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HIGHER ORDER CARTAN CONNECTIONS

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To Ivan Kolář, on the occasion of his 60th birthday.

Abstract. A Cartan connection associated with a pair $P(M,G') \subset P(M,G)$ is defined in the usual manner except that only the injectivity of $\omega : T(P') \to T(G)_e$ is required. For an r-th order connection associated with a bundle morphism $\Phi : P' \to P$ the concept of Cartan order $q \le r$ is defined, which for $q = r = 1, \Phi : P' \subset P$, and $\dim M = \dim G/G'$ coincides with the classical definition. Results are obtained concerning the Cartan order of r-th order connections that are the product of r first order (Cartan) connections.

1. Preliminaries

All manifolds are assumed smooth and finite dimensional. Following [7], the category of principal bundles $P(M,G)$ for a fixed manifold M will be denoted by $\mathcal{PB}(M)$. Thus a typical morphism $(\Phi, \Phi_G) : P'(M,G') \to P(M,G)$ of $\mathcal{PB}(M)$, is given by a fibre preserving map $\Phi : P' \to P$ and a homomorphism $\Phi_G : G' \to G$ such that $\Phi(h'g') = \Phi(h')\Phi_G(g')$, for any $h' \in P', g' \in G'$. We shall write sometimes simply $\Phi : P' \to P$ instead of the explicit $(\Phi, \Phi_G) : P'(M,G') \to P(M,G)$. Also, $\mathcal{FM}(M)$ will denote the category of fibred manifolds over M and fibre preserving maps.

If $p : E \to M$ is a fibred manifold denote by $J^r E$ the space of holonomic r-jets of its local sections which is again a fibred manifold $\alpha : J^r E \to M$. By iteration of J^1 one obtains the fibred manifold $\widetilde{J}^r E$ of non-holonomic jets of sections and its submanifold $\bar{J}^r E$ of semi-holonomic ones (c.f.[1]).

If $p = \mathrm{pr}_M : M \times N \to M$, where N is another manifold, we write $J^r(M,N)$ instead of $J^r(M \times N)$, and $J^r_x(M,N)_y \subset J^r(M,N)$ for the submanifold of jets with source $x \in M$ and target $y \in N$. Similarly $\widetilde{J}^r(M,N)$ and $\bar{J}^r(M,N)$. We shall use the symbol $\circ$ to denote composition of jets, ie. if $Z = j^r_x f \in J^r(M,N)$ and $Y = j^r_y g \in J^r(N,Q), y = f(x)$, then $Y \circ Z = j^r_x(g \circ f) \in J^r(M,Q)$ with an appropriate extension to non-holonomic and semi-holonomic jets (c.f. (1.5) below). Also, $j^r_x(t \mapsto f(t))$ will sometimes stand for $j^r_x f$, and we shall use the abbreviated notation $j^r_x = j^r_x(t \mapsto t)$ and $j^r_x[c] = j^r_x(t \mapsto c)$ for the jets of the identity and constant maps respectively.

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There is a functor $\mathbf{J} : \mathcal{FM}(M) \to \mathcal{FM}(M)$ which assignes J^1E to E and $j^1_x s \mapsto j^1_x(f \circ s)$ to $f : E \to F$. By iteration one obtains the functor $\mathbf{J}^r : \mathcal{FM}(M) \to \mathcal{FM}(M)$ which assigns $\widetilde{J}^r E$ to E. Also, there are natural transformations $\pi^r_s : \mathbf{J}^r \to \mathbf{J}^s$ for $0 \le s \le r$, where $\mathbf{J}^0 = \mathrm{id}_{\mathcal{FM}(M)}$, satisfying

$$\pi^r_s \circ \widetilde{J}^r(f) = \widetilde{J}^s(f) \circ \pi^r_s \text{ for } 0 \le s \le r \text{ and any } f \in \mathcal{FM}(M). \tag{1.1}$$

More generally, given $E \in \mathcal{FM}(M)$ and a pair $s \le r$ there are $r - s + 1$ projections (c.f. .[8])

$$\pi^{r\to i}_s = \mathbf{J}(\pi^{i-1}_{s-1}) \circ \pi^r_i \qquad i = s, s+1, \dots, r. \tag{1.2}$$

Note that $\pi^{r\to s}_s = \pi^r_s$ and $Z \in \widetilde{J}^r E$ is semi-holonomic iff for any $1 \le s \le r$

$$\pi^r_s(Z) = \pi^{r\to i}_s(Z) \in \widetilde{J}^s E \text{ whenever } i = s+1, s+2, \dots, r. \tag{1.3}$$

An element $X \in \widetilde{J}^r_x(M, N)$ can be represented by its coordinates $(X^\alpha_{\iota_1,\dots,\iota_r}) \in \widetilde{J}^r_0(\mathbb{R}^m, \mathbb{R}^n)$, where $\iota_1, \dots, \iota_r = 0, 1, 2, \dots, m; \quad \alpha = 1, \dots, n$ (c.f. [9]) which gives the coordinate expression

$$\pi^{r\to i}_s : (X^\alpha_{\iota_1,\dots,\iota_r}) \mapsto (X^\alpha_{\iota_1,\dots,\iota_{s-1},0,\dots,0,\iota_s,0,\dots,0}), \tag{1.4}$$
where ι_s is in the i-th place.

Recall also the rule for the composition of non-holonomic jets (c.f. [2]) defined recurrently as follows. If $Z = j^1_x\rho \in \widetilde{J}^r_x(M, N)$ and $W = j^1_y\sigma \in \widetilde{J}^r_y(N, Q)$, where $\rho : M \rightsquigarrow \widetilde{J}^{r-1}(M, N)$ and $\sigma : N \rightsquigarrow \widetilde{J}^{r-1}(N, Q)$ are local sections in a neighbourhood of $x \in M$ and $y = \pi^r_0(Z) \in N$ respectively, then their composition $W \circ Z$ is given by (c.f. [1])

$$W \circ Z = j^1_x(u \mapsto \sigma(\pi^{r-1}_0 \rho(u)) \circ \rho(u)). \tag{1.5}$$

In coordinates, this rule is best expressed recurrently as follows: The coordinate $U^\gamma_{k_1,\dots,k_r}$ of $U = W \circ Z$ is obtained by formally applying the differential operator D_r to the function $U^\gamma_{k_1,\dots,k_{r-1}}(W^\beta_{j_1,\dots,j_{r-1}}, Z^\alpha_{i_1,\dots,i_{r-1}})$ and writing

$Z^\alpha_{i_1,\dots,i_{r-1},0}$ instead of 'the value of $Z^\alpha_{i_1,\dots,i_{r-1}}$',
$Z^\alpha_{i_1,\dots,i_{r-1},i_r}$ instead of $D_r Z^\alpha_{i_1,\dots,i_{r-1}}$,
$W^\beta_{j_1,\dots,j_{r-1},0}$ instead of 'the value of $W^\beta_{j_1,\dots,j_{r-1}}$', and
$\sum_{s=1}^n W^\beta_{j_1,\dots,j_{r-1},s} Z^s_{0,\dots,0,i_r}$ instead of $D_r W^\beta_{j_1,\dots,j_{r-1}}$.

