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PRODUCT PRESERVING FUNCTORS OF INFINITE DIMENSIONAL MANIFOLDS

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To Ivan Kolář, on the occasion of his 60th birthday.

Abstract. The theory of product preserving functors and Weil functors is partly extended to infinite dimensional manifolds, using the theory of C^∞-algebras.

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1. Introduction

In competition to the theory of jets of Ehresmann André Weil in [21] explained a construction which is on the one hand more restricted than that of jets since it allows only for covariant constructions in the sense of category theory (contravariant in the sense of differential geometry), but is more flexible since it uses more input: a finite dimensional formally real algebra. Later it was realized that Weil's construction describes all product preserving bundle functors on the category of finite dimension manifolds. This was developed independently by [1], [4], less completely by [13], and exposed in detail in chapter VIII of [6]. A jet-like approach to Weil's construction was given in [18] and used by Kolář in [5] to discuss natural transformations. The purpose of this paper is to present some versions of this theory in the realm of infinite dimensional manifolds, in the setting of convenient calculus as in [2], [10], [8].

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2. Infinite dimensional manifolds

2.1. Calculus in infinite dimensions. A locally convex vector space E is called *convenient* if for each smooth curve the Riemann integrals over compact intervals exist (this is a weak completeness condition). The final topology on E with respect to all smooth curves is called the c^∞-topology. A mapping between (c^∞-) open subsets of convenient vector spaces is called smooth if it maps smooth curves to smooth curves. Multilinear mappings are smooth if and only if they are bounded. This gives a meaningful theory which up to Fréchet spaces coincides with any reasonable theory of smooth mappings. The main additional property is cartesian closedness: If U, V are c^∞-open subsets in and if G is a convenient vector space, then $C^\infty(V,G)$ is again a convenient vector space (with the locally convex topology of convergence of compositions with smooth curves in V, uniformly on compact intervals, in all derivatives separately), and we have

$$C^\infty(U, C^\infty(V,G)) \cong C^\infty(U \times V, G).$$

Expositions of this theory can be found in [2], [10], [8], e.g. Real analytic and holomorphic versions of this theory are also available, [11], [7], [10].

2.2. Manifolds. A chart (U,u) on a set M is a bijection $u : U \to u(U) \subseteq E_U$ from a subset $U \subseteq M$ onto a c^∞-open subset of a convenient vector space E_U.

For two charts (U_α, u_α) and (U_β, u_β) on M the mapping $u_{\alpha\beta} := u_\alpha \circ u_\beta^{-1} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ for α, $\beta \in A$ is called the *chart changing*, where $U_{\alpha\beta} := U_\alpha \cap U_\beta$. A family $(U_\alpha, u_\alpha)_{\alpha\in A}$ of charts on M is called an *atlas* for M, if the U_α form a cover of M and all chart changings $u_{\alpha\beta}$ are defined on c^∞-open subsets.

An atlas $(U_\alpha, u_\alpha)_{\alpha\in A}$ for M is said to be a C^∞-*atlas*, if all chart changings $u_{\alpha\beta} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ are smooth. Two C^∞-atlases are called C^∞-*equivalent*, if their union is again a C^∞-atlas for M. An equivalence class of C^∞-atlases is called a C^∞-*structure* on M. The union of all atlases in an equivalence class is again an atlas, the *maximal* atlas for this C^∞-structure. A C^∞-*manifold* M is a set together with a C^∞-structure on it.

2.3. A mapping $f : M \to N$ between manifolds is called smooth if for each $x \in M$ and each chart (V,v) on N with $f(x) \in V$ there is a chart (U,u) on M with $x \in U$, $f(U) \subseteq V$, such that $v \circ f \circ u^{-1}$ is smooth. This is the case if and only if $f \circ c$ is smooth for each smooth curve $c : \mathbb{R} \to M$.

We will denote by $C^\infty(M,N)$ the space of all C^∞-mappings from M to N.

