k = \int_0^1 e^{-2\pi iks}(\phi_s)_* Y\,ds$ *satisfies* $[Y_k, X] = 2\pi ik Y_k$ *for some* $k\in Z$ $\}$ = $\{$ *Y; such that* $(\phi_t)_* Y$ *is periodic with period 1* $\}$

(4) *A is the identity on* $\ker ad(X)$ *and*

$$\ker(I-\phi_*) = \ker ad(X)\oplus \ker A \tag{2.11}$$

(topological direct sum).

Proof. (1) $ad(X)$- invariance follows by differentiation of $(\phi_t)_* Y$ at $t=0$ for Y in the subspace. Then the invariance under A follows from (2.10).

The assertion (2) is a direct consequence of (2.10).

The first equality in (3) can be verified as follows

$$[Y_k, X] = \int_0^1 e^{-2\pi iks}((\phi_s)_* Y)'\,ds = (\phi_* - I)Y + 2\pi ik Y_k,$$

where we integrated by parts. So Y is in $\ker(\phi_* - I)$ if and only if Y_k is an eigenvector of $ad(X)$ with eigenvalue $-2\pi ik$. Now, the second set in (3) comes from the implication

$$\phi_* Y = Y \Rightarrow (\phi_{t+1})_* Y = (\phi_t)_* Y$$

for all t.

To prove (4), let Y be in $\ker ad(X)$, so $[X,Y]=0$. It follows that Y is a fixed point of all isomorphisms $(\phi_t)_*$. Therefore $AY = Y$. As to the remaining part of (4), let Y be in $\ker(I-\phi_*)$, so the map $t\to(\phi_t)_* Y$ is periodic with period 1, and let

$$(\phi_t)_* Y = Y_0 + \sum_{k=1}^{\infty} Y_k e^{2\pi ikt}, \tag{2.12}$$

$$Y_0 = \int_0^1 (\phi_s)_* Y\,ds, \qquad Y_k = \int_0^1 e^{-2\pi isk}(\phi_s)_* Y\,ds \tag{2.13}$$

be its Fourier series. We see that $Y_0 = AY$, hence $Y\in\ker A$ if and only if the first Fourier coefficient is zero. On the other hand we easily get $ad(X)Y_0 = Y - \phi_* Y = 0$, so Y_0 is in $\ker ad(X)$. This proves the decomposition (2.11). Since $ad(X)$ and A are closed linear operators on $\Gamma(TM)$ it follows that $\ker ad(X)$ and $\ker A$ are closed (disjoint) subspaces; hence the sum (2.11) is topological. $\square$

Additionally we indicate the following property of Fourier coefficients Y_k, $k \in Z$, from (2.13).

$$(\phi_t)_* Y_k = e^{2\pi i k} Y_k, \tag{2.14}$$

which is equivalent to

$$[Y_k, X] = 2\pi i k Y_k. \tag{2.15}$$

The equivalence verifies as follows: (2.14) implies (2.15) by differentiation at $t = 0$.

For the inverse, we pass from (2.15) to

$$[(\phi_t)_* Y_k, X] = \frac{d}{dt}(\phi_t)_* Y_k = 2\pi i k (\phi_t)_* Y_k.$$

Thus $(\phi_t)_* Y_k$ satisfies differential equation $Y' = 2\pi i k Y$ with initial condition $Y(0) = Y_k$, whose solution is $Y(t) = Y_k e^{2\pi i k t}$. Hence (2.14).

Theorem 1. *The derivative $D \exp X$ is injective if and only if equations* (2.14) *or* (2.15) *have only trivial solutions for $k \neq 0$. The non-trivial solutions Y_k span over R the kernel of $D \exp X$.*

Proof. Imposing $(\phi_{-t})_*$ on (2.12) and using (2.14) we get easily

$$Y = (\phi_{-t})_* Y_0 + \sum_{k \geq 1} Y_k = Y_0 + \sum_{k \geq 1} Y_k,$$

since Y_0 is a fixed point of the adjoint operator. Moreover, again using (2.14) we compute $AY_k = \int_0^1 (\phi_t)_* Y_k dt = \int_0^1 e^{2\pi i k t} Y_k dt = 0$. This completes the proof. $\square$

Remark. Since $(\exp X)_* X = X$ for every X, the operator $I - \phi_*$ is never injective in the entire space $\Gamma(TM)$. Therefore it is reasonable to consider the above spectral properties on closed invariant subspaces, which we mentioned in Lemma 1.