In particular, we obtain the following

Lemma 1.1. *Let $Z \in \widetilde{J}_x^{r+1}(M,N), W \in \widetilde{J}_y^{r+1}(N,Q)$ and let $\pi_r^{r+1}(Z) = j_x^r[y]$. Let Z have coordinates $Z^\alpha_{i_1,\dots,i_r,i_{r+1}}, i_s = 0,1,\dots,m; \quad \alpha = 1,\dots,n$ and let W have coordinates $W^\beta_{j_1,\dots,j_{r+1}}, j_k = 0,1,\dots,n; \quad \beta = 1,\dots,q$. Then the coordinates of $U = W \circ Z$ are given by*

$$U^\beta_{i_1,\dots,i_r,i_{r+1}} = \sum_{j=1}^{n} W^\beta_{0,\dots,0,j,0,\dots,0} Z^j_{i_1,\dots,i_r,i_{r+1}},$$

where the subscript j is in the place of the first non-zero index among $i_1,\dots,i_r,i_{r+1}$.

Note that by our assumption $Z^\alpha_{i_1,\dots,i_r,0} = 0$ hence also $U^\beta_{i_1,\dots,i_r,0} = 0$.

One verifies easily that

$$\pi^s_q \circ \pi^{r\to i}_s = \pi^r_q \qquad \text{for } 0 \le q < s \le i \le r \tag{1.6}$$

and

$$\pi^{r\to i}_s(A \circ B) = \pi^{r\to i}_s(A) \circ \pi^{r\to i}_s(B) \tag{1.7}$$

for any two non-holonomic r-jets for which the composition $A \circ B$ is defined.

2. First order Cartan connections

Recall the standard definition as given in e.g. [3]. Given a Lie group G, a subgroup $G' \subset G$ and a principal bundle $P'(M,G')$ — giving rise to a reduction $P'(M,G') \subset P(M,G)$, with $P(M,G)$ the extension by $G' \subset G$ — a Cartan connection for this pair is a one-form ω on P' with values in the Lie algebra $T(G)_e$ satisfying $\omega(A^*) = A$ for every $A \in T(G')_e, (R_a)^*\omega = \mathrm{ad}(a^{-1})\omega$ for every $a \in G'$ and such that $\omega(Y) = 0$ implies $Y = 0 \in T(P')$. It follows then that $\dim G/G' \ge \dim M$. Note that in [3] and elsewhere one assumes equality of these dimensions. If that is the case we shall speak of a *classical* Cartan connection giving rise to an absolute parallelism on P'.

Standard examples of classical Cartan connections are

(i) an *affine* connection on M: here $P' = PM$, the standard frame bundle of M, and P is the affine bundle, ie. the extension of the structure group $GL(m,\mathbb{R})$ of PM by the affine group;

(ii) a *conformal* connection on M: here $P = PM$ and P' is a conformal structure on M, ie. a reduction of $GL(m,\mathbb{R})$ to $CO(M) = \{A \in GL(m,\mathbb{R}) : {}^tAA = cI \text{ for some } c > 0\}$. Such a Cartan connection is equivalent to one associated with the pair given by $P = P^2M$, the bundle of second order holonomic frames of M, and a reduction of it to a certain subgroup of its structure group G^2_m (c.f. [3]);

(iii) a *projective* connection on M: here $P = P^2M$ as above, and P' is another reduction of it to a suitable subgroup of its structure group G^2_m (c.f. [3]).

A connection in a principal bundle $P(M,G)$ can also be seen as a morphism $C : P \to J^1P$ of $\mathcal{FM}(M)$ satisfying $\pi^1_0 \circ C = \mathrm{id}_P$ and $C(hg) = C(h) \cdot j^1_{ph}[g]$ for any $h \in P$, $g \in G$. Here $\cdot$ denotes the jet-prolongation of the action of G on P (c.f. [1], [2] and [6]). This can be generalized to Cartan connections.

Proposition 2.1. *For any reduction of principal bundles $P'(M,G') \subset P(M,G)$ there is a canonical one-to-one correspondence between Cartan connections $\omega : T(P') \to T(G)_e$ and morphisms $\Gamma : P' \to J^1P$ of $\mathcal{FM}(M)$ satisfying*

$$\pi_0^1 \circ \Gamma = \mathrm{id}_{P'} \tag{2.1}$$

$$\Gamma(h'g') = \Gamma(h') \cdot j^1_{p'h'}[g'] \tag{2.2}$$

(2.3) *If $Y \in J_0^1(\mathbb{R}, P')_{h'}$ and $\Gamma(h') \circ j^1_{h'}p' \circ Y = Y$ then necessarily $Y = j_0^1[h']$, where $p' : P' \to M$ is the projection.*

Proof. Let $h' \in P'$ be fixed and let $\omega : T(P') \to T(G)_e$ have the above properties, in particular $\omega \circ^* = \mathrm{id}_{T(G')_e}$. Then, as in the case of $P' = P$, one can easily see that $\Gamma(h') : T(M)_x \to T(P)_{h'}$, defined as $\Gamma(h')X = Y - (^*\circ\omega)(Y)$, where Y is any element of $T(P')_{h'}$ such that $T(p')Y = X$, represents an element of J^1P with the required properties: (2.1) follows easily from the fact that $\Gamma(h') \circ j^1_{h'}p' \circ Y = Y$ means $\omega(Y) = 0$, hence $Y = 0$ by assumption. Conversely, if $\Gamma : P' \to J^1P$ has the listed properties then viewing again $\Gamma(h')$ as a linear map $T(M)_x \to T(P)_{h'}$, one defines $\omega(Y) =^{*-1} (Y - \Gamma(h')T(p')Y)$ for $Y \in T(P')_{h'}$ and any $h' \in P'$. The required properties of ω follow again easily from those of Γ.

