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PRODUCT PRESERVING BUNDLE FUNCTORS ON FIBERED MANIFOLDS

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To Ivan Kolář, on the occasion of his 60th birthday.

Abstract. The complete description of all product preserving bundle functors on fibered manifolds in terms of natural transformations between product preserving bundle functors on manifolds is given.

0. About 1985, Eck [1], Luciano [4] and Kainz and Michor [2] obtained the complete description of all product preserving bundle functors on manifolds in terms of the Weil bundles [6] by Morimoto [5].

In this note we present the complete description of all product preserving bundle functors on fibered manifolds in terms of natural transformations between product preserving bundle functors on manifolds.

The category of smooth manifolds and their smooth maps will be denoted by $\mathcal{M}$. The category of smooth fibered manifolds and their smooth fibered maps will be denoted by $\mathcal{FM}$. The definition of bundle functors on a category over manifolds can be found in the fundamental monograph by Kolář, Michor and Slovák [3].

All manifolds are assumed to be finite dimensional, without boundaries and smooth, i.e. of class $\mathcal{C}^\infty$. Maps between manifolds are assumed to be smooth, i.e. of class $\mathcal{C}^\infty$.

1. This item consists of some examples concerning (not necessarily product preserving) bundle functors. In these examples we build a "machinary" which we use (in Item 2) to give the above mentioned full description. All constructions presented in the examples are canonical.

In the first example we show how a natural transformation $\mu : G \to H$ between bundle functors on manifolds induces canonically a bundle functor $G \times_\mu H$ on fibered manifolds.

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1.1. Example. Let $\mu = \{\mu_M\} : G \to H$ be a natural transformation between two bundle functors $G, H : \mathcal{M} \to \mathcal{FM}$. We define a bundle functor $G \times_\mu H : \mathcal{FM} \to \mathcal{FM}$ as follows.

Given a fibered manifold $\pi : X \to Y$ we have two maps

$$G(Y) \xrightarrow{\mu_Y} H(Y) \xleftarrow{H(\pi)} H(X)$$

between manifolds. They are transversal as $H(\pi)$ is a surjective submersion. Hence we have the pull-back diagram

$$\begin{array}{ccc} G(Y) \times_{\mu_Y, H(Y), H(\pi)} H(X) & \xrightarrow{p_2} & H(X) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle H(\pi)} \\ G(Y) & \xrightarrow{\mu_Y} & H(Y) \end{array}$$

with smooth maps. We put

$$(G \times_\mu H)(\pi) := G(Y) \times_{\mu_Y, H(Y), H(\pi)} H(X) \ . \tag{1.1}$$

Let $p_\pi^{G \times_\mu H} = p_X^H \circ p_2 : (G \times_\mu H)(\pi) \to X$, where $p_X^H : H(X) \to X$ is the bundle projection of H.

It remains to define $G \times_\mu H$ on fibered maps.

Let $f : X \to \overline{X}$ be a fibered map from $\pi : X \to Y$ into $\overline{\pi} : \overline{X} \to \overline{Y}$. Let $\underline{f} : Y \to \overline{Y}$ denote the corresponding base map. Since μ is a natural transformation and H is a functor, we have the following commutative diagrams of maps:

$$\begin{array}{ccc} G(Y) & \xrightarrow{\mu_Y} & H(Y) \\ {\scriptstyle G(\underline{f})}\downarrow & & \downarrow{\scriptstyle H(\underline{f})} \\ G(\overline{Y}) & \xrightarrow{\mu_{\overline{Y}}} & H(\overline{Y}) \ , \end{array} \qquad \begin{array}{ccc} H(Y) & \xleftarrow{H(\pi)} & H(X) \\ {\scriptstyle H(\underline{f})}\downarrow & & \downarrow{\scriptstyle H(f)} \\ H(\overline{Y}) & \xleftarrow[H(\overline{\pi})]{} & H(\overline{X}) \ . \end{array}$$

Hence we have the induced by $G(\underline{f})$ and $H(f)$ smooth (pull-back) map $(G \times_\mu H)(f) : (G \times_\mu H)(\pi) \to (G \times_\mu H)(\overline{\pi})$,

$$(G \times_\mu H)(f) = \text{ the restriction of } G(\underline{f}) \times H(f) \ . \tag{1.2}$$

The correspondence $G \times_\mu H$ as above is a bundle functor $\mathcal{FM} \to \mathcal{FM}$. If G and H are product preserving, then so is $G \times_\mu H$.