Recall by the way that $f \in \mathcal{D}(M)$ is an *Anosov diffeomorphism* if and only if $f_* - I$ is an automorphism on $\Gamma(TM)$ (cf. Mather [7]); hence the

Corollary. An Anosov diffeomorphism never imbeds into a flow.

It is worth to remark that the set of all Anosov diffeomorphisms on Riemannian compact manifold is open in $Diff^1(M)$, so they are not so few.

Proposition 1. *Let $E = \Gamma_c(TM)$ be the space of all compactly supported vector fields on a smooth, connected and noncompact manifold M. Suppose that a vector field X on M is complete and defines a flow ϕ_t which has no relatively compact trajectories except fixed points, which are hyperbolic. Then $D\exp(X)$, as well as $ad(X)$, are injective on E.*

Proof. It is enough to show that equations (2.14) have only trivial solutions. It is easy to see that the subspace E of $\Gamma(TM)$ is invariant under the adjoint action of any flow. We write (2.14) in the form

$$(T\phi_t.Y)(\phi_t^{-1}(x)) = e^{2\pi ikt}Y(x) \tag{2.16}$$

for some integer k. If the trajectory $\phi_t(x)$ is not compact, then there is a $T = T(x)$ such that the point $\phi_T^{-1}(x)$ is outside the support of Y. Then the LHS is zero, so is also the RHS; hence $Y(x) = 0$.

Suppose now that $x = a$ is a critical hyperbolic point of X. Then for all $t \in R$, $\phi_t(a) = a$, $T\phi_t(a) = e^{tD_aX}$ and this linear operator in the tangent space T_aM has no eigenvalue with modulus equal to 1. In this case (2.16) writes

$$e^{tD_aX}Y(a) = e^{2\pi ikt}Y(a) \tag{2.17}$$

which implies immediately $Y(a) = 0$. □

Recall that an *Anosov flow* on a complete Riemannian manifold M is a flow ϕ_t whose induced flow $T\phi_t$ on TM is hyperbolic in the following sense: The tangent bundle TM can be written as the Whitney sum of 3 invariant subbundles, $TM = E_+ \oplus E_- \oplus E_o$ where on E_+, $T\phi_t$ is contracting, on E_-, $T\phi_t$ is expanding and E_0 is the one-dimensional bundle defined by the infinitesimal generator of ϕ_t. An important class of examples of Anosov flows are the geodesic flows on the tangent bundles of Riemannian manifolds of negative curvature.

Proposition 2. *If a vector field X on a complete Riemannian manifold M generates an Anosov flow ϕ_t, then the derivative of the exponential map at X is an isomorphism of $\Gamma(TM)$.*

Proof. By standard arguments (cf. Mather [7]) it follows that for the restriction of the adjoint operator ϕ_* to $\Gamma(E_+ + E_-)$ there exists a continuous inverse $(I - \phi_*)^{-1}$. We let p_o, p_1 denote respectively the projections of $\Gamma(TM)$ onto $\Gamma(E_0)$ and $\Gamma(E_+ + E_-)$. Then using (2.9) and (2.10) we verify directly that, given a vector field Z on M, the equation

$$D\exp(X)Y = Z$$

has a unique solution

$$Y = (ad(X) \circ (I - \phi_*)^{-1} \circ p_1 + p_o)Z \circ \phi, \tag{2.18}$$

where $\phi = \exp X$ and Y depends continuously on Z. □

Proposition 3. *For $a \in M$ and $\phi_t = \exp tX$, the linear map $Y \to D(\phi_t)_*Y(a)$ from $\Gamma(TM)$ into the tangent space $T_{\phi_t(a)}M$ is surjective except uniquely when $X(a) = 0$ and the operator D_aX in T_aM has an eigenvalue $\lambda = \frac{2\pi ik}{t}$ for some $k \in Z \setminus \{0\}$.*