Remark. For the cannonical Cartan connection associated with the homogeneous space G/G' we have $P'(M,G') = G(G/G',G')$ and $P(M,G) = G/G' \times G$ with the one-form $\omega : T(G)_g \to T(G)_e$ defined by $\omega(Y) = T(L_{g^{-1}})Y$. The associated $\Gamma(g) \in J^1(G/G' \times G)$ defined by Proposition 2.1 becomes simply $\Gamma(g) = (j^1_x, j^1_x[g])$, where $x = gG'$.

The choice of source $0 \in \mathbb{R}$ in (2.3) is rather arbitrary in the sense that if (2.1) and (2.2) are satisfied then (2.3) is equivalent to

$$W \in J_a^1(V, P')_{h'} \text{ and } \Gamma(h') \circ j^1_{h'} \circ W = W \text{ implies } W = j_a^1[h'], \tag{2.4}$$

where V is any manifold and $a \in V$. In fact, assuming (2.4), let $Y \in J_0^1(\mathbb{R}, P')_{h'}$ be such that $\Gamma(h') \circ j^1_{h'}p' \circ Y = Y$. Then for any $Z \in J_a^1(V, \mathbb{R})_0$ we have $\Gamma(h') \circ j^1_{h'}p' \circ Y \circ Z = Y \circ Z$ and thus by assumption $Y \circ Z = j_a^1[h']$. As Z was arbitrary, one concludes from the chain rule that $Y = 0 \in T(P')_{h'}$, ie. $Y = j_0^1[h']$. Conversely, assuming (2.4), one obtains $W \circ Z = 0 \in J_a^1(V, P')_{h'}$, for any $Z \in J_0^1(\mathbb{R}, V)_a$ whence again $W = j_a^1[h']$.

3. The general case

From now on all higher order jets, connections etc. will be assumed non-holonomic unless otherwise stated. Recall (c.f. [6]) that an r-th order connection in $P(M,G)$ is a morphism $\Gamma : P \to \widetilde{J}^rP$ of $\mathcal{FM}(M)$ which satisfies $\pi_0^r \circ \Gamma = \mathrm{id}_P$ and $\Gamma(hg) = \Gamma(h) \cdot j^r_{ph}[g]$ for any $g \in G$.

Let $(\Phi, \Phi_G) : P'(M, G') \to P(M, G)$ be a fixed morphism of $\mathcal{PB}(M)$. An r-th order *Φ-connection* (or *relative connection*) is a morphism $\Gamma : P' \to \widetilde{J}^r P$ of $\mathcal{FM}(M)$ which satisfies $\pi^r_0 \circ \Gamma = \Phi$ and $\Gamma(h'g') = \Gamma(h') \cdot j^r_{p'h'}[\Phi_G(g')]$ for any $g' \in G'$. Both an r-th order connection in a principal bundle as well as a first order Cartan connection for $P' \subset P$ are special cases of a relative connection. Also, if ξ is an r-th order connection in $P'(M, G')$ then $\mathbf{J}^r(\Phi) \circ \xi$ is an r-th order Φ-connection, and if η is an r-th order connection in $P(M, G)$ then $\eta \circ \Phi$ is again an r-th order Φ-connection. Note that Prop. 6.1 of [6], Ch. II says that for any first order connection ξ in P' there is a unique first order connection η in P such that the two Φ-connections $\mathbf{J}(\Phi) \circ \xi$ and $\eta \circ \Phi$ coincide. This can be extended to connections of arbitrary order $r \geq 1$.

Of course, not every Φ-connection can be written as $\mathbf{J}^r(\Phi) \circ \xi$ for some connection $\xi : P' \to \widetilde{J}^r P'$; if it can, the Φ-connection will be called *straight*. On the other hand, $\Gamma = \eta \circ \Phi$ defines a one-to-one correspondence between Φ-connections Γ and connections η in P. To see this, first assume $\eta_1 \circ \Phi = \eta_2 \circ \Phi$. Then for any $h \in P$ there is an $h' \in P'$ and a $g \in G$ such that $h = \Phi(h')g$. Thus $\eta_1(h) = \eta_1(\Phi(h')) \cdot j^r_x[g] = \eta_2(h)$. Hence there is at most one η such that $\Gamma = \eta \circ \Phi$. One verifies easily, that $\eta(h) = \Gamma(h') \cdot j^r_x[g]$, defines the required connection in P.

The following is obvious.

Proposition 3.1. *Let $\Phi_1 : P_2 \to P_1$ and $\Phi_2 : P_3 \to P_2$ be two morphisms of $\mathcal{PB}(M)$. Let further $\Gamma_1 : P_2 \to \widetilde{J}^r P_1$ be an r-th order Φ_1-connection, and $\Gamma_2 : P_3 \to \widetilde{J}^s P_2$ be an s-th order Φ_2-connection. Then*

$$\Gamma_1 * \Gamma_2 := \mathbf{J}^s(\Gamma_1) \circ \Gamma_2 : P_3 \to \widetilde{J}^{r+s} P_1 \tag{3.1}$$

is an $(r+s)$-th order $(\Phi_1 \circ \Phi_2)$-connection, (called their *product*), *and*

$$\mathbf{J}^s(\Phi_1) \circ \Gamma_2 : P_3 \to \widetilde{J}^s P_1 \tag{3.2}$$

is an s-th order $(\Phi_1 \circ \Phi_2)$-connection, (called the *extension of* Γ_2 *by* Φ_1*).*

It is also easily verified, that any Φ-connection Γ of order $r \geq 1$ gives rise to $r - s + 1$ Φ-connections of order s where $1 \leq s \leq r$, namely (c.f. (1.2) and (1.6))

$$\pi^{r\to i}_s \circ \Gamma = \mathbf{J}(\pi^{i-1}_{s-1}) \circ \pi^r_i \circ \Gamma : P' \to \widetilde{J}^s P \qquad \text{for } i = s, s+1, \dots, r, \tag{3.3}$$

in particular to r first order Φ-connections

$$\pi^{r\to i}_1 \circ \Gamma = \mathbf{J}(\pi^{i-1}) \circ \pi^r_i \circ \Gamma : P' \to J^1 P \qquad \text{for } i = 1, \dots, r. \tag{3.4}$$

It follows from Proposition 3.1 that if C is a c-th order Φ-connection, ξ is an a-th order connection in P' and η a b-th order connection in P then $\eta * C * \xi$ is an $(a + b + c)$-th order Φ -connection. We shall be interested only in the special case where C is a first order Φ-connection, $\xi = \xi_1 * \ldots * \xi_a$ and $\eta = \eta_1 * \cdots * \eta_b$ with $\xi_1, \ldots, \xi_a$ and $\eta_1, \ldots, \eta_b$ first order connections in P' and P respectively.