2.4. The topology of a manifold. The *natural topology* on a manifold M is the identification topology with respect to some (smooth) atlas $(u_\alpha : M \supseteq U_\alpha \to u_\alpha(U_\alpha) \subseteq E_\alpha)$, where a subset $W \subseteq M$ is open if and only if $u_\alpha(U_\alpha \cap W)$ is c^∞-open in E_α for all α. This topology depends only on the structure, since diffeomorphisms are homeomorphisms for the c^∞-topologies. It is also the final topology with respect to all inverses of chart mappings in one atlas. It is also the final topology with respect to all smooth curves. For a (smooth) manifold we will require certain properties for the natural topology, which will be specified when needed, like:

(1) *Smoothly Hausdorff*: The smooth functions in $C^\infty(M,\mathbb{R})$ separate points in M.
(2) *Smoothly regular*: For each neighborhood U of a point $x \in M$ there exists a smooth function $f: M \to \mathbb{R}$ with $f(x) = 1$ and carrier $f^{-1}(\mathbb{R} \setminus \{0\})$ contained in U; equivalently the initial topology with respect to $C^\infty(M,\mathbb{R})$ equals the natural topology.
(3) *Smoothly real compact*: Any bounded algebra homomorphism $C^\infty(M,\mathbb{R}) \to \mathbb{R}$ is given by a point evaluation; equivalently, the natural mapping $M \to \operatorname{Hom}(C^\infty(M,\mathbb{R}),\mathbb{R})$ is surjective.

2.5. Submanifolds. A subset N of a manifold M is called a *submanifold*, if for each $x \in N$ there is a chart (U,u) of M such that $u(U \cap N) = u(U) \cap F_U$, where F_U is a c^∞-closed linear subspace of the convenient model space E_U. Then clearly N is itself a manifold with $(U \cap N, u \restriction U \cap N)$ as charts, where (U,u) runs through all *submanifold charts* as above.

A submanifold N of M is called a *splitting submanifold* if there is a cover of N by submanifold charts (U,u) as above such that the $F_U \subset E_U$ are complemented (i.e. splitting) linear subspaces. Then obviously every submanifold chart is splitting.

2.6. Products. Let M and N be smooth manifolds described by smooth atlases $(U_\alpha, u_\alpha)_{\alpha\in A}$ and $(V_\beta, v_\beta)_{\beta\in B}$, respectively. Then the family $(U_\alpha \times V_\beta, u_\alpha \times v_\beta : U_\alpha \times V_\beta \to E_\alpha \times F_\beta)_{(\alpha,\beta)\in A\times B}$ is a smooth atlas for the cartesian product $M \times N$. Beware, however, the manifold topology of $M \times N$ may be finer than the product topology, see [10]. If M and N are metrizable, then it is the product topology, by [10] again. Clearly the projections

$$M \xleftarrow{pr_1} M \times N \xrightarrow{pr_2} N$$

are also smooth. The *product* $(M \times N, pr_1, pr_2)$ has the following universal property:

For any smooth manifold P and smooth mappings $f : P \to M$ and $g : P \to N$ the mapping $(f,g) : P \to M \times N$, $(f,g)(x) = (f(x), g(x))$, is the unique smooth mapping with $pr_1 \circ (f,g) = f$, $pr_2 \circ (f,g) = g$.

2.7. Lemma. [10] *For a convenient vector space E and any smooth manifold M the set $C^\infty(M,E)$ of smooth E-valued functions on M is again a convenient vector space with the locally convex topology of uniform convergence on compact subsets of compositions with smooth curves in M, in all derivatives separately. Moreover, with this structure, for two manifolds M, N, the exponential law holds:*

$$C^\infty(M, C^\infty(N,E)) \cong C^\infty(M \times N, E).$$

3. Weil functors on infinite dimensional manifolds

3.1. A real commutative algebra A with unit 1 is called *formally real* if for any $a_1, \dots, a_n \in A$ the element $1 + a_1^2 + \dots + a_n^2$ is invertible in A. Let $E = \{e \in A : e^2 = e, e \neq 0\} \subset A$ be the set of all nonzero idempotent elements in A. It is not empty since $1 \in E$. An idempotent $e \in E$ is said to be *minimal* if for any $e' \in E$ we have $ee' = e$ or $ee' = 0$.