Proof. In view of (2.6) it suffices to consider the surjectivity of the map $Y \to \int_0^t (\phi_s)_* Y(a) ds$ valued in $T_a M$. Using the Taylor expansion of second order in t, with $\frac{d}{dt}(\phi_t)_* Y = (\phi_t)_*[Y, X]$, we obtain

$$\int_0^t (\phi_s)_* Y ds = tY + \int_0^t (t-s)(\phi_s)_*[Y, X] ds. \tag{2.19}$$

Since for small t the problem can be considered in a local chart on M around a, we can choose Y to be locally constant $= Y_o$. Then $[Y_o, X] = D_x X.Y_o$ and setting $P_s = T\phi_s \circ D_x X \circ \phi_s^{-1}$ it follows from (2.19)

$$\int_0^t (\phi_s)_* Y_o ds = (tI + \int_0^t (t-s) P_s(a) ds) Y_o \tag{2.20}$$

For sufficiently small t the operator at Y_o on the RHS is invertible. Thus there is a $t_o > 0$ such that $Y \to \int_0^t (\phi_s)_* Y(a) ds$ is surjective onto $T_a M$ for all $0 < t \leq t_o$.

Suppose now that $X(a) \neq 0$, so $\phi_s(a) \neq a$ for small s. We take $t_1 > t_o$ such that the trajectory $\phi_{t_1}(a)$ is not periodic or that t_1 is not greater than the minimal period in the opposite case. We choose an open subset U such that $\phi_s^{-1}(x) \in U$ for $0 < s < t_o$ and $\phi_s^{-1}(x)$ is not in U for $t_o \leq s < t_1$. Then for any vector field Y with support in U

$$\int_0^t (\phi_s)_* Y ds = \int_0^{t_o} (\phi_s)_* Y ds$$

for $t_o < t \leq t_1$. For suitable t_o this means that the surjectivity in question prolongs to all t from the interval $[0, t_1]$. If the trajectory is periodic it follows from above that $Y \to \int_0^t (\phi_s)_* Y ds$ is surjective for the minimal period and consequently for arbitrary t.

Let in turn $X(a) = 0$. Then $\phi_s(a) = a$ for all s, $T\phi_s(a) = e^{sD_a X}$ and

$$\int_0^t (\phi_s)_* Y(a) ds = \int_0^t e^{sD_a X} Y(a) ds$$

The eigenvalues of the operator $\int_0^t e^{sD_a X} ds$ are of the form $\int_0^t e^{s\lambda} ds = \frac{e^{t\lambda}-1}{\lambda}$ where λ is an eigenvalue of $D_a X$. Therefore the operator above is invertible if and only if $\lambda \neq \frac{2\pi i k}{t}$ for k being non-zero integers. $\square$

3. Differentiation of some types of flows

3.1. General case and evolution flows. Let $\mathcal{X}$ be a vector Con-space, F a smooth convenient manifold, and $L : \mathcal{X} \to \Gamma(TF)$ a C^1 map such that each vector field $L(X)$ integrates uniquely to a smooth flow $Fl^X : F \times R \to F$. Then Fl_t^X is a 1-parameter group of diffeomorphisms of F and we have

$$\frac{d}{dt} Fl_t^X = L(X) \circ Fl_t^X, \qquad FL_0^X = id_F, \tag{3.1}$$

$$T(Fl_t^X) \cdot L(X) = L(X) \circ Fl_t^X, \tag{3.2}$$

where $T(Fl_t^X) : TF \to TF$ is the tangent map. Applying a similar procedure as in the case of the flow $\exp tX$, which corresponds to $L(X) = X$, we can derive the formula for the X-derivative of Fl^X. We start with

$$T(Fl_{-t}^X) \cdot DFl_t^X \cdot Y = T(Fl_{-t}^X) \left.\frac{d}{ds}\right|_{s=0} Fl_t^{X+sY}$$
$$= \left.\frac{d}{ds}\right|_{s=0} \int_0^t \frac{d}{du} Fl_{-u}^X Fl_u^{X+sY} du$$

and similarly transform it using both formulae (3.1) and (3.2):

$$= \int_0^t T(Fl_{-u}^X) \left.\frac{d}{ds}\right|_{s=0} \{L(X+sY) - L(X)\} \circ Fl_u^{X+sY} du$$
$$= \int_0^t T(Fl_{-u}^X) \left.\frac{d}{ds}\right|_{s=0} \int_0^s DL(X+vY)Y\,dv \circ Fl_u^{X+sY} du$$
$$= \int_0^t T(Fl_{-u}) DL(X)Y \circ Fl_u^X du = \int_0^t (Fl_s^X)^* DL(X)Y\,ds$$