Proposition 3.2. *Put $r = a + b + 1$, where $a \geq 0$ and $b \geq 0$ are some integers, and let*

$$\Gamma = \eta_1 * \cdots * \eta_b * C * \xi_1 * \cdots * \xi_a \tag{3.5}$$

be an r-th order Φ connection as above. Then

$$\begin{aligned} \pi_1^{r \to i} \circ \Gamma &= \eta_i \circ \Phi && \text{for } i = 1, \dots, b \\ &= C && \text{for } i = b + 1 \\ &= \mathbf{J}(\Phi) \circ \xi_{i-b-1} && \text{for } i = b + 2, \dots, b + a + 1 = r. \end{aligned} \tag{3.6}$$

Proof. First note that $\pi_{r-1}^r \circ \Gamma = \pi_{r-1}^r \circ \mathbf{J}(\eta_1 * \cdots * \eta_b * C * \xi_1 * \ldots * \xi_{a-1}) \circ \xi_a = \eta_1 * \ldots * \eta_b * C * \xi_1 * \ldots * \xi_{a-1}$ (or $\eta_1 * \ldots * \eta_b$ if $a = 0$), hence $\pi_i^r \circ \Gamma$ will be of the form (3.5) truncated to the first i terms only. Explicitly, $\pi_i^r \circ \Gamma$ equals

$$\begin{aligned} &\eta_1 * \cdots * \eta_i \circ \Phi && \text{for } i = 1, \dots, b \\ &\eta_1 * \cdots * \eta_b * C && \text{for } i = b + 1 \\ &\eta_1 * \cdots * \eta_b * C * \xi_1 * \cdots * \xi_{i-b-1} && \text{for } i = b + 2, \dots, r. \end{aligned} \tag{3.7}$$

Applying now $\mathbf{J}(\pi_0^{i-1})$ to these products we get the last connection preceded by $\mathbf{J}(\Phi)$ iff the other $i - 1$ terms contained C. This gives exactly (3.6) as required.

We shall say that the r-th order Φ-connection $\Gamma : P' \to \widetilde{J}^r P$ has *Cartan order at least* q, where $0 \leq q \leq r$, if for each $h' \in P'$

$$\begin{aligned} &\Gamma(h') \circ \mathbf{J}^r(p')Y = \mathbf{J}^r(\Phi)Y \text{ for some } Y \in \widetilde{J}_0^r(\mathbb{R}, P')_{h'} \\ &\text{implies } \pi_q^r Y = j_0^q[h']. \end{aligned} \tag{3.8}$$

Here $\mathbf{J}^r(p')Y = j_{h'}^r p' \circ Y$. Thus we use the same notation $\mathbf{J}$ for this endofunctor on any category of fibred manifolds over a fixed base given from the context (in this case $\mathbb{R}$).

The Φ-connection Γ is said to have *Cartan order* q if $q \leq r$ is the largest integer satisfying (3.8), and Γ is called a *Cartan Φ-connection* if its Cartan order is r. In view of Proposition 2.1, a first order Cartan connection for the pair $P' \subset P$ is the same thing as a first order Cartan ι-connection, where ι is the inclusion $P' \subset P$.

Remark. In the same sense as (2.3) was equivalent to (2.4), also (3.8) is equivalent to

$$\begin{aligned} &\Gamma(h') \circ \mathbf{J}^r(p')Y = \mathbf{J}^r(\Phi)Y \text{ for some } Y \in \widetilde{J}_a^r(V, P')_{h'} \\ &\text{implies } \pi_q^r Y = j_a^q[h'], \text{ where } V \text{ is any manifold and } a \in V. \end{aligned} \tag{3.9}$$

This is true in particular for $V = M$ and $a = x = p'h'$. Note, however, that in this case the condition in (3.9) can never be satisfied by $Y \in \widetilde{J}^r P'$ with $q > 0$ since $j_x^1[h'] \notin J^1 P'$. On the other hand, if Φ is an immersion then (3.9) is always satisfied with $Y \in \widetilde{J}^r_x(M, P'_x)_{h'}$ and $q = r$. In fact, now $\mathbf{J}^r(p')Y = j^r_x[x]$, so the relation in (3.9) becomes $j^r_x[h] = \mathbf{J}^r(\Phi)Y \in \widetilde{J}^r_x(M, P_x)_h$, where $h = \Phi(h')$. A simple application of the Rank theorem shows that Φ has a local left inverse whence $Y = j^r_x[h']$.

Conversely, if Φ has Cartan order at least one then Φ must be injective. In fact, let $g : \mathbb{R} \rightsquigarrow \ker \Phi_G$ be smooth in a neighbourhood of $0, g(0) = e$. If $\ker \Phi_G \subseteq G'$ is non-trivial then g can be chosen so that $j^1_0(t \mapsto h'g(t)) \neq j^1_0[h']$. This means that $Y = j^r_0(t \mapsto h'g(t))$ will satisfy the condition in (3.8) but $\pi^r_1 Y \neq j^1_0[h']$, and so the Cartan order of Φ is 0.