Let $\overline{\mu} : \overline{G} \to \overline{H}$ be an another natural transformation between bundle functors $\overline{G}, \overline{H} : \mathcal{M} \to \mathcal{FM}$ and let (ν, ρ) be a pair of natural transformations $\nu = \{\nu_M\} : G \to \overline{G}$ and $\rho = \{\rho_M\} : H \to \overline{H}$ such that the following diagram

$$\begin{array}{ccc} G(M) & \xrightarrow{\nu_M} & \overline{G}(M) \\ {\scriptstyle \mu_M}\downarrow & & \downarrow{\scriptstyle \overline{\mu}_M} \\ H(M) & \xrightarrow{\rho_M} & \overline{H}(M) \end{array} \tag{1.3}$$

is commutative for any manifold M. Then we define a natural transformation $\nu \times_{\mu,\overline{\mu}} \rho : G \times_\mu H \to \overline{G} \times_{\overline{\mu}} \overline{H}$ as follows.

Given a fibered manifold $\pi : X \to Y$ we have the following commutative diagrams of maps:

$$\begin{array}{ccc} G(Y) & \xrightarrow{\mu_Y} & H(Y) \\ {\scriptstyle \nu_Y}\downarrow & & \downarrow{\scriptstyle \rho_Y} \\ \overline{G}(Y) & \xrightarrow{\overline{\mu}_Y} & \overline{H}(Y)\ , \end{array} \qquad \begin{array}{ccc} H(Y) & \xleftarrow{H(\pi)} & H(X) \\ {\scriptstyle \rho_Y}\downarrow & & \downarrow{\scriptstyle \rho_X} \\ \overline{H}(Y) & \xleftarrow[\overline{H}(\pi)]{} & \overline{H}(X)\ . \end{array}$$

(The first diagram is commutative by the assumption and the second one is commutative as ρ is a natural transformation.) Hence we have the induced by ν_Y and ρ_X smooth (pull-back) map $(\nu \times_{\mu,\overline{\mu}} \rho)_\pi : (G \times_\mu H)(\pi) \to (\overline{G} \times_{\overline{\mu}} \overline{H})(\pi)$,

$$(1.4) \qquad (\nu \times_{\mu,\overline{\mu}} \rho)_\pi = \text{ the restriction of } \nu_Y \times \rho_X\ .$$

The family $\nu \times_{\mu,\overline{\mu}} \rho = \{(\nu \times_{\mu,\overline{\mu}} \rho)_\pi\} : G \times_\mu H \to \overline{G} \times_{\overline{\mu}} \overline{H}$ is a natural transformation.

In second example we show how a bundle functor F on fibered manifolds induces canonically a natural transformation $\mu^F : G^F \to H^F$ between bundle functors on manifolds. We need some preparations.

From now on we fix a one-point manifold pt.

Any manifold M determines fibered manifolds $id_M : M \to M$ (the identity map) and $pt_M : M \to pt$. Any map $f : M \to N$ can be considered as fibered maps ($\mathcal{FM}$-morphisms) $f : id_M \to id_N$, $f : pt_M \to pt_N$ and $f : id_M \to pt_N$. Consequently, we have two functors $i_{(1)}, i_{(2)} : \mathcal{M} \to \mathcal{FM}$ given by

$$(1.5) \qquad i_{(1)}(M) := id_M\ ,\ i_{(1)}(f) := f \qquad i_{(2)}(M) := pt_M\ ,\ i_{(2)}(f) := f$$

for any manifold M and any map $f : M \to N$, and a natural transformation $id : i_{(1)} \to i_{(2)}$ consisting of $\mathcal{FM}$-morphisms

$$(1.6) \qquad id_M : id_M \to pt_M$$

for any manifold M. Of course, functors $i_{(1)}$ and $i_{(2)}$ are product preserving.

1.2. Example. Let $F : \mathcal{FM} \to \mathcal{FM}$ be a bundle functor. We define a natural transformation $\mu^F : G^F \to H^F$ between two bundle functors on manifolds as follows.