Hence it follows finally

$$D(Fl_t^X).Y = T(Fl_t^X) \int_0^t (Fl_s^X)^* \circ DL(X)Y\,ds$$

$$= \int_0^t (Fl_s^X)_* \circ DL(X)Y\,ds \circ Fl_t^X \tag{3.3}$$

Formally, for $t = 1$, (3.3) follows from (2.1) by the chain rule applied to $D(\exp \circ L)$

We deliberately repeated the procedure to see explicitely that it works also in the case of *evolution flows* which we consider in the following generalized sense:

Let $X, Y \in C^\infty(J, \mathcal{X})$, for a closed interval $J = [-a, a] \subset R$, or $J = R$, denote smooth curves (or 1-parameter smooth families) in $\mathcal{X}$. Thus $X = (X_t)_{t \in J}$.

Definition. With $L : \mathcal{X} \to \Gamma(TF)$ as above we define the (evolution) integral of $L(X)$ to be the unique smooth curve $g \in C^\infty(F, F)$, if such exists, satisfying the ordinary differential equation

$$\frac{d}{dt} g(t) = L(X_t) \circ g(t), \tag{3.4}$$

for $t \in J$, with initial condition $g(0) = id_F$. If we denote the solution $g(t)$ by Fl_t^X (or $Fl_t^{L(X)}$ if necessary), we shall call the family Fl^X the *evolution flow* generated by $L(X)$ if

(i) Fl_t^X leaves the family $L(X)$ invariant :

$$T(Fl_t^X).L(X_t) = L(X_t) \circ Fl_t^X. \tag{3.5}$$

(ii) If $L(X)$ has an evolution integral then so does $-L(X)$.

Lemma 2. *For each $t \in J$ the map Fl_t^X is an diffeomorphism of M and $Fl_t^{-L(X)} = (Fl_t^{L(X)})^{-1}$.*

Proof. We have from (3.4) and (3.5)

$$\frac{d}{dt}(Fl_t^{L(X)} \circ Fl_t^{-L(X)}) = L(X_t) \circ Fl_t^{L(X)} \circ Fl_t^{-L(X)} - T(FL_t^{L(X)}).L(X_t) \circ Fl_t^{-L(X)} = 0.$$

Since the initial value at $t = 0$ is the id_F, the result follows.

Observe that in the case of $L(X)$ independent of t, the unique solution of the Cauchy problem (3.1) satisfies $Fl_{t+s}^X = Fl_t^X \circ Fl_s^X$, which in turn implies the invertibility of Fl_t^X and the invariance condition (3.2), equivalent to $(Fl_t^X)_* L(X) = L(X)$. This is not the case if $L(X)$ depends on t.

As the relations (3.1) and (3.2) were sufficient to derive (3.3), an analogous formula holds also for evolution flows.

$$DFl_t^X \cdot Y = T(Fl_t^X) \int_0^t (Fl_s^X)^* \circ DL(X_s) Y_s ds. \tag{3.6}$$

(3.3) is a particular case of (3.6) if X_t is constant.