Lemma. *Let A be a real commutative algebra with unit which is formally real and finite dimensional as a real vector space.*

Then there is a decomposition $1 = e_1 + \dots + e_k$ into all minimal idempotents. Furthermore $A = A_1 \oplus \dots \oplus A_k$, where $A_i = e_i A = \mathbb{R} \cdot e_i \oplus N_i$, and N_i is a nilpotent ideal.

This is standard, see [6], 35.1, for a proof.

3.2. Definition. A *Weil algebra* A is a real commutative algebra with unit which is of the form $A = \mathbb{R} \cdot 1 \oplus N$, where N is a finite dimensional ideal of nilpotent elements.

So by lemma 3.1 a formally real and finite dimensional unital commutative algebra is the direct sum of finitely many Weil algebras.

3.3. Chart description of Weil functors. Let $A = \mathbb{R} \cdot 1 \oplus N$ be a Weil algebra. We want to associate to it a functor $T_A : \mathcal{M}f \to \mathcal{M}f$ from the category $\mathcal{M}f$ of all smooth manifolds modelled on convenient vector spaces into itself.

Step 1. If $f \in C^\infty(\mathbb{R}, \mathbb{R})$ and $\lambda 1 + n \in \mathbb{R} \cdot 1 \oplus N = A$, we consider the Taylor expansion $j^\infty f(\lambda)(t) = \sum_{j=0}^\infty \frac{f^{(j)}(\lambda)}{j!} t^j$ of f at λ and we put

$$T_A(f)(\lambda 1 + n) := f(\lambda) 1 + \sum_{j=1}^\infty \frac{f^{(j)}(\lambda)}{j!} n^j,$$

which is finite sum, since n is nilpotent. Then $T_A(f) : A \to A$ is smooth and we get $T_A(f \circ g) = T_A(f) \circ T_A(g)$ and $T_A(\mathrm{Id}_\mathbb{R}) = \mathrm{Id}_A$.

Step 2. If $f \in C^\infty(\mathbb{R}, F)$ for a convenient vector space F and $\lambda 1 + n \in \mathbb{R} \cdot 1 \oplus N = A$, we consider the Taylor expansion $j^\infty f(\lambda)(t) = \sum_{j=0}^\infty \frac{f^{(j)}(\lambda)}{j!} t^j$ of f at λ and we put

$$T_A(f)(\lambda 1 + n) := 1 \otimes f(\lambda) + \sum_{j=1}^\infty n^j \otimes \frac{f^{(j)}(\lambda)}{j!},$$

which is finite sum, since n is nilpotent. Then $T_A(f) : A \to A \otimes F =: T_A F$ is smooth.

Step 3. For $f \in C^\infty(E, F)$, where E, F are convenient vector spaces, we want to define the value of $T_A(f)$ at an element of the convenient vector space $T_A E = A \otimes E$. Such an element may be uniquely written as $1 \otimes x_1 + \sum_j n_j \otimes x_j$, where 1 and the $n_j \in N$ form a fixed finite linear basis of A, and where the $x_i \in E$. Let again $j^\infty f(x_1)(y) = \sum_{k\ge 0} \frac{1}{k!} d^k f(x_1)(y^k)$ be the Taylor expansion of f at $x_1 \in E$ for $y \in E$. Then we put

$$\begin{aligned} T_A(f)(1\otimes x_1 + \sum_j n_j \otimes x_j) :=& \\ &= 1\otimes f(x_1) + \sum_{k\ge 0} \frac{1}{k!} \sum_{j_1,\dots,j_k} n_{j_1}\dots n_{j_k} \otimes d^k f(x_1)(x_{j_1},\dots,x_{j_k}) \end{aligned}$$

which is again a finite sum. A change of basis in N induces the transposed change in the x_i, namely $\sum_i(\sum_j a_i^j n_j)\otimes \bar x_i = \sum_j n_j \otimes (\sum_i a_i^j \bar x_i)$, so the value of $T_A(f)$ is independent of the choice of the basis of N. Since the Taylor expansion of a composition is the composition of the Taylor expansions we have $T_A(f\circ g) = T_A(f)\circ T_A(g)$ and $T_A(\mathrm{Id}_E) = \mathrm{Id}_{T_A E}$.