Composing functor F with functors $i_{(1)}$ and $i_{(2)}$ we obtain (as easily seen) two bundle functors on $\mathcal{M}$

$$(1.7) \qquad G^F := F \circ i_{(1)}\ , \qquad H^F := F \circ i_{(2)}\ .$$

If F is product preserving, then so are G^F and H^F.

Lifting the natural transformation (1.6) to F we obtain a natural transformation $\mu^F : G^F \to H^F$

$$\mu^F_M := F(id_M) : G^F(M) \to H^F(M) \tag{1.8}$$

for any manifold M.

Let $\overline{F} : \mathcal{FM} \to \mathcal{FM}$ be another bundle functor and let $\eta = \{\eta_\pi\} : F \to \overline{F}$ be a natural transformation. We define two natural transformations $\nu^\eta = \{\nu^\eta_M\} : G^F \to G^{\overline{F}}$ and $\rho^\eta = \{\rho^\eta_M\} : H^F \to H^{\overline{F}}$ by

$$\nu^\eta_M := \eta_{i_{(1)}(M)} : G^F(M) \to G^{\overline{F}}(M)\ ,\ \rho^\eta_M := \eta_{i_{(2)}(M)} : H^F(M) \to H^{\overline{F}}(M) \tag{1.9}$$

for any manifold M. The following diagram

$$\begin{array}{ccc} G^F(M) & \xrightarrow{\nu^\eta_M} & G^{\overline{F}}(M) \\ {\scriptstyle \mu^F_M}\downarrow & & \downarrow{\scriptstyle \mu^{\overline{F}}_M} \\ H^F(M) & \xrightarrow{\rho^\eta_M} & H^{\overline{F}}(M) \end{array} \tag{1.10}$$

is commutative for any manifold M. (For, $\overline{F}(id_M) \circ \eta_{i_{(1)}(M)} = \eta_{i_{(2)}(M)} \circ F(id_M)$ as η is a natural transformation.)

In the next example we construct a natural transformation $F \to G^F \times_{\mu^F} H^F$.

1.3. Example. Let $F : \mathcal{FM} \to \mathcal{FM}$ be a bundle functor. Let $\mu^F : G^F \to H^F$ be the natural transformation as in Example 1.2. Let $G^F \times_{\mu^F} H^F : \mathcal{FM} \to \mathcal{FM}$ be the bundle functor as in Example 1.1 for μ^F instead of μ. We define a natural transformation $\Theta = \{\Theta_\pi\} : F \to G^F \times_{\mu^F} H^F$ as follows.

Let $\pi : X \to Y$ be a fibered manifold. Then

$$(G^F \times_{\mu^F} H^F)(\pi) = F(id_Y) \times_{F(id_Y), F(pt_Y), F(\pi)} F(pt_X)\ , \tag{1.11}$$

where the $\mathcal{FM}$-morphisms $id_Y : id_Y \to pt_Y$ and $\pi : pt_X \to pt_Y$ are determined by id_Y and π respectively. From the functoriality of F it follows that the image of the system of maps $(F(\pi), F(id_X)) : F(\pi) \to F(id_Y) \times F(pt_X)$, where the $\mathcal{FM}$-morphisms $\pi : \pi \to id_Y$ and $id_X : \pi \to pt_X$ are determined by π and id_X respectively, is contained in $(G^F \times_{\mu^F} H^F)(\pi)$. (For, the composition of $\pi : \pi \to id_Y$ with $id_Y : id_Y \to pt_Y$ is equal to the composition of $id_X : \pi \to pt_X$ with $\pi : pt_X \to pt_Y$.) We set

$$\Theta_\pi := (F(\pi), F(id_X)) : F(\pi) \to (G^F \times_{\mu^F} H^F)(\pi) \subset F(id_Y) \times F(pt_X)\ , \tag{1.12}$$

where the $\mathcal{FM}$-morphisms $\pi : \pi \to id_Y$ and $id_X : \pi \to pt_X$ are determined by π and id_X respectively.

The family $\Theta = \{\Theta_\pi\} : F \to G^F \times_{\mu^F} H^F$ is a natural transformation.

In the next example we present a relationships between a natural transformation $\mu : G \to H$ and the natural transformation corresponding to $G \times_\mu H$.