Let $F = G$ be a convenient Lie group with Lie algebra $\mathcal{X} = g$, and let $L(X_t) = R_{X_t}$ be the right invariant vector field generated by $X_t \in g$. Then $L(X_t) \circ g(t) = T_e(\mu^{g(t)}) X_t$ where $\mu : G \times G \to G$ is the product in G and μ^a is the right translation. In this case the solution $g(t)$ is called (see [6]) the *right evolution* of X and denoted by $Evol_G^r(X)(t)$. It follows readily that

$$Fl_t^{R_X} = \mu_{Evol_G^r(X)(t)} : G \to G$$

is the evolution flow generated by R_X with $X \in C^\infty(R, g)$. Recall that G is a *regular Lie group* in the sense of Kriegl-Michor if the right evolution exists for every smooth curve X_t in g. Then the map $evol^r : C^\infty(R, g) \to G$ defined by $evol^r(X) := g(1)$ generalizes the exponential mapping $\exp : g \to G$. In particular, the formula (3.6) computed at the unity e reads

$$DEvol_G^r(X)(t).Y = T_e(\mu_{Evol_G^r(X)(t)}) \int_0^t (Evol_G^r(X)(s))^* Y(s) ds \tag{3.7}$$

since $Fl_t^{R_X}(e) = \mu_{Evol_G^r(X)(t)}(e) = Evol_G^r(X)(t)$ and $R_Y(e) = Y$.

Comment. Let us note that although $Evol_G^r(X) \in C^\infty(R, G)$ and $Fl^{R_X} \in C^\infty(G, C^\infty(R, G))$, they are in 1-1 correspondence. It seems also that the flow version formula (3.6) is a bit more general than (3.7) containing de facto Fl^{R_X} in the first term on the right.

Remark. Let F be a linear space and suppose that Fl_t^X is a 1-parameter group of linear operators on F. Then $T(Fl_t^X) = Fl_t^X$. If, moreover, the map L is linear then $DL(X)Y = L(Y)$, and in this case (3.6) writes

$$DFl_t^X \cdot Y = Fl_t^X \int_0^t Fl_{-s}^X \circ L(Y) \circ Fl_s^X \, ds. \tag{3.8}$$

3.2. Flows induced by action of a regular Lie group. Let G be a regular Lie group (cf. [6]) acting smoothly on a convenient manifold F with action $\rho : G \times F \to F$. For $a \in G$ and $f \in F$ let ρ_a be the translation of F and ρ^f the orbital projection $G \to F$.

We let g denote the Lie algebra of G and for $X \in g$ we set

$$L(X)(f) := T_e\rho^f \cdot X \quad Fl_t^X(f) := \rho_{\exp tX}(f),$$

so that $L(X)$ is the fundamental vector field on F induced by X and Fl_t^X the flow generated by $L(X)$. As for finite Lie groups the following identity holds

$$T_f\rho_a L(Y)(f) = L(Ad(a)Y)(\rho_a(f)).$$

For $a = \exp tX$ this translates into

$$T(Fl_t^X) \cdot L(Y)(f) = L((\exp tX)_* Y)(Fl_t^X(f)) \tag{3.9}$$

which gives

$$(Fl_t^X)_* \circ L(Y) = L((\exp tX)_* Y) \tag{3.10}$$

and similarly

$$(Fl_t^X)^* \circ L(Y) = L((\exp tX)^* Y) \tag{3.11}$$

Proposition 4. *The "fundamental vector field map" L commutes with adjoint actions $(\exp tX)_*$ in g and $(Fl_t^X)_*$ in $\Gamma(TF)$, and the differentiation formula (3.8) reads now*

$$D\rho_{\exp tX}(f).Y = L(\int_0^t (\exp sX)_* Y \, ds)(\rho_{\exp tX}(f)) \tag{3.12}$$

$$= T(\rho_{\exp tX})(f).L(\int_0^t (\exp sX)^* Y \, ds)(f)$$

where $f \in F$.

The first assertion follows from (3.9). To get (3.12) we commuted the integral with the linear operator L in (3.8) and used (3.10) and (3.11).

3.3. The case $L(X)$ = the Lie derivative. Now we consider the case when the diffeomorphism group of a smooth compact manifold M acts smootly on a convenient manifold N with action ρ as above. We let X^ρ denote the fundamental vector field on N induced by the vector field X on M.

As we know the space $F := C^\infty(M, N)$ is a smooth convenient manifold. The action ρ induces naturally a smooth action ρ^* of $\mathcal{D}(M)$ on F defined by

$$\rho^*(\phi, f) = \rho(\phi, f \circ \phi^{-1}).$$

The flow Fl_t^X, defined now by $\rho^*_{\exp tX}$, is generated by the Lie derivative (strictly: the negative of)

$$L(X)(f) = X^\rho \circ f - Tf \circ X$$

valued in the tangent space at f to F, i.e., in $\Gamma(f^*TN)$. Obviously $L(X)$ is the fundamental vector field on $C^\infty(M, N)$ induced by X.