If $F = G/\Phi_G(G')$ then G acts to the left on F and one obtains the associated with P bundle $E = (P \times F)/G$. For each $x \in M$ the element $e(x) = [\Phi(h'), e\Phi_G(G')] \in E_x, x = p'h'$, is independent of the choice of $h' \in P'_x$, and so we have a distinguished section $e : M \to E$. In case of a (classical) first order Cartan connection, the absolute differential of this section defines a soldering of E along the section e. This can again be generalised. First note that each $h \in P$ can be seen as a diffeomorphism $\{h\} : F \to E_{ph}$ assigning to $\xi \in F$ the element $[h, \xi]$ giving rise to a composition $P \times F \to E$. If $r > 1$ then its prolongation is the composition $\widetilde{J}^{r-1}P \times \widetilde{J}^{r-1}(M, F) \to \widetilde{J}^{r-1}E, (Z, \Xi) \mapsto [Z \cdot \Xi]$, which again for a fixed $Z \in \widetilde{J}^{r-1}P$ is a diffeomorphism $\widetilde{J}^{r-1}(M, F) \to \widetilde{J}^{r-1}(M, E_x)$ and so we also have a composition $\widetilde{J}^{r-1}P \times \widetilde{J}^{r-1}E \to \widetilde{J}^{r-1}(M, F), (Z, S) \mapsto Z^{-1} \cdot S$. Thus we can write the absolute differential with respect to $\Gamma(h') = j^1_x\sigma \in \widetilde{J}^r_x P$ of e at x (c.f. [2] and [5]) as

$$\nabla e(x) = j^1_x(u \mapsto \sigma(x) \cdot (\sigma(u)^{-1} \cdot j^{r-1}_u e)) \in \widetilde{J}^r_x(M, E_x)_{e(x)}. \tag{3.10}$$

In particular, we get a map

$$\begin{aligned} \widetilde{J}^r_0(\mathbb{R}, M)_x &\to \widetilde{J}^r_0(\mathbb{R}, E_x)_{e(x)} \\ X &\mapsto \nabla e(x) \circ X. \end{aligned} \tag{3.11}$$

Note that the formula (3.10) can also be written as

$$\nabla e(x) = j^1_x(u \mapsto [\sigma(x) \cdot g(u), j^{r-1}_u[e\Phi_G(G')]]) \tag{3.12}$$

where $g(u) \in \widetilde{J}^{r-1}_u(M, G)_e$ is such that $j^{r-1}_u(\Phi \circ \rho) = \sigma(u) \cdot g(u)$ for some section $\rho : M \rightsquigarrow P', \rho(x) = h'$. To see this first assume $r = 1$ and let ρ be an arbitrary smooth section as above. Then $\Phi(\rho(u)) = \sigma(u)g(u)$ for some smooth $g : M \rightsquigarrow G$ and so $\sigma(u)^{-1} \cdot e(u) = g(u) \cdot \Phi(\rho(u))^{-1} \cdot e(u) = g(u)\Phi_G(G')$. Thus $\sigma(x) \cdot (\sigma(u)^{-1} \cdot e(u)) = [\sigma(x)g(u), e\Phi_G(G')]$ as required. Note that $g(u)$ depends on $\rho(u)$, however not so the equivalence class. If $r > 1$, observe that the composition $P \times G \to P$ — both $(h, g) \mapsto hg$ as well as $(h, g) \mapsto hg^{-1}$ — can be prolonged to a multiplication $\widetilde{J}^{r-1}_x P \times \widetilde{J}^{r-1}_x(M, G) \to \widetilde{J}^{r-1}_x P$ and so we conclude that there is an element $g(u) \in \widetilde{J}^{r-1}_u(M, G)_e$ with the required property. A prolongation of the formulae obtained for $r = 1$ leads to (3.12) for a general $r \geq 1$.

Note also that g in (3.12) was chosen so that $\mathbf{J}^r(\Phi)j^r_x\rho = \Gamma(h') \cdot \tilde{g}, \tilde{g} = j^r_x g \in \widetilde{J}^r_x(M, G)_e$, and though $\tilde{g}$ depends on the choice of $\rho, \Gamma(h')$ uniquely determines its equivalence class $[\tilde{g}] \in \widetilde{J}^r_x(M, G)_e / \widetilde{J}^r_x(M, \Phi_G(G'))_e$. Thus we can also write

$$\nabla e(x) = [j^1_x[\sigma(x)] \cdot \tilde{g}, j^r_x[e\Phi_G(G')]]. \tag{3.13}$$

Proposition 3.3. *If the r-th order Φ-connection Γ has Cartan order $q \leq r$ then* (3.11) *is injective in the sense that* $\nabla e(x) \circ X = j_0^r[e(x)]$ *with* $X \in \widetilde{J}_0^r(\mathbb{R}, M)_x$ *implies* $\pi_q^r X = j_0^q[x]$.

Proof. The condition $\nabla e(x) \circ X = j_0^r[e(x)]$ can be written as $\nabla e(x) \circ X = \nabla e(x) \circ j_0^r[x]$. By (3.13) we have $\nabla e(x) \circ X = [j_0^1[\sigma(x)] \cdot (\tilde{g} \circ X), j_0^r[e\Phi_G(G')]]$ and similarly with $j_0^r[e(x)]$ instead of X. Since the action of $\widetilde{J}_0^r(\mathbb{R}, G)$ on $\widetilde{J}_0^r(\mathbb{R}, P)$ is free we conclude that $\tilde{g} \circ X = \tilde{g} \circ j_0^r[x]$, ie.$\tilde{g} \circ X = j_0^r[e]$ since $\pi_0^r \tilde{g} = e$. On the other hand, $\mathbf{J}^r(\Phi) j_x^r \rho = \Gamma(h') \cdot \tilde{g}$ gives $\mathbf{J}^r(\Phi) Z = \Gamma(h') \circ X \cdot \tilde{g} \circ X$, where $Z = j_x^r \rho \circ X \in \widetilde{J}_0^r(\mathbb{R}, P')_{h'}$ and so $\mathbf{J}^r(p')Z = X$. Thus we get $\mathbf{J}^r(\Phi)Z = \Gamma(h') \circ \mathbf{J}^r(p')Z \cdot j_0^r[e]$ or $\mathbf{J}^r(\Phi)Z = \Gamma(h') \circ \mathbf{J}^r(p')Z$ which implies $\pi_q^r Z = j_0^q[h']$ by the Cartan property of Γ. Applying $\mathbf{J}^q(p')$ to this relation we obtain $\pi_q^r X = j_0^q[x]$ as required.