1.4. Example. Let $\mu : G \to H$ be a natural transformation between two bundle functors $G, H : \mathcal{M} \to \mathcal{FM}$. Let $G \times_\mu H : \mathcal{FM} \to \mathcal{FM}$ be the corresponding bundle functor as in Example 1.1. Let $\overset{o}{\mu} : \overset{o}{G} \to \overset{o}{H}$ be the corresponding natural transformation as in Example 1.2 for $F = G \times_\mu H$. Then

$$\overset{o}{G}(M) = \{(a, \mu(a)) \mid a \in G(M)\} \subset G(M) \times H(M) , \tag{1.13}$$

$$\overset{o}{H}(M) = \{(b, c) \in G(pt) \times H(M) \mid \mu_{pt}(b) = H(pt_M)(c)\} \subset G(pt) \times H(M) , \tag{1.14}$$

$$\overset{o}{\mu}_M = \text{ the restriction of } G(pt_M) \times H(id_M) . \tag{1.15}$$

For any manifold M we define $\mathcal{O}_M : \overset{o}{G}(M) \to G(M)$,

$$\mathcal{O}_M := \text{ the restriction of the usual projection.} \tag{1.16}$$

The family $\mathcal{O} = \{\mathcal{O}_M\} : \overset{o}{G} \to G$ is a natural equivalence.

For any manifold M we define $\mathcal{Q}_M : \overset{o}{H}(M) \to H(M)$,

$$\mathcal{Q}_M := \text{ the restriction of the usual projection.} \tag{1.17}$$

The family $\mathcal{Q} = \{\mathcal{Q}_M\} : \overset{o}{H} \to H$ is a natural transformation. It is a natural equivalence if and only if $\mu_{pt} : G(pt) \to H(pt)$ is a diffeomorphism. (For, if $\mathcal{Q}_{pt}$ is a diffeomorphism, then so is μ_{pt}. If μ_{pt} is a diffeomorphism, then (of course) so is $\mathcal{Q}_M$.)

The following diagram

$$\begin{array}{ccc} \overset{o}{G}(M) & \xrightarrow{\mathcal{O}_M} & G(M) \\ {\scriptstyle \overset{o}{\mu}_M}\big\downarrow & & \big\downarrow{\scriptstyle \mu_M} \\ \overset{o}{H}(M) & \xrightarrow{\mathcal{Q}_M} & H(M) \end{array} \tag{1.18}$$

is commutative for any manifold M.

Now, we present a very importrant modification of Example 1.1, a basic model.

1.5. Example. Given a natural transformation $\mu : G \to H$ between bundle functors on $\mathcal{M}$ such that $\mu_{pt} : G(pt) \to H(pt)$ is a diffeomorphism we modify the bundle functor $G \times_\mu H$ of Example 1.1 as follows.

For any fibered manifold $\pi : X \to Y$ we put

$$(1.19)\qquad T^\mu(\pi) = \begin{cases} G(M) & \text{if } \pi = id_M \text{ for a manifold } M \\ H(M) & \text{if } \pi = pt_M \text{ for a manifold } M \\ (G \times_\mu H)(\pi) & \text{otherwise .} \end{cases}$$

Then $T^\mu(\pi)$ is a fibered manifold over X. We also define $I_\pi : T^\mu(\pi) \to (G \times_\mu H)(\pi)$ by

$$(1.20)\qquad I_\pi = \begin{cases} \mathcal{O}_M^{-1} & \text{if } \pi = id_M \text{ for a manifold } M \\ \mathcal{Q}_M^{-1} & \text{if } \pi = pt_M \text{ for a manifold } M \\ id_{(G\times_\mu H)(\pi)} & \text{otherwise ,} \end{cases}$$

where $\mathcal{O}_M : \overset{o}{G}(M) = (G \times_\mu H)(id_M) \to G(M)$ and $\mathcal{Q}_M : \overset{o}{H}(M) = (G \times_\mu H)(pt_M) \to H(M)$ are as in Example 1.4.

There exists one and only one bundle functor $T^\mu : \mathcal{FM} \to \mathcal{FM}$ such that the family $I = \{I_\pi\} : T^\mu \to G \times_\mu H$ is a natural equivalence.

If G and H are product preserving, then so is T^μ.