If $\varphi : A \to B$ is a homomorphism between two Weil algebras we have $(\varphi\otimes F)\circ T_A f = T_B f \circ (\varphi \otimes E)$ for $f \in C^\infty(E,F)$.

Step 4. Let $\pi = \pi_A : A \to A/N = \mathbb{R}$ be the projection onto the quotient field of the Weil algebra A. This is a surjective algebra homomorphism, so by step 3 the following diagram commutes for $f \in C^\infty(E,F)$:

$$\begin{tikzcd} A\otimes E \arrow[r, "T_A f"] \arrow[d, "\pi\otimes E"'] & A\otimes F \arrow[d, "\pi\otimes F"] \\ E \arrow[r, "f"] & F \end{tikzcd}$$

If $U \subset E$ is a c^∞-open subset we put $T_A(U) := (\pi\otimes E)^{-1}(U) = (1\otimes U)\times(N\otimes E)$, which is an c^∞-open subset in $T_A(E) := A\otimes E$. If $f : U \to V$ is a smooth mapping between c^∞-open subsets U and V of E and F, respectively, then the construction of step 3, applied to the Taylor expansion of f at points in U, produces a smooth mapping $T_A f : T_A U \to T_A V$, which fits into the following commutative diagram:

$$\begin{tikzcd} U\times(N\otimes E) \arrow[r, equal] \arrow[dr, "pr_1"'] & T_A U \arrow[r, "T_A f"] \arrow[d, "\pi\otimes E"] & T_A V \arrow[d, "\pi\otimes F"'] & V\times(N\otimes F) \arrow[l, equal] \arrow[dl, "pr_1"] \\ & U \arrow[r, "f"] & V & \end{tikzcd}$$

We have $T_A(f\circ g) = T_A f \circ T_A g$ and $T_A(\mathrm{Id}_U) = \mathrm{Id}_{T_A U}$, so T_A is now a covariant functor on the category of c^∞-open subsets of convenient vector spaces and smooth mappings between them.

Step 5. Let M be a smooth manifold, let $(U_\alpha, u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subset E_\alpha)$ be a smooth atlas of M with chart changings $u_{\alpha\beta} := u_\alpha \circ u_\beta^{-1} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$. Then the smooth mappings

$$\begin{tikzcd} T_A(u_\beta(U_{\alpha\beta})) \arrow[r, "T_A(u_{\alpha\beta})"] \arrow[d, "\pi\otimes E_\beta"'] & T_A(u_\alpha(U_{\alpha\beta})) \arrow[d, "\pi\otimes E_\alpha"] \\ u_\beta(U_{\alpha\beta}) \arrow[r, "u_{\alpha\beta}"] & u_\alpha(U_{\alpha\beta}) \end{tikzcd}$$

form again a cocycle of chart changings and we may use them to glue the c^∞-open sets $T_A(u_\alpha(U_\alpha)) = u_\alpha(U_\alpha) \times (N \otimes E_\alpha) \subset T_A E_\alpha$ in order to obtain a smooth manifold which we denote by $T_A M$. By the diagram above we see that $T_A M$ will be the total space of a fiber bundle $T(\pi_A, M) = \pi_{A,M} : T_A M \to M$, since the atlas $(T_A(U_\alpha), T_A(u_\alpha))$ constructed just now is already a fiber bundle atlas. So if M is Hausdorff then also $T_A M$ is Hausdorff, since two points x_i can be separated in one chart if they are in the same fiber, or they can be separated by inverse images under $\pi_{A,M}$ of open sets in M separating their projections.

This construction does not depend on the choice of the atlas. For two atlases have a common refinement and one may pass to this.

If $f \in C^\infty(M, M')$ for two manifolds M, M', we apply the functor T_A to the local representatives of f with respect to suitable atlases. This gives local representatives which fit together to form a smooth mapping $T_A f : T_A M \to T_A M'$. Clearly we again have $T_A(f \circ g) = T_A f \circ T_A g$ and $T_A(\mathrm{Id}_M) = \mathrm{Id}_{T_A M}$, so that $T_A : \mathcal{M}f \to \mathcal{M}f$ is a covariant functor.