Example. If $P' = M \times G'$ and $\Phi = \mathrm{id}_M \times \Phi_G$ then an r-th order Φ-connection is in fact a map $\Gamma : M \times G' \to \widetilde{J}_0^r(M, G)$ satisfying $\pi_0^r \Gamma(x, g') = \Phi_G(g')$ and $\Gamma(x, g'g'') = \Gamma(x, g') \cdot j_x^r[\Phi_G(g'')]$. Clearly, it has Cartan order at least $q \leq r$ if

$$\begin{aligned} &\Gamma(x, g') \circ X = \mathbf{J}^r(\Phi)Y, \; X \in \widetilde{J}_0^r(\mathbb{R}, M)_x, \; Y \in \widetilde{J}_0^r(\mathbb{R}, G')_{g'} \\ &\text{implies } \pi_q^r X = j_0^q[x] \text{ and } \pi_q^r Y = j_0^q[g']. \end{aligned} \tag{3.14}$$

Let now $M = \mathbb{R}^m, G' = GL(m, \mathbb{R}), G = A(m)$, the affine group seen as a subgroup of $GL(m+1, \mathbb{R}), \Phi_G(g') = \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix}$. Put

$$\begin{aligned} \Gamma(x, g') &= j_x^r F = j_x^r \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \\ &= j_x^r \Big(u \mapsto \begin{pmatrix} \sum_{i=1}^m (u^i - x^i + 1) g' & u - x \\ 0 & 1 \end{pmatrix}\Big). \end{aligned} \tag{3.15}$$

It is easily verified that this defines a holonomic Φ-connection. We claim that its Cartan order is r. So let $X \in \widetilde{J}_0^r(\mathbb{R}, \mathbb{R}^m)_x, Y \in \widetilde{J}_0^r(\mathbb{R}, Gl(m\mathbb{R})_{g'}$. The condition in (3.14) says

$$j_x^r F \circ X = (j_{g'}^r \Phi_G) \circ Y.$$

Since the second and higher order derivatives of F at x and of Φ_G at g' are all zero, it follows from the coordinate expression of the composition of non-holonomic jets (c.f. end of Section 1) that (3.16) in the $\iota_1, \iota_2, \ldots, \iota_r$ coordinate gives

$$\sum_{\alpha=1}^m D_\alpha F(x) X^\alpha_{\iota_1, \ldots \iota_r} = \sum_{(\alpha,\beta)=(1,1)}^{(m,m)} D_{(\alpha,\beta)} \Phi_G(g') Y^{(\alpha,\beta)}_{\iota_1, \ldots \iota_r} \tag{3.17}$$

unless, of course, $\iota_1 = \iota_2 = \ldots = \iota_r = 0$. Since

$$D_\alpha F(x) = \begin{pmatrix} g' & \delta_\alpha \\ 0 & 0 \end{pmatrix} \text{ and } D_{(\alpha,\beta)} \Phi_G(g') = \begin{pmatrix} \Delta_{(\alpha,\beta)} & 0 \\ 0 & 0 \end{pmatrix},$$

where the ik entry in $\Delta(\alpha, \beta)$ is $\delta^i_\alpha \delta^k_\beta$ we conclude easily that $X^\alpha_{\iota_1, \ldots, \iota_r} = Y^{(\alpha,\beta)}_{\iota_1, \ldots \iota_r} = 0$ for all $\alpha, \beta = 1, \ldots, m$ and $\iota_1, \ldots \iota_r$ that are not all zero. Thus $X = j_0^r[x]$ and $Y = j_0^r[g']$ showing that Γ defined in (3.15) has indeed Cartan order r.

Proposition 3.4. *If $\Gamma : P' \to \widetilde{J}^r P$ is a Φ-connection such that for some $1 \le q < s \le i \le r$ the Φ-connection $\pi_s^{r\to i} \circ \Gamma : P' \to \widetilde{J}^s P$ has Cartan order at least q, then so does Γ.*

Proof. Let $h' \in P'$ be fixed and assume that $\Gamma(h') \circ \mathbf{J}^r(p')Y = \mathbf{J}^r(\Phi)Y$ for some $Y \in \widetilde{J}_0^r(\mathbb{R}, P')_{h'}$. Then by (1.7) we have also $(\pi_s^{r\to i} \circ \Gamma(h') \circ \mathbf{J}^s(p') \circ \pi_s^{r\to i})Y = (\mathbf{J}^s(\Phi) \circ \pi_s^{r\to i})Y$ and so, by assumtion, $(\pi_q^s \circ \pi_s^{r\to i})Y = j_0^q[h']$ which by (1.6) implies $\pi_q^r Y = j_0^q[h']$ as required.

Remark. If $q = s$, ie. if $\pi_s^{r\to i} \circ \Gamma : P' \to \widetilde{J}^s P$ is Cartan then (1.6) does not work and Proposition 3.4 must be applied with $q = s - 1$. Except when $s = i$ in which case (1.6) is not needed. Thus we get

Corollary 3.4a. *If $\Gamma : P' \to \widetilde{J}^r P$ is a Φ-connection such that for some $1 \le s \le i \le r$ the Φ-connection $\pi_s^{r\to i} \circ \Gamma : P' \to \widetilde{J}^s P$ is Cartan then Γ has Cartan order at least $s - 1$. If $\pi_s^r \circ \Gamma$ is Cartan, then Γ has Cartan order at least s.*

In particular, if $\pi_1^r \circ \Gamma$ is Cartan, then the Cartan order of Γ must be at least one.

Proposition 3.5. *If the Φ-connection $\Gamma : P' \to \widetilde{J}^r P$ is such that for some $0 < s \le r$ the Φ-connection $\pi_s^r \circ \Gamma : P' \to \widetilde{J}^s P$ has Cartan order less than s, then so has Γ.*

Proof. Let $Z \ne j_0^s[h'] \in \widetilde{J}_0^s(\mathbb{R}, P')_{h'}$ be such that $(\pi_s^{r\to i} \circ \Gamma)(h') \circ \mathbf{J}^s(p')Z = \mathbf{J}^s(\Phi)Z$ and put $Y = j_0^{r-s}[Z]$. Then $\mathbf{J}^r(p')Y = j_0^{r-s}[\mathbf{J}^s(p')Z], \Gamma(h') \circ \mathbf{J}^r(p')Y = j_0^{r-s}[(\pi_s^r \circ \Gamma)(h') \circ \mathbf{J}^s(p')Z], \mathbf{J}^r(\Phi)Y = j_0^{r-s}[\mathbf{J}^s(\Phi)Z]$ so Y satisfies the condition in (3.8) but $\pi_s^r Y \ne j_0^s[h']$ as required.