All the results of the previous section apply in this particular case.

We may take $N = E$ and $F = \Gamma(E)$ where E is a natural bundle over M. Then f is a section of E and $\phi_* f = \rho^*_\phi(f \circ \phi^{-1})$ is the usual action of diffeomorphisms on sections.

In particular, let $E = TM$ and $f = Y$ a vector field on M, then

$$Fl_t^X(Y) = (\exp tX)_* Y, \qquad L(X)Y = [Y, X].$$

Substituting it to the differentiation formula (3.12) we obtain

$$D((\exp tX)_* Y).Z = [(\exp tX)_* Z, \int_0^t (\exp sX)_* Y\, ds] \tag{3.13}$$

$$= ad(\int_0^t (\exp sX)_* Y\, ds)((\exp tX)_* Z).$$

3.4. Groups of bounded operators.

Definition. A one parameter family $T_t, t \in R$, of bounded linear operators on a Con-space F is a *differentiable group of bounded operators*, or briefly a C_1 group, if it satisfies

(i) $T_0 = I$
(ii) $T_{t+s} = T_t T_s$ for $t \in R$
(iii) $Lf := \lim_{t\to 0} \frac{T_t f - f}{t}$ exists for every $f \in F$.

The linear operator $L : F \to F$ is the *infinitesimal generator* of the group.

Lemma 3. *The map $t \to T_t$ from R into $L(F, F)$ is bounded on compact intervals.*

Proof. Suppose that for every interval $[0, \epsilon]$ its image by T is unbounded. Then there is a sequence $t_n \to 0$ whose image $\{T_{t_n}\}$ is unbounded. From the uniform boundedness theorem it follows that for some $f \in F$ the set $\{T_{t_n} f\}$ is unbounded, contrary to (iii) which implies that $T_{t_n} f$ is convergenent to f. Thus we proved that there is a $\delta > 0$ such that the subset $T_{[0,\delta]} \subset L(F,F)$ is bounded. Now, for every $f \in F$ and $a \in R$ the set

$$T_{[a,a+\delta]}f = T_a T_{[0,\delta]}f$$

is bounded in F since the operator T_a is bounded. Again using the pointwise boundedness argument we conclude the proof of the lemma. $\square$

Lemma 4. *If T_t is a C_1 group then for every $f \in F$, $t \to T_t f$ is a continuous function from R into F.*

Proof. Let $t, h \in R$, then

$$T_{t+h}f - T_t f = T_t(T_h f - f).$$

Since $T_t : F \to F$ is a bounded linear operator, it is also continuous (even smooth), so the term on the right hand side tends to zero, as $T_h f \to f$. $\square$

Lemma 5. *Let T_t be a C_1 group and let L be its infinitesimal generator, then for $f \in F$ we have:*

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T_s f ds = T_t f. \tag{1}$$

$$L(\int_0^t T_s f ds) = T_t f - f. \tag{2}$$

$$\frac{d}{dt} T_t f = L T_t f = T_t L f. \tag{3}$$

Proof.(1) follows directly from the continuity of $t \to T_t f$. For $f \in F$ and $h > 0$

$$\begin{aligned} \frac{T_h - I}{h} \int_0^t T_s f ds &= \frac{1}{h} \int_0^t (T_{s+h} f - T_s f) ds \\ &= \frac{1}{h} \int_0^{t+h} T_s f ds - \frac{1}{h} \int_0^t T_s f ds \end{aligned}$$

and the RHS tends to $T_t f - f$ as $h \to 0$, which proves (2).
For (3) we have

$$\frac{T_h - I}{h} T_t f = T_t \frac{T_h - I}{h} f \to T_t f$$

as $t \to 0$. Hence the commutativity and the first equality in (3). $\square$

Theorem 2. *If L is the infinitesimal generator of a C_1 group T_t in a convenient (resp. Fréchet) space then*
(a) *L is a closed (resp. bounded) linear operator.*
(b) *L determines the group uniquely.*
(c) *The map $t \to T_t$ is smooth.*