3.4. Remark. If we apply the construction of 3.3, step 5 to the algebra $A = 0$, which we did not allow ($1 \neq 0 \in A$), then $T_0 M$ depends on the choice of the atlas. If each chart is connected, then $T_0 M = \pi_0(M)$, computing the connected components of M. If each chart meets each connected component of M, then $T_0 M$ is one point.

3.5. Theorem. Main properties of Weil functors. *Let $A = \mathbb{R} \cdot 1 \oplus N$ be a Weil algebra, where N is the maximal ideal of nilpotents. Then we have:*

1. The construction of 3.3 defines a covariant functor $T_A : \mathcal{M}f \to \mathcal{M}f$ such that $(T_A M, \pi_{A,M}, M)$ is a smooth fiber bundle with standard fiber $N \otimes E$ if M is modelled on the convenient space E. For any $f \in C^\infty(M, M')$ we have a commutative diagram

$$\begin{tikzcd} T_A M \arrow[r, "T_A f"] \arrow[d, "\pi_{A,M}"'] & T_A M' \arrow[d, "\pi_{A,M'}"] \\ M \arrow[r, "f"] & M'. \end{tikzcd}$$

So (T_A, π_A) is a bundle functor on $\mathcal{M}f$, which gives a vector bundle on $\mathcal{M}f$ if and only if N is nilpotent of order 2.

2. The functor $T_A : \mathcal{M}f \to \mathcal{M}f$ is multiplicative: it respects products. It maps the following classes of mappings into itself: immersions, splitting immersions, embeddings, splitting embeddings, closed embeddings, submersions, splitting submersions, surjective submersions, fiber bundle projections. It also respects transversal pullbacks. For fixed manifolds M and M' the mapping $T_A : C^\infty(M, M') \to C^\infty(T_A M, T_A M')$ is smooth, so it maps smoothly parametrized families into smoothly parametrized families.

3. If (U_α) is an open cover of M then $T_A(U_\alpha)$ is also an open cover of $T_A M$.

4. Any algebra homomorphism $\varphi : A \to B$ between Weil algebras induces a natural transformation $T(\varphi, \quad) = T_\varphi : T_A \to T_B$. If φ is injective, then $T(\varphi, M) : T_A M \to T_B M$ is a closed embedding for each manifold M. If φ is surjective, *then $T(\varphi, M)$ is a fiber bundle projection for each M. So we may view T as a co-covariant bifunctor from the category of Weil algebras times $\mathcal{M}f$ to $\mathcal{M}f$.*

Proof. 1. The main assertion is clear from 3.3. The fiber bundle $\pi_{A,M} : T_A M \to M$ is a vector bundle if and only if the transition functions $T_A(u_{\alpha\beta})$ are fiber linear $N \otimes E_\alpha \to N \otimes E_\beta$. So only the first derivatives of $u_{\alpha\beta}$ should act on N, so any product of two elements in N must be 0, thus N has to be nilpotent of order 2.

2. The functor T_A respects finite products in the category of c^∞-open subsets of convenient vector spaces by 3.3, step 3 and 5. All the other assertions follow by looking again at the chart structure of $T_A M$ and by taking into account that f is part of $T_A f$ (as the base mapping).

3. This is obvious from the chart structure.

4. We define $T(\varphi, E) := \varphi \otimes E : A \otimes E \to B \otimes E$. By 3.3, step 3, this restricts to a natural transformation $T_A \to T_B$ on the category of c^∞-open subsets of convenient vector spaces and by gluing also on the category $\mathcal{M}f$. Obviously T is a co-covariant bifunctor on the indicated categories. Since $\pi_B \circ \varphi = \pi_A$ (φ respects the identity), we have $T(\pi_B, M) \circ T(\varphi, M) = T(\pi_A, M)$, so $T(\varphi, M) : T_A M \to T_B M$ is fiber respecting for each manifold M. In each fiber chart it is a linear mapping on the typical fiber $N_A \otimes E \to N_B \otimes E$.