If $\overline{\mu} : \overline{G} \to \overline{H}$ is another natural transformation such that $\overline{\mu}_{pt}$ is a diffeomorphism and (ν, ρ) is a pair of natural transformations $\nu = \{\nu_M\} : G \to \overline{G}$ and $\rho = \{\rho_M\} : H \to \overline{H}$ such that the diagram (1.3) is commutative for any manifold M, then we define a natural transformation $(\widetilde{\nu, \rho}) = \{(\widetilde{\nu, \rho})_\pi\} : T^\mu \to T^{\overline{\mu}}$ given by the compositions

$$(1.21)\qquad (\widetilde{\nu, \rho})_\pi \; : \; T^\mu(\pi) \xrightarrow{I_\pi} (G \times_\mu H)(\pi) \xrightarrow{(\nu \times_{\mu,\overline{\mu}} \rho)_\pi} (\overline{G} \times_{\overline{\mu}} \overline{H})(\pi) \xrightarrow{I_\pi^{-1}} T^{\overline{\mu}}(\pi)$$

for any fibered manifold π.

Functor T^μ has the following very improtrant property.

The natural transformation $\mu^F : G^F \to H^F$ corresponding to $F = T^\mu$ in the sense of Example 1.2 is equal to μ, i.e.

$$(1.22)\qquad \mu^F = \mu \; , \qquad \text{if } F = T^\mu \; .$$

(It follows from the fact that the diagram (1.18) is commutative.) In other words, T^μ is an "extension" of μ.

1.6. Example. Let $F : \mathcal{FM} \to \mathcal{FM}$ be a bundle functor. Let $\mu^F : G^F \to H^F$ be the corresponding natural transformation as in Example 1.2. Assume that μ^F_{pt} is a diffeomorphism. Then we have the bundle functor $T^{\mu^F} : \mathcal{FM} \to \mathcal{FM}$ and the natural equivalence $I : T^{\mu^F} \to G^F \times_{\mu^F} H^F$ as in Example 1.5 for $\mu = \mu^F$. On the other hand in Example 1.3 we have constructed the natural transformation $\Theta : F \to G^F \times_{\mu^F} H^F$. Therefore we have a natural transformation $\kappa = \{\kappa_\pi\} : F \to T^{\mu^F}$ given by the compositions

$$(1.23)\qquad \kappa_\pi \; : \; F(\pi) \xrightarrow{\Theta_\pi} G^F \times_{\mu^F} H^F \xrightarrow{I_\pi^{-1}} T^{\mu^F}(\pi)$$

for any fibered manifold π.

2. In this item we restrict ourselves to product preserving bundle functors only. We give the full description of product preserving bundle functors on fibered manifolds in terms of natural transformations between product preserving bundle functors on manifolds.

We start with the proof of the following classification theorem.

2.1. Theorem. (1) *Let $F:\mathcal{FM}\to\mathcal{FM}$ be a product preserving bundle functor. Then the described in Example 1.6 natural transformation $\kappa=\{\kappa_\pi\}:F\to T^{\mu^F}$ is a natural equivalence.*

(2) *If $\mu:G\to H$ is a natural transformation between two product preserving bundle functors $G,H:\mathcal{M}\to\mathcal{FM}$ and κ is the natural transformation as in Example 1.6 for $F=T^\mu$, where $T^\mu:\mathcal{FM}\to\mathcal{FM}$ is as in Example 1.5, then $\kappa=\{\kappa_\pi\}:T^\mu\to T^\mu$ and $\kappa_\pi=id_{T^\mu(\pi)}$ for any fibered manifold π.*

(3) *If $\mu:G\to H$ is a natural transformation between product preserving bundle functors $G,H:\mathcal{M}\to\mathcal{FM}$, then the described in Example 1.5 bundle functor $T^\mu:\mathcal{FM}\to\mathcal{FM}$ is the unique up to natural equivalence product preserving bundle functor on fibered manifolds such that the described in Example 1.2 natural transformation $\mu^F:G^F\to H^F$ corresponding to $F=T^\mu$ is equal to μ.*

Proof. (ad 1) Let $\pi:X\to Y$ be a fibered manifold. It is sufficient to show that the described in Example 1.3 natural transformation $\Theta_\pi:F(\pi)\to(G^F\times_{\mu^F}H^F)(\pi)$ is a diffeomorphism for any fibered manifold π.

