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VECTOR FORM BRACKETS IN LIE ALGEBROIDS

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To Ivan Kolář, on the occasion of his 60th birthday.

Abstract. A brief exposition of Lie algebroids, followed by a discussion of vector forms and their brackets in this context - and a formula for these brackets in "deformed" Lie algebroids.

1. Introduction

The sections in the tangent bundle of a (smooth) manifold can be defined as the derivations on the ring of (smooth) functions on the manifold, and thus are seen to form a Lie algebra. A *Lie algebroid* **[3]** is a direct generalization: it consists of a triple, say $(A, [,]^A, a)$, where A is a vector bundle over a base manifold, say B, and $[,]^A$ is a Lie algebra product on $\Gamma(A)$, the (smooth) section in A. Further, $a : A \to TB$, the *anchor*, is a bundle map to the tangent bundle of B, which establishes a homomorphism between the Lie algebras $\Gamma(A)$ and $\Gamma(TB)$:

$$a([u,v]^A) = [au, av], \qquad u, v \in \Gamma(A),$$

and satisfies the product rule

$$[u, fv]^A = f[u,v]^A + (a(u) \cdot f)v$$

where $f \in F$ (the ring of functions on B).

Given a Lie algebroid A and its dual bundle A^*, all tensor bundles can be constructed as a straightforward generalization of the structures based on TB and T^*B. This note shows some less obvious constructions.

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2. A handy notation. **[4]** Denote by $S_{m,n}$ the set of (m,n)-shuffles, that is, the permutations $\sigma = (i_1, \dots, i_{m+n})$ of $m+n$ symbols such that $i_1 < \dots < i_m$ and $i_{m+1} < \dots < i_{m+n}$. (Any other selection from the cosets $S_{m+n}/S_m \times S_n$

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will serve equally well.) Let $\sigma = (\sigma_1, \sigma_2)$, where $\sigma_1 = (i_1, \dots, i_m)$ and $\sigma_2 = (i_{m+1}, \dots, i_{m+n})$. Let V be a vector space, and α, β multilinear maps of V to V, say $\alpha \in \mathrm{Hom}(\bigwedge^a V, V)$ $\beta \in \mathrm{Hom}(\bigwedge^b V, V)$, then define $\alpha \mathbin{\bar\wedge} \beta \in \mathrm{Hom}(\bigwedge^{a+b-1}, V)$ by

$$(2.1) \qquad (\alpha \mathbin{\bar\wedge} \beta)(v_1, \dots, v_{a+b-1}) = \sum_{(\sigma_1,\sigma_2)\in S_{b,a-1}} \mathrm{sgn}(\sigma_1, \sigma_2)\alpha(\beta(v_{\sigma_1}), v_{\sigma_2}).$$

It is well known that, for $\gamma \in \mathrm{Hom}(\bigwedge^c V, V)$ we have

$$(2.2) \quad (\alpha \mathbin{\bar\wedge} \beta) \mathbin{\bar\wedge} \gamma - \alpha \mathbin{\bar\wedge} (\beta \mathbin{\bar\wedge} \gamma) = (-1)^{(b-1)(c-1)}((\alpha \mathbin{\bar\wedge} \gamma) \mathbin{\bar\wedge} \beta - \alpha \mathbin{\bar\wedge} (\gamma \mathbin{\bar\wedge} \beta)),$$

while, if $\alpha \in \mathrm{Hom}(V, V)$ then both sides of (2.2) vanish.

Based on $\bar\wedge$ we define a commutator bracket;

$$(2.3) \qquad [\alpha, \beta]^{\bar\wedge} = \alpha \mathbin{\bar\wedge} \beta - (-1)^{(a-1)(b-1)}\beta \mathbin{\bar\wedge} \alpha.$$

It is easy to show that $[\,,]^{\bar\wedge}$ defines a graded Lie algebra structure with reduced grading:

$$(2.4) \qquad \sum_{cycl} (-1)^{(c-1)(a-1)}[[\alpha, [\beta, \gamma]^{\bar\wedge}]^{\bar\wedge} = 0.$$

Our application is to the case when $V = \Gamma(A) \oplus F$; i.e., when the entries in α, β, etc. are sections in a Lie algebroid A or functions on the base space, or formal sums of the two. In the latter case, the linearity permits a decomposition of $\alpha(v_1, \dots, v_a)$ into *pure* terms, in which each entry is either a section in A or a function.