So if φ is injective, $T(\varphi, M)$ is fiberwise injective and linear in each canonical fiber chart, so it is a closed embedding.

If φ is surjective, let $N_1 := \ker \varphi \subseteq N_A$, and let $V \subset N_A$ be a linear complement to N_1. Then if M is modeled on convenient vector spaces E_α and for the canonical charts we have the commutative diagram:

$$\begin{tikzcd}
T_A M \arrow[r, "T(\varphi, M)"] & T_B M \\
T_A(U_\alpha) \arrow[u] \arrow[r, "T(\varphi, U_\alpha)"] \arrow[d, "T_A(u_\alpha)"'] & T_B(U_\alpha) \arrow[u] \arrow[d, "T_B(u_\alpha)"'] \\
u_\alpha(U_\alpha) \times (N_A \otimes E_\alpha) \arrow[r, "\mathrm{Id} \times (\varphi | N_A \otimes E_\alpha)"] \arrow[d, equal] & u_\alpha(U_\alpha) \times (N_B \otimes E_\alpha) \arrow[d, equal] \\
u_\alpha(U_\alpha) \times (N_1 \otimes E_\alpha) \times (V \otimes E_\alpha) \arrow[r, "\mathrm{Id} \times 0 \times Iso"] & u_\alpha(U_\alpha) \times 0 \times (N_B \otimes E_\alpha)
\end{tikzcd}$$

So $T(\varphi, M)$ is a fiber bundle projection with standard fiber $E_\alpha \otimes \ker \varphi$. □

3.6. Theorem. *Let A and B be Weil algebras. Then we have:*

(1) *We get the algebra A back from the Weil functor T_A by $T_A(\mathbb{R}) = A$ with addition $+_A = T_A(+_\mathbb{R})$, multiplication $m_A = T_A(m_\mathbb{R})$ and scalar multiplication $m_t = T_A(m_t) : A \to A$.*

(2) *The natural transformations $T_A \to T_B$ correspond exactly to the algebra homomorphisms $A \to B$.*

Proof. (1) is obvious. (2) For a natural transformation $\varphi : T_A \to T_B$ its value $\varphi_{\mathbb{R}} : T_A(\mathbb{R}) = A \to T_B(\mathbb{R}) = B$ is an algebra homomorphisms. The inverse of this mapping is already described in theorem 3.5.4. □

3.7. Proposition. *For two manifolds M_1 and M_2, with M_2 smoothly real compact and smoothly regular, the mapping*

$$\begin{aligned} C^\infty(M_1, M_2) &\to \operatorname{Hom}(C^\infty(M_2,\mathbb{R}), C^\infty(M_1,\mathbb{R})) \\ f &\mapsto (f^* : g \mapsto g \circ f) \end{aligned}$$

is bijective.

Proof. Let $x_1 \in M_1$ and $\varphi \in \operatorname{Hom}(C^\infty(M_2,\mathbb{R}), C^\infty(M_1,\mathbb{R}))$. Then $ev_{x_1} \circ \varphi$ is in $\operatorname{Hom}(C^\infty(M_2,\mathbb{R}),\mathbb{R})$, so by 2.4 there is a $x_2 \in M_2$ such that $ev_{x_1} \circ \varphi = ev_{x_2}$ since M_2 is smoothly real compact, and x_2 is unique since M_2 is smoothly Hausdorff. If we write $x_2 = f(x_1)$, then $f : M_1 \to M_2$ and $\varphi(g) = g \circ f$ for all $g \in C^\infty(M_2,\mathbb{R})$. This implies that f is smooth, since M_2 is smoothly regular. □

3.8. Remark. If M is a smoothly real compact and smoothly regular manifold we consider the set $D_A(M) := \operatorname{Hom}(C^\infty(M,\mathbb{R}), A)$ of all bounded homomorphisms from the convenient algebra of smooth functions on M into a Weil algebra A. Obviously we have a natural mapping $T_A M \to D_A M$ which is given by $X \mapsto (f \mapsto T_A(f).X)$, using 3.5 and 3.6.