In particular if $\pi_1^r \circ \Gamma$ is not Cartan then the Cartan order of Γ must be zero. A first order connection in a principal bundle can, of course, never be a Cartan connection. It follows now that neither can an r-th order connection, where $r \ge 1$. More generally, we have

Proposition 3.6. *The Cartan order of a straight Φ-connection of order $r \ge 1$ is always zero.*

Proof. Let $\Gamma = \mathbf{J}^r(\Phi) \circ \xi$. We have seen that ξ has Cartan order zero, ie. there is an $Y \in \widetilde{J}_0^s(\mathbb{R}, P')_{h'}, \pi_1^r Y \ne j_0^1[h']$ such that $\xi(h') \circ \mathbf{J}^r(p')Y = Y$. Hence $\mathbf{J}^r(\Phi)\xi(h') \circ \mathbf{J}^r(p')Y = \mathbf{J}^r(\Phi)Y$ with $\pi_1^r Y \ne j_0^1[h']$ showing that the Cartan order of Γ is less than one.

Proposition 3.7. *If Γ is an arbitrary r-th order Φ-connection and if ξ is a first order connection in P' then the Cartan order of the $(r+1)$-st order Φ-connection $\Gamma * \xi$ is less than $r + 1$.*

Proof. Again, since the Cartan order of ξ is zero, there exists a $Y = j_0^1 y \in J_0^1(\mathbb{R}, P')_{h'} \ne j_0^1[h']$ such that $\xi(h') \circ \mathbf{J}(p')Y = Y$ which implies $j_{h'}^1\Gamma \circ \{\xi(h') \circ \mathbf{J}(p')Y\}^{[r]} = j_{h'}^1\Gamma \circ Y^{[r]}$. Here we have defined $Y^{[r]} = j_0^1(t \mapsto j_x^r[y(t)]) \in \widetilde{J}_0^{r+1}(\mathbb{R}, P')$. Explicitly,

$$j_{h'}^1\Gamma \circ j_0^1(t \mapsto j_t^r[c(p'(y(t)))]) = j_0^1(t \mapsto \Gamma(y(t)) \circ j_t^r[y(t)], \tag{3.18}$$

where we have written $\xi(h') = j^1_x c$. The left hand side in (3.18) is easily seen to be $j^1_{h'}\Gamma \circ \xi(h')^{[r]} \circ \mathbf{J}^{r+1}(p')Y^{[r]}$ — these are all composition of $(r+1)$-jets — or $\{(J(\Gamma)\circ\xi)(h')\} \circ \mathbf{J}^{r+1}(p')Y^{[r]} = (\Gamma * \xi)(h') \circ \mathbf{J}^{r+1}(p')Y^{[r]}$, whereas the right-hand-side is $j^1_0(t \mapsto j^r_t[\pi^r_0\Gamma(y(t))]) = j^1_0(t \mapsto j^r_t[\Phi(y(t))]) = \mathbf{J}^{r+1}(p')Y^{[r]}$. Thus we have shown that

$$(\Gamma * \xi)(h') \circ \mathbf{J}^{r+1}(p')Y^{[r]} = \mathbf{J}^{r+1}(\Phi) \text{ with } Y^{[r]} \neq j^r_0[h'], \tag{3.19}$$

and so the Cartan order of $\Gamma * \xi$ is less than $r+1$.

A slight modification of the proof gives immediately

Proposition 3.7a. *If η is an arbitrary r-th order connection in P and if C is a first order Φ-connection that is not Cartan, then the Cartan order of the $(r+1)$-st order Φ-connection $\eta * C$ is less than $r+1$.*

Proposition 3.8. *Let η be an r-th order connection in P, where the Φ-connection $\eta\circ\Phi$ is Cartan. Assume also that the r first order connections $\pi^{r\to i}_1 \circ \eta \circ \Phi$ are Cartan. Let further C be a first order Cartan Φ-connection. Then the $(r+1)$-st order Φ-connection $\eta * C$ is also Cartan.*

Proof. Since the Cartan property is local, we can assume $P' = M \times G'$, $P = M\times G$ and $\check{\mathrm{D}}\Phi(x,g') = (x, \Phi_G(g'))$. Then, as in (3.14), we have to show that

$$\begin{aligned} &(\eta * C)(x,g') \circ X = \mathbf{J}^{r+1}(\Phi)Y, \\ &X \in \widetilde{J}^{r+1}_0(\mathbb{R}, M)_x, Y \in \widetilde{J}^{r+1}_0(\mathbb{R}, G')_{g'} \\ &\text{implies } X = j^{r+1}_0[x] \text{ and } Y = j^{r+1}_0[g']. \end{aligned} \tag{3.20}$$

We have $\pi^{r+1}_r \circ (\eta * C) = \eta\circ\Phi$ and so by our assumption and Corollary 3.4a we know that $\pi^{r+1}_r(X) = j^r_0[x]$ and $\pi^{r+1}_r(Y) = j^r_0[g']$. If $X^j_{i_1,\dots i_r, i_{r+1}}$, $j = 1,\dots,m; i_s = 0$ or 1 and $Y^\alpha_{i_1,\dots i_r,i_{r+1}}, \alpha = 1,\dots,q' = \dim G'; i_s = 0$ or 1 are the coordinates of X and Y respectively, then this means that $X^j_{i_1,\dots,i_r,0} = 0$ as well as $Y^\alpha_{i_1,\dots,i_r,0} = 0$. The coordinates $K^\alpha_{j_1,\dots j_r,j_{r+1}}, \alpha = 1,\dots,q = \dim G; j_s = 0,1,\dots,m$ of $(\eta * C)(x,g') \in \widetilde{J}^{r+1}_x(M,G)_g, g = \Phi_G(g')$, are obtained from those of η and C as follows:

If the coordinates of $C(x,g') \in J^1(M,G)$ are $C^\alpha_i, \alpha = 1,\dots,q; i = 0,1,\dots,m$ and those of $\eta : M\times G \to \widetilde{J}^r_x(M,G)_g$ are the functions $H^\alpha_{j_1,\dots j_r}, \alpha = 1,\dots,q = \dim G; j_s = 0,1,\dots,m$ then

$$\begin{aligned} K^\alpha_{j_1,\dots j_r,0} &= H^\alpha_{j_1,\dots j_r,0}(x,g), \text{ and for } j_{r+1} \neq 0 \\ K^\alpha_{j_1,\dots j_r,j_{r+1}} &= D_{j_{r+1}}(u \mapsto H^\alpha_{j_1,\dots j_r}(u, C(u)) \\ &= \sum_{\gamma=1}^{q}(D_\gamma H^\alpha_{j_1,\dots j_r})(x,g)C^\gamma_{j_{r+1}} + (D_{j_{r+1}}H^\alpha_{j_1,\dots j_r})(x,g). \end{aligned} \tag{3.21}$$