Each A-(differential) form or A-vector form ω is identified with an element of $\mathrm{Hom}(\bigwedge V, V)$, also denoted ω, which takes the same values when evaluated on A-sections, and vanishes when any one entry is a function. As a result, all expressions of the form $L \mathbin{\bar\wedge} \omega$ vanish when L is an A-form or a A-vector form and ω an A-form.

The structure of Lie algebroid is incorporated in an element $\mu \in \mathrm{Hom}(\bigwedge^2 V, V)$, the *multiplication* map, as follows. (It is not a vector form!)

$$(2.5) \qquad \begin{aligned} &\mu(u, v) = [u, v]^A \quad \text{for} \quad u, v \in \Gamma(A); \\ &\mu(u, f) = -\mu(f, u) = a(u) \cdot f \quad \text{for} \quad u \in \Gamma(A), f \in F; \end{aligned}$$

where a is the anchor. Finally, $\mu(f, g) = 0$ for $f, g \in F$.

Lemma 1. *If μ is the multiplication map of a Lie algebroid, then $[\mu, \mu]^{\bar\wedge} = 0$ and $\mu(u, fv) = f\mu(u, v) + \mu(u, f)v$. Conversely, if $[\,,]^A$ is any alternating product on $\Gamma(A)$, $a : A \to TB$ any bundle map, μ defined by (3.5) and $[\mu, \mu]^{\bar\wedge} = 0$, then $(A, [\,,]^A, a)$ is a Lie algebroid with multiplication μ.*

We may write (A, μ) instead of $(A, [\,,]^A, a)$.

Proof. There are three cases to be considered for the first formula, depending on how many of the variables in $[\mu,\mu]^{\bar\wedge}(\ldots)$ are A-sections and how many are functions. Note that $[\mu,\mu]^{\bar\wedge} = 2\mu \mathbin{\bar\wedge} \mu$.

$$(\mu \mathbin{\bar\wedge} \mu)(u,v,w) = \sum_{cycl} \mu(\mu(u,v),w) = \sum_{cycl} [[u,v]^A, w]^A = 0,$$

$$\begin{aligned}(\mu \mathbin{\bar\wedge} \mu)(u,v,f) &= \mu(\mu(u,v),f) + \mu(\mu(v,f),u) + \mu(\mu(f,u),v)\\ &= \mu([u,v]^A, f) + \mu(a(v)\cdot f, u) + \mu(-a(u)\cdot f, v)\\ &= a([u,v]^A)\cdot f - a(u)\cdot a(v)\cdot f + a(v)\cdot a(u)\cdot f\\ &= a([u,v]^A)\cdot f - [a(u),a(v)]\cdot f = 0,\end{aligned}$$

Finally, $\mu \mathbin{\bar\wedge} \mu(\ldots)$ is easily seen to vanish when two or more of the variables are functions.

The second formula is just a re-write of the product rule in a Lie algebroid. □

Consider an A-form $\omega \in \Gamma(\bigwedge^p A^*)$, then for its A-exterior derivative we find

$$\begin{aligned}(d^A\omega)(u_0,\ldots,u_p) &= \sum_{i=0}^{p}(-1)^i a(u_i)\cdot\omega(u_0,\ldots,\hat{i},\ldots,u_p)\\ &\quad+\sum_{i<j}(-1)^{i+j}\omega([u_i,u_j]^A, u_0,\ldots,\hat{i},\ldots,hat_j,\ldots,u_p)\\ &=\sum_{i=0}^{p}(-1)^i\mu(u_i,\omega(u_0,\ldots,\hat{i},\ldots,u_p)\\ &\quad+\sum_{i<j}(-1)^{i+j}\omega(\mu(u_i,u_j),u_0,\ldots,\hat{i},\ldots,\hat{j},\ldots,u_p)\\ &=(-1)^{p+1}(\mu \mathbin{\bar\wedge} \omega)(u_0,\ldots,u_p) - (\omega \mathbin{\bar\wedge} \mu)(u_0,\ldots,u_p)\\ &= -[\omega,\mu]^{\bar\wedge}(u_0,\ldots,u_p).\end{aligned}$$

We have just shown

Lemma 2. $d^A\omega = -[\omega,\mu]^{\bar\wedge}$.

The Jacobi indentity (2.4) easily implies that $(d^A)^2 = 0$.