Using the standard argument with fibered manifold charts one can assume that $\pi:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}^n$, the projection onto the first factor.

Since the bundle functors $G^F\times_{\mu^F}H^F$ and F are product preserving and the fibered manifold $\pi:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}^n$ is the (multi)product of fibered manifolds $id_{\mathbf{R}}$ and $pt_{\mathbf{R}}$, we can assume that $\pi=id_{\mathbf{R}}$ or $\pi=pt_{\mathbf{R}}$.

At first we assume that $\pi=id_{\mathbf{R}}$. From (1.11) it follows that the manifold $(G^F\times_{\mu^F}H^F)(id_{\mathbf{R}})$ is a submanifold in the product $F(id_{\mathbf{R}})\times F(pt_{\mathbf{R}})$ of manifolds and

$$(G^F\times_{\mu^F}H^F)(id_{\mathbf{R}})=\{(a,F(id_{\mathbf{R}})(a))\in F(id_{\mathbf{R}})\times F(pt_{\mathbf{R}})\mid a\in F(id_{\mathbf{R}})\}\ ,$$

where $id_{\mathbf{R}}:id_{\mathbf{R}}\to pt_{\mathbf{R}}$ (on the righ-hand side of the formula) is the $\mathcal{FM}$-morphism determined by $id_{\mathbf{R}}$. By (1.12),

$$\Theta_{id_{\mathbf{R}}}=(F(id_{\mathbf{R}}),F(id_{\mathbf{R}})):F(id_{\mathbf{R}})\to(G^F\times_{\mu^F}H^F)(id_{\mathbf{R}})\ ,$$

the system of maps, where $id_{\mathbf{R}}:id_{\mathbf{R}}\to id_{\mathbf{R}}$ and $id_{\mathbf{R}}:id_{\mathbf{R}}\to pt_{\mathbf{R}}$ are the $\mathcal{FM}$-morphisms determined by $id_{\mathbf{R}}$. Since $id_{\mathbf{R}}:id_{\mathbf{R}}\to id_{\mathbf{R}}$ is the identity morphism on $id_{\mathbf{R}}$, we have

$$\Theta_{id_{\mathbf{R}}}(a)=(a,F(id_{\mathbf{R}})(a)) \tag{2.1}$$

for any $a \in F(id_{\mathbf{R}})$. Hence $\Theta_{id_{\mathbf{R}}}$ is a diffeomorphism.

It remains to assume that $\pi = pt_{\mathbf{R}}$. The proof in this case is similar as for $\pi = id_{\mathbf{R}}$. We leave the details to the reader.

(ad 2) The fact $\kappa : T^\mu \to T^\mu$ follows from (1.22). Now by the reasons as in the proof of part (1) one can assume that $\pi = id_{\mathbf{R}}$ or $\pi = pt_{\mathbf{R}}$. We leave the details to the reader.

(ad 3) The existence part of the assertion follows from (1.22).

If $F : \mathcal{FM} \to \mathcal{FM}$ is another product preserving bundle functor such that $\mu^F = \mu$, then $\kappa : F \to T^{\mu^F} = T^\mu$ is an equivalence because of the part (1) of the theorem. □

Remark. The described in Example 1.2 correspondence "$F \to \mu^F$" is not a bijection between the product preserving bundle functors on fibered manifolds and the natural transformations of product preserving bundle functors on manifolds. Similarly, the described in Example 1.5 correspondence "$\mu \to T^\mu$" is not a bijection between the natural transformations of product preserving bundle functors on manifolds and the product preserving bundle functors on fibered manifolds.

It remains to discuss relationships between natural transformations of bundle functors on fibered manifolds and commutative diagrams of natural transformations of bundle functors on manifolds. To all be clear it is sufficient to prove the following theorem.