Note that because of $(\pi^r_0 \circ \eta)(u,a) = a$, ie. $H^\alpha_{0,\dots 0}(u,a) = a$, we have

$$D_\gamma H^\alpha_{0,\dots 0} = \delta^\alpha_\gamma \text{ and } D_j H^\alpha_{0,\dots 0} = 0 \text{ for } \gamma = 1,\dots,q; \text{ and } j = 1,\dots,m. \tag{3.22}$$

We can now apply Lemma 1.1 to the coordinate version of the relation in (3.20) to obtain

$$\sum_{j=1}^{m} K^{\alpha}_{0,\dots,0,j,0,\dots,0} X^{j}_{i_1,\dots i_r,i_{r+1}} = \sum_{\gamma=1}^{q} (D_\gamma \Phi^{\alpha}_{G})(x,g') Y^{\gamma}_{i_1,\dots,i_r,i_{r+1}}. \tag{3.23}$$

Substituting from (3.21) and observing (3.22) we get

$$K^{\alpha}_{0,\dots,0,j,0,\dots,0} = H^{\alpha}_{0,\dots,0,j,0,\dots,0}(x,g) \text{ and } K^{\alpha}_{0,\dots,0,j} = C^{\alpha}_{j}. \tag{3.24}$$

Consequently, (3.23) says

$$\sum_{j=1}^{m} H^{\alpha}_{0,\dots,0,j,0,\dots,0}(x,g) X^{j}_{i_1,\dots i_r,i_{r+1}} = \sum_{\gamma=1}^{q} (D_\gamma \Phi^{\alpha}_{G})(x,g') Y^{\gamma}_{i_1,\dots,i_r,i_{r+1}} \tag{3.25}$$

if $i_1 = \ldots = i_r = 0$ and only $i_{r+1} \neq 0$, or

$$\sum_{j=1}^{m} C^{\alpha}_{j} X^{j}_{i_1,\dots i_r,i_{r+1}} = \sum_{\gamma=1}^{q} (D_\gamma \Phi^{\alpha}_{G})(x,g') Y^{\gamma}_{i_1,\dots,i_r,i_{r+1}} \tag{3.26}$$

otherwise. It follows from (1.4) that $H^{\alpha}_{0,\dots,0,j,0,\dots,0}(x,g)$ are the coordinates of $(\pi^{r\to j}_{1} \circ \eta \circ \Phi)(x,g')$ and so (3.25) implies $X^{j}_{0,\dots 0,i_{r+1}} = 0$ as well as $Y^{\gamma}_{0,\dots 0,i_{r+1}} = 0$ because $\pi^{r\to j}_{1} \circ \eta \circ \Phi$ were assumed Cartan. Similarly (3.26) implies $X^{j}_{i_1,\dots i_r,i_{r+1}} = 0$ and $Y^{\gamma}_{i_1,\dots i_r,i_{r+1}} = 0$ because C was assumed Cartan. This completes the proof.

Proposition 3.9. *Let C be a first order Φ-connection, $\xi_1,\dots,\xi_a$ first order connections in P' and $\eta_1,\dots,\eta_b$ first order connections in P. If $\eta_1 \circ \Phi,\dots,\eta_b \circ \Phi$ and C are all Cartan connections then the Cartan order of the r-th order Φ-connection*

$$\Gamma = \eta_1 * \cdots * \eta_b * C * \xi_1 * \cdots * \xi_a \tag{3.5}$$

is $b+1$.

Proof. Proposition 3.8 guarantees that the Cartan order of the $(b+1)$-st order Φ-connection $\pi^{r}_{b+1} \circ \Gamma = \eta_1 * \cdots * \eta_b * C$ is $b+1$. By Corollary 3.4a the Cartan order of Γ is thus at least $b+1$. If $a > 0$ then Proposition 3.7 says that the Cartan order of $\pi^{r}_{b+2} \circ \Gamma = \eta_1 * \cdots * \eta_b * C * \xi_1$ is less than $b+2$ and so by Proposition 3.5 also the Cartan order of Γ is less than $b+2$.

More generally,

Proposition 3.10. *Let $\xi_1,\dots,\xi_a; \eta_1,\dots,\eta_b$ and $C = \eta_{b+1} \circ \Phi$ be first order connections as above. Let $0 \le s \le b+1$ be such that the sequence $\eta_1 \circ \Phi,\dots,\eta_s \circ \Phi$ consists of Cartan connections but $\eta_{s+1} \circ \Phi$ is not Cartan. Then the Cartan order of the r-th order Φ-connection (3.5) is exactly s.*

Proof. Proposition 3.9 guarantees that the Cartan order of $\pi^{r}_{s} \circ \Gamma = \eta_1 * \ldots * \eta_s \circ \Phi$ is s and Corollary 3.4a that that of Γ is at least s. Since $\eta_{s+1} \circ \Phi$ is not Cartan it follows from Proposition 3.7a that $\pi^r_{s+1} \circ \Gamma = \eta_1 * \ldots * \eta_{s+1} \circ \Phi$ has Cartan order less than $s+1$. So by Proposition 3.5 also the Cartan order of Γ is less than $s+1$, hence equals s as required.

A special case is that of a $\Gamma = \eta * \ldots * \eta \circ \Phi$, ($\eta$ repeated r-times), where $\eta \circ \Phi : P' \to J^1 P$ is a single Cartan connection. Proposition 3.9 guarantees that this Γ is an r-th order Cartan Φ-connection. In case of the Cartan ι-connection $C = \eta \circ \iota$ cannonically associated with the homogeneous space G/G', with $\iota : G' \to G$ the inclusion map, (see Remark after Propostion 2.1) the corresponding r-th prolongation $\Gamma = \eta * \ldots * \eta \circ \iota : G \to J^r(G/G' \times G)$ can easily be seen to be given by $\Gamma(g) = (j^r_x, j^r_x[g])$, where $x = gG'$, which is self-evidently Cartan of order r as expected.

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