3. Derivations on differential forms. The classical theory of derivations on the graded ring of differential forms (i.e., the case of the standard tangent Lie algebroid), see **[1]**, states that every derivation is uniquely a sum of one of type i_* and one of type d_*. A derivation is of type i_* if it vanishes on functions, and is of the form $\omega \mapsto i_L\omega = \omega \mathbin{\bar\wedge} L$, where L is a vector form, $L \in \Gamma(TB \otimes \bigwedge T^*B)$, and it is of type d_* if it commutes with the exterior derivative d; in this case it is of the form $\omega \mapsto d_L\omega = [i_L, d] = (i_L d + (-1)^q d i_L)\omega$, where $L \in \Gamma(TB \otimes \bigwedge^q T^*B)$.

In a general Lie algebroid A the A-forms also admit derivations of types i_*^A (the same as i_* above), and d_*^A (in obvious generalization of the above), but these need not span all derivations.

For example, consider the Lie algebroid A with bundle space TB but trivial bracket and anchor. Then the classical d is a derivation, but is not a sum of derivations of types i_*^A and d_*^A, because the first vanish on functions, and the second are trivial (zero).

The commutator relations for derivations of types i_* and d_* for the standard tangent Lie algebroid are (see **[1]**)

$$[i_L, i_M] = i_{[M,L]^{\overline{\wedge}}}; \tag{3.1}$$

$$[i_L, d_M] = d_{M \,\overline{\wedge}\, L} + (-1)^m i_{[L,M]}; \tag{3.2}$$

$$[d_L, d_M] = d_{[L,M]}. \tag{3.3}$$

We generalize these formulas to Lie algebroids, as follows. (3.1) is "the same" (see above); (3.2) is seen as a definition of $[L, M]$, and requires a proof that $[i_l, i_M] - d_{M \,\overline{\wedge}\, L}$ is indeed of type i_*^A. Then (3.3) easily follows from (3.2) as a consequence of the Jacobi identity for derivations.

In what follows, the A for Lie algebroids has been suppressed in the formulas. – Define $\mu(L, M)$ by

$$\mu(L,M)(u_1, \dots, u_{q+m}) = \sum_{(\sigma_1,\sigma_2)\in S_{q,m}} \operatorname{sgn}(\sigma_1,\sigma_2)\mu(L(u_{\sigma_1}), M(u_{\sigma_2})). \tag{3.4}$$

Note that $\mu(L, M)$, though not an A-vector form, vanishes if any one entry is a function; in particular, $\mu(L, M) \,\overline{\wedge}\, \omega = 0$ for A-forms ω.

Lemma 3. *We have the following*

$$\mu(L,M) = (-1)^{q(m-1)}((\mu \,\overline{\wedge}\, L) \,\overline{\wedge}\, M - \mu \,\overline{\wedge}\, (L \,\overline{\wedge}\, M)); \tag{3.5}$$

$$d_L\omega = (-1)^{q+1}\omega \,\overline{\wedge}\, [L,\mu]^{\overline{\wedge}} + (-1)^{pq+q}(\mu \,\overline{\wedge}\, L) \,\overline{\wedge}\, \omega; \tag{3.6}$$

$$[M \,\overline{\wedge}\, L, \mu]^{\overline{\wedge}} = M \,\overline{\wedge}\, [L,\mu]^{\overline{\wedge}} - (-1)^q [M,\mu]^{\overline{\wedge}} \,\overline{\wedge}\, L + (-1)^{q+1}\mu(L,M); \tag{3.7}$$

$$[\omega \,\overline{\wedge}\, L, \mu]^{\overline{\wedge}} = \omega \,\overline{\wedge}\, [L,\mu]^{\overline{\wedge}} - (-1)^q[\omega,\mu]^{\overline{\wedge}} \,\overline{\wedge}\, L + (-1)^{qp+1}(\mu \,\overline{\wedge}\, L) \,\overline{\wedge}\, \omega. \tag{3.8}$$

Proof. For (3.5) see page 104 of **[4]**. (Note that the proof of (2.2) in **[1]** contains two canceling errors, and would give an incorrect sign in (3.5).) The other formulas require simple calculations using the definitions and (2.2). □

Theorem 1. *The derivations of types i_* and d_* in a Lie algebroid satisfy (3.1-3). The bracket $[L,M]$ is given by*

$$[L,M] = \mu(L,M) + (-1)^{m(q-1)} L \mathbin{\bar\wedge} [M,\mu]^{\bar\wedge} + (-1)^{q+1} M \mathbin{\bar\wedge} [L,\mu]^{\bar\wedge}. \tag{3.9}$$