2.2. Theorem. *Let $F, \overline{F} : \mathcal{FM} \to \mathcal{FM}$ be two product preserving bundle functors. Let $\mu^F : G^F \to H^F$ and $\mu^{\overline{F}} : G^{\overline{F}} \to H^{\overline{F}}$ be the corresponding natural transformations as in Example 1.2. Let (ν, ρ) be a pair of natural transformations $\nu = \{\nu_M\} : G^F \to G^{\overline{F}}$ and $\rho = \{\rho_M\} : H^F \to H^{\overline{F}}$ such that the diagram*

$$\begin{array}{ccc} G^F(M) & \xrightarrow{\nu_M} & G^{\overline{F}}(M) \\ {\scriptstyle \mu^F_M}\downarrow & & \downarrow{\scriptstyle \mu^{\overline{F}}_M} \\ H^F(M) & \xrightarrow{\rho_M} & H^{\overline{F}}(M) \end{array} \tag{2.2}$$

is commutative for any manifold M. Then $\eta = \{\eta_\pi\} : F \to \overline{F}$ given by the compositions

$$\eta_\pi : F(\pi) \xrightarrow{\kappa_\pi} T^{\mu^F}(\pi) \xrightarrow{\tilde{(\nu,\rho)}_\pi} T^{\mu^{\overline{F}}}(\pi) \xrightarrow{\kappa^{-1}_\pi} \overline{F}(\pi) \tag{2.3}$$

for any fibered manifold π (see (1.21) and (1.23)) is the unique natural transformation $F \to \overline{F}$ such that $\nu^\eta = \nu$ and $\rho^\eta = \rho$, where $\nu^\eta : G^F \to G^{\overline{F}}$ and $\rho^\eta : H^F \to H^{\overline{F}}$ are corresponding to η as in Example 1.2.

Proof. It is sufficient to verify that $\rho^\eta_M = \rho_M$ and $\nu^\eta_M = \nu_M$ for any manifold M. Since the bundle functors $G^F, H^F, G^{\overline{F}}, H^{\overline{F}}$ are product preserving we can assume that $M = \mathbf{R}$. So, it remains to prove that $\eta_{id_{\mathbf{R}}} = \nu_{\mathbf{R}}$ and $\eta_{pt_{\mathbf{R}}} = \rho_{\mathbf{R}}$.

If $a \in F(id_{\mathbf{R}})$, then

$$\begin{aligned}
\eta_{id_{\mathbf{R}}}(a) &= (\Theta^{-1}_{id_{\mathbf{R}}} \circ (\nu \times_{\mu,\overline{\mu}} \rho)_{id_{\mathbf{R}}} \circ \Theta_{id_{\mathbf{R}}})(a) && \text{(by (1.23) and (1.21))}\\
&= (\Theta^{-1}_{id_{\mathbf{R}}} \circ (\nu \times_{\mu,\overline{\mu}} \rho)_{id_{\mathbf{R}}})(a, F(id_{\mathbf{R}})(a)) && \text{(by (2.1))}\\
&= \Theta^{-1}_{id_{\mathbf{R}}}(\nu_{\mathbf{R}}(a), \rho_{\mathbf{R}}(F(id_{\mathbf{R}})(a))) && \text{(by (1.4))}\\
&= \Theta^{-1}_{id_{\mathbf{R}}}(\nu_{\mathbf{R}}(a), \overline{F}(id_{\mathbf{R}})(\nu_{\mathbf{R}}(a))) && \text{(by (2.2) as } F(id_{\mathbf{R}}) = \mu^F_{\mathbf{R}})\\
&= \nu_{\mathbf{R}}(a) && \text{(by (2.1))}\ ,
\end{aligned}$$

i.e. $\eta_{id_{\mathbf{R}}} = \nu_{\mathbf{R}}$. Similarly, $\eta_{pt_{\mathbf{R}}} = \rho_{\mathbf{R}}$.

Now we prove the uniqueness part of the theorem. Suppose that $\tilde{\eta} : F \to \overline{F}$ is another natural transformation such that $\nu^{\tilde{\eta}} = \nu$ and $\rho^{\tilde{\eta}} = \rho$. Then (in particular) $\tilde{\eta}_{id_{\mathbf{R}}} = \nu_{\mathbf{R}} = \eta_{id_{\mathbf{R}}}$ and $\tilde{\eta}_{pt_{\mathbf{R}}} = \rho_{\mathbf{R}} = \eta_{pt_{\mathbf{R}}}$. Hence $\tilde{\eta}_\pi = \eta_\pi$ for any fibered manifold π (because of the same reasons as in the proof of the part (2) of Theorem 2.1). □