Proof. L is evidently linear and defined on the whole of F. Since F is a complete linear metric space, to have L bounded it is enough to show that L is a closed operator, by the closed graph theorem. To show that L is closed, let $f_n \to f$ and $Lf_n \to g$ as $n \to \infty$. By integration of (3) we have

$$T_t f_n - f_n = \int_0^t T_t L f_n \, ds.$$

The integrand on the right converges to $T_s g$ uniformly on bounded intervals. Therefore letting $n \to \infty$ gives

$$T_t f - f = \int_0^t T_s g ds.$$

Dividing it by $t \neq 0$ and letting $t \to 0$ we see by using (1) that $Lf = g$, which was to be proved.

In order to prove (b) let S_t be also a C_1 group of bounded linear operators with infinitesimal generator L. From (3) it follows readily that the function $s \to T_{t-s} S_s f$ is differentiable and that

$$\frac{d}{ds} T_{t-s} S_s f = -L T_{t-s} S_s f + T_{t-s} L S_s = 0.$$

Therefore $s \to T_{t-s} S_s f$ is constant and so its values at $s = 0$ and $s = t$ are the same, which means $T_t f = S_t f$ for every $f \in F$.

Now, (c) follows from (3) of Lemma 5, from which we derive successively

$$(T_t f)^{(k)} = L^k T_t f = T_t L^k f.$$

Since by lemma 4 the map $t \to T_t f$ is continuous, it follows that all the derivatives $(T_t f)^{(k)}$ exist and are continuous, so $t \to T_t f$ is smooth for every $f \in F$. To get (c) we apply the smooth uniform boundedness theorem. This completes the proof of the theorem. □

Comment. It is not only for the sake of simplicity that we consider C_1 groups instead of C_0 groups of linear bounded operators. For C_0 groups the regularity condition (iii) of the definition is replaced by : $\lim_{t \to 0} T_t f = f$ for every $f \in F$. Consequently, the infinitesimal generator L is densely defined in F and in general L is unbounded if F is a Banach space (cf. Pazy [9], from where we adapted some simple relations). In practice, in spaces of smooth functions, the infinitesimal generators are linear differential operators, which are bounded in appropriate Fréchet spaces (contrary to as it is in Banach spaces). So we may start from the C_1 level. Anyway the Montgomery-Zippin theorem on the smoothness of a continuous action of the reals on a smooth manifold does not work here.

Suppose that F is a convenient vector space and L a linear map from $\mathcal{X}$ into End(E) such that every operator $L(X)$ is the infinitesimal generator of a C_1 group of bounded linear operators $T_t^X, t \in R$, on F.
As a direct consequence of the results above the following theorem can be stated.

Theorem 3. *Let T_t^X, $X \in \mathcal{X}$, be a family of C_1 groups of bounded linear operators in a Con-space F, with infinitesimal generators $L(X)$, then*

$$L(X)\int_0^t T_s^X f ds = (T_t^X - I)f,$$

$$L(X)T_t^X f = T_t^X L(X)f = \frac{d}{dt}T_t^X f,$$

$$\frac{d^n}{dt^n}T_t^X = L(X)^n T_t^X,$$

$$DT_t^X Y = T_t^X \int_0^t (T_s^X)^* L(Y) ds = \int_0^t T_{t-s}^X L(Y) T_s^X ds,$$

(Taylor expansion)

$$T_t^X f = f + tL(X)f + \frac{t^2}{2!}L(X)^2 f + \cdots + \frac{t^k}{k!}L(X)^k f + \int_0^t \frac{(t-s)^k}{k!} L(X)^{k+1} T_s^X f ds.$$

The map $t \longrightarrow T_t^X f$ is smooth for every X and f. Thus the composite $T \circ L$ is an element of the function space $C^\infty(\mathcal{X}, C^\infty(F \times R, F))$, which by the property of cartesian closedness can be canonically identified with $C^\infty(\mathcal{X} \times F \times R, F)$, so $T \circ L$ is smooth with respect to the triple of variables (X, f, t).

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