Proof. In the following calculations, the $\bar\wedge$ on $[\,,\,]^{\bar\wedge}$ will be suppressed. Use Lemma 3.

$$\begin{aligned}
&(-1)^m [i_L, d_M]\omega - (-1)^m d_{L \bar\wedge M}\omega \\
&= (-1)^m (d_M\omega) \mathbin{\bar\wedge} L - (-1)^{qm} d_M(\omega \mathbin{\bar\wedge} L) - (-1)^m d_{M \bar\wedge L}\omega \\
&= (-1)^m ((-1)^{m+1} \omega \mathbin{\bar\wedge} [M,\mu] + (-1)^{m(p-1)} (\mu \mathbin{\bar\wedge} M) \mathbin{\bar\wedge} \omega) \mathbin{\bar\wedge} L) \\
&\quad - (-1)^{qm} ((-1)^{m+1} (\omega \mathbin{\bar\wedge} L) \mathbin{\bar\wedge} [M,\mu] + (-1)^{(p+q)m} (\mu \mathbin{\bar\wedge} M) \mathbin{\bar\wedge} (\omega \mathbin{\bar\wedge} L)) \\
&\quad - (-1)^m ((-1)^{q+m} \omega \mathbin{\bar\wedge} [M \mathbin{\bar\wedge} L, \mu] + (-1)^{(p-1)(q+m-1)} (\mu \mathbin{\bar\wedge} (M \mathbin{\bar\wedge} L)) \mathbin{\bar\wedge} \omega).
\end{aligned}$$

We first combine the first terms in each line.

$$\begin{aligned}
&- (\omega \mathbin{\bar\wedge} [M,\mu]) \mathbin{\bar\wedge} L - (-1)^{qm+p+q} (\omega \mathbin{\bar\wedge} L) \mathbin{\bar\wedge} [M,\mu] - (-1)^q \omega \mathbin{\bar\wedge} [M \mathbin{\bar\wedge} L, \mu] \\
&= -(\omega \mathbin{\bar\wedge} [M,\mu]) \mathbin{\bar\wedge} L \\
&\quad + ((-1)^{(q-1)m} (\omega \mathbin{\bar\wedge} (L \mathbin{\bar\wedge} [M,\mu] + (\omega \mathbin{\bar\wedge} [M,\mu]) \mathbin{\bar\wedge} L - \omega \mathbin{\bar\wedge} ([M,\mu] \mathbin{\bar\wedge} L)) \\
&\quad - (-1)^q \omega \mathbin{\bar\wedge} (M \mathbin{\bar\wedge} [L,\mu] - (-1)^q [M,\mu] \mathbin{\bar\wedge} L + (-1)^{q+1} \mu(L,M)) \\
&= \omega \mathbin{\bar\wedge} ((-1)^{(q-1)m} L \mathbin{\bar\wedge} [M,\mu] + (-1)^{q+1} M \mathbin{\bar\wedge} [L,\mu] + \mu(L,M)).
\end{aligned}$$

This proves (3.9), after we show the vanishing of the remaining terms:

$$\begin{aligned}
&(-1)^{pm} (((\mu \mathbin{\bar\wedge} M) \mathbin{\bar\wedge} \omega) \mathbin{\bar\wedge} L - (\mu \mathbin{\bar\wedge} M) \mathbin{\bar\wedge} (\omega \mathbin{\bar\wedge} L) \\
&\quad + (-1)^{(p-1)(q-1)} (\mu \mathbin{\bar\wedge} (M \mathbin{\bar\wedge} L)) \mathbin{\bar\wedge} \omega) \\
&= (-1)^{pm} (-(\mu \mathbin{\bar\wedge} M) \mathbin{\bar\wedge} (L \mathbin{\bar\wedge} \omega) + (-1)^{(p-1)(q-1)} (((\mu \mathbin{\bar\wedge} M) \mathbin{\bar\wedge} L) \mathbin{\bar\wedge} \omega \\
&\quad - (\mu \mathbin{\bar\wedge} (M \mathbin{\bar\wedge} L)) \mathbin{\bar\wedge} \omega))) = 0 + (-1)^{pm+(p+m-1)(q-1)} \mu(M,L) \mathbin{\bar\wedge} \omega = 0.
\end{aligned}$$