Remark. Theorem 2.2 shows that given two product preserving bundle functors $F, \overline{F}$ on fibered manifolds the described in Example 1.2 correspondence "$\eta \to (\nu^\eta, \rho^\eta)$" is a bijection between the natural transformations $F \to \overline{F}$ and the pairs (ν, ρ) of natural transformations $\nu : G^F \to G^{\overline{F}}$ and $\rho : H^F \to H^{\overline{F}}$ such that the diagram (1.3) (with $\mu^F : G^F \to H^F$ and $\mu^{\overline{F}} : G^{\overline{F}} \to H^{\overline{F}}$ instead of $\mu : G \to H$ and $\overline{\mu} : \overline{G} \to \overline{H}$ respectively) is commutative for any manifold M.

To present a corollary of Theorem 2.1 we need a preparation.

We say that two bundle functors $F, \overline{F}$ on fibered manifolds are *equivalent* if there exists a natural equivalence $F \to \overline{F}$.

We say that two natural transformations $\mu : G \to H$ and $\overline{\mu} : \overline{G} \to \overline{H}$ between bundle functors on manifolds are *equivalent* if there exist two natural equivalences $\nu = \{\nu_M\} : G \to \overline{G}$ and $\rho : H \to \overline{H}$ such that the diagram (1.3) is commutative for any manifold M.

2.3. Corollary. *The described in Example 1.2 correspondence "$F \to \mu^F$" induces a bijection between the equivalence classes of product preserving bundle functors on fibered manifolds and the equivalence classes of natural transformations of product preserving bundle functors on manifolds. The inverse bijection is induced by the described in Example 1.5 correspondence "$\mu \to T^\mu$".*

Proof. The correspondence "$[F] \to [\mu^F]$" is well-defined. For, if $\eta : F \to \overline{F}$ is a natural equivalence, then so are the defined in Example 1.2 natural transformations ν^η, ρ^η.

The correspondence "$[\mu] \to [T^\mu]$" is well-defined. For, if μ is equivalent to $\overline{\mu}$ and a pair (ν, ρ) realizes this equivalence, then the described in Example 1.5 natural transformation $\widetilde{(\nu, \rho)} : T^\mu \to T^{\overline{\mu}}$ is a natural equivalence.

From Theorem 2.1 (1) it follows that $[F] = [T^{\mu^F}]$ if F is product preserving. From Theorem 2.1 (3) it follows that $[\mu] = [\mu^F]$ if $F = T^\mu$. □

We end this paper with the following application of Corollary 2.3.

We say that a bundle functor $F : \mathcal{FM} \to \mathcal{FM}$ has *the manifold property* if for every manifold M the manifold $F(id_M)$ is diffeomorphic to M.

For example, all vertical Weil bundle functors $V_A : \mathcal{FM} \to \mathcal{FM}$, cf. [3], are product preserving bundle functors with the manifold property.

Conversely, if $F : \mathcal{FM} \to \mathcal{FM}$ is a product preserving bundle functor with the manifold property and $\mu^F : G^F \to H^F$ is the corresponding natural transformation as in Example 1.2, then G^F is naturally equivalent to the defined by (1.5) functor $i_{(1)} : \mathcal{M} \to \mathcal{FM}$ and H^F is (by the well-known description of product preserving bundle functors on manifolds, cf. [1],[4],[2]) naturally equivalent to the Weil functor $T_A : \mathcal{M} \to \mathcal{FM}$ for Weil algebra $A = H^F(\mathbf{R})$. Consequently, $\mu^F : G^F \to H^F$ is equivalent to the unique natural transformation $i_{(1)} \to T_A$. Similarly, $\mu^{V_A} : G^{V_A} \to H^{V_A}$ is equivalent to $i_{(1)} \to T_A$. Hence $[\mu^F] = [\mu^{V_A}]$. Therefore $[F] = [V_A]$.

Thus we have proved the following corollary.

2.4. Corollary. *Let $F : \mathcal{FM} \to \mathcal{FM}$ be a product preserving bundle functor. Then the following conditions are equivalent:*

(i) F has the manifold property;

(ii) F is naturally equivalent with a vertical Weil functor $V_A : \mathcal{FM} \to \mathcal{FM}$.

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