□

4. The "deformed" Lie algebroid. [2] The operator i_L, so far defined as acting on $\Gamma(\bigwedge A^*)$, is extended to act on $\mathrm{Hom}(\bigwedge V, V)$ (V as defined in section 2) in the case when L is an A-vector 1-form. In that case we prefer the notation h, k, etc. over L, etc., and set

$$i_h\alpha = \alpha \mathbin{\bar\wedge} h - h \mathbin{\bar\wedge} \alpha. \tag{4.1}$$

According to (2.2) and the line following, i_h satisfies a product rule with respect to $\bar\wedge$:

$$i_h(\alpha \mathbin{\bar\wedge} \beta) = (i_h\alpha) \mathbin{\bar\wedge} \beta + \alpha \mathbin{\bar\wedge} i_h\beta.$$

If A is a Lie algebroid with multiplicatiom μ, and h an A-vector 1-form, a new, *deformed* multiplication μ_h is given by

$$\mu_h(u,v) = \mu(hu,v) + \mu(u,hv) - h\mu(u,v),$$

i.e., by $\mu_h = i_h\mu$. (This implies (see (2.5)) that a *deformed* anchor map a_h is given by $a_h(u) = a(hu)$.) In general, μ_h does not define a Lie algebroid structure on the bundle space A.

Lemma 4. *If $[h,h] = 0$ then μ_h defines a Lie algebroid structure (A,μ_h), and h is a homomorphism to (A,μ).*

Proof. The product rule $\mu_h(u,fv) = \cdots$ follows by a simple calculation, using just the F-linearity of h and the fact that h acts trivially on functions.

Again, we suppress the $\bar{\wedge}$ on $[,]^{\bar{\wedge}}$ below. The formula (3.9) with $L = M = h$, and the observation that $\mu(h,h)(u,v) = 2\mu(hu,hv)$, yields

$$\mu(hu,hv) = -h \mathbin{\bar{\wedge}} [h,\mu](u,v) = h\mu_h(u,v),$$

so h gives the homomorphism of μ_h to μ.

Formulas (3.7) and (3.9), with $L = M = h$, using $[h,\mu] = -i_h\mu = -\mu_h$ give rise to

$$[h^2,\mu] = -h \mathbin{\bar{\wedge}} \mu_h - \mu_h \mathbin{\bar{\wedge}} h + \mu(h,h), \qquad 0 = \mu(h,h) - 2h \mathbin{\bar{\wedge}} \mu_h .$$

Elimination of $\mu(h,h)$ by subtraction yields $[h^2,\mu] = -[h,\mu_h]$. Bracketing with μ, combined with the product rule for i_h yields

$$[\mu,[h^2,\mu]] = -[\mu,i_h\mu_h] = -i_h[\mu,\mu_h] + [i_h\mu,\mu_h] = i_h[\mu,[h,\mu]] + [\mu_h,\mu_h].$$

Now, $[\mu,[\mu,k]] = -\frac{1}{2}[k,[\mu,\mu]] = 0$ for any vector 1-form k (Jacobi identity), so we find $[\mu_h,\mu_h] = 0$. Hence, μ_h satisfies the Jacobi identity. □

Given a Lie algebroid (A,μ) and a deformed Lie algebroid (A,μ_h), we define $[L,M]_h$, the A-vector form bracket with respect to μ_h by replacing in (3.9) all μ by μ_h. (Formulas (3.1-3) will then be valid, after the same substitution, see Lemma 2.)

Theorem 2. *Let $[L,M]_h$ denote the vector form bracket in a deformed Lie algebroid (A,μ_h), then*

$$[L,M]_h = i_h[L,M] - [i_hL,M] - [L,i_hM]. \tag{4.2}$$

Proof. Let $P(X,Y,\ldots)$ be a polynomial, linear in each of $X,Y,\ldots$, with $X,Y,\ldots \in \operatorname{Hom}(\bigwedge V,V)$, and with the (non-commutative, non-assocative) product $\bar{\wedge}$, with respect to which i_h is a derivation, then

$$i_hP(X,Y,\ldots) = P(i_hX,Y,\ldots) + P(X,i_hY,\ldots) + \ldots .$$

Apply this to $P(L,M,\mu) = [L,M]$ (see (3.9)), and observe that

$$[L,M]_h = P(L,M,i_h\mu).$$

The result follows immediately. □

References

The following articles contain suitable introductions to the relevant topics, as well as references to further information.

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