Let $\mathbb{D}$ be the algebra of Study numbers $\mathbb{R}.1 \oplus \mathbb{R}.\delta$ with $\delta^2 = 0$. Then $T_{\mathbb{D}} M = TM$, the tangent bundle, and $D_{\mathbb{D}}(M)$ is the smooth bundle of all operational tangent vectors, i.e. bounded derivations at a point x of the algebra of germs $C^\infty_x(M,\mathbb{R})$ see [10]. We want to point out that even on Hilbert spaces there exist derivations which are differential operators of order 2 and 3, respectively, see [10].

It would be nice if $D_A(M)$ were a smooth manifold, not only for $A = \mathbb{D}$. We do not know whether this is true. The obvious method of proof hits severe obstacles, which we now explain.

Let $A = \mathbb{R}.1 \oplus N$ for a nilpotent finite dimensional ideal N, let $\pi : A \to \mathbb{R}$ be the corresponding projection. Then for $\varphi \in D_A(M) = \operatorname{Hom}(C^\infty(M,\mathbb{R}), A)$ the character $\pi \circ \varphi = \mathrm{ev}_x$ for a unique $x \in M$, since M is smoothly real compact. Moreover $X := \varphi - \mathrm{ev}_x .1 : C^\infty(M,\mathbb{R}) \to N$ satisfies the *expansion property* at x:

$$X(fg) = X(f).g(x) + f(x).X(g) + X(f).X(g). \tag{1}$$

Conversely a bounded linear mapping $X : C^\infty(M,\mathbb{R}) \to N$ with property (1) is called an *expansion at x*. Clearly each expansion at x defines a bounded homomorphism φ with $\pi \circ \varphi = \mathrm{ev}_x$. So we view $D_A(M)_x$ as the set of all expansions at x. Note first that for an expansion $X \in D_A(M)_x$ the value $X(f)$ depends only on the germ of f at x: If $f|U = 0$ for a neighborhood U of x, choose a smooth function h with $h = 1$ off U and $h(x) = 0$. Then $h^k f = f$ and $X(f) = X(h^k f) = 0 + 0 + X(h^k)X(f) = \dots = X(h)^k X(f)$ which is 0 for k larger than the nilpotence index of N.

Suppose now that $M = U$ is a c^∞-open subset of a convenient vector space E. We can ask whether $D_A(U)_x$ is a smooth manifold. We have no proof of this. Let us sketch the difficulty. A natural way to prove that would be by induction on the nilpotence index of N. Let $N_0 := \{n \in N : n.N = 0\}$, which is an ideal in A. Consider the short exact sequence

$$0 \to N_0 \to N \xrightarrow{p} N/N_0 \to 0$$

and a linear section $s : N/N_0 \to N$. For $X : C^\infty(U, \mathbb{R}) \to N$ we consider $\bar{X} := p \circ X$ and $X_0 := X - s \circ \bar{X}$. Then X is an expansion at $x \in U$ if and only if

$\bar{X}$ is an expansion at x with values in N/N_0 and X_0 satisfies

$$X_0(fg) = X_0(f)g(x) + f(x)X_0(g) + s(\bar{X}(f)).s(\bar{X}(g)) - s(\bar{X}(f).\bar{X}(g)). \tag{2}$$

Note that (2) is an affine equation in X_0 for fixed $\bar{X}$. By induction the $\bar{X} \in D_{A/N_0}(U)_x$ form a smooth manifold, and the fiber over a fixed $\bar{X}$ consists of all $X = X_0 + s \circ \bar{X}$ with X_0 in the closed affine subspace described by (2), whose model vector space is the space of all derivations at x. If we were able to find a (local) section $D_{A/N_0}(U) \to D_A(U)$ and if these sections would fit together nicely we could then conclude that $D_A(U)$ were the total space of a smooth affine bundle over $D_{A/N_0}(U)$, so it would be smooth. But this translates to a lifting problem as follows: A homomorphism $C^\infty(U, \mathbb{R}) \to A/N_0$ has to be lifted in a 'natural way' to $C^\infty(U, \mathbb{R}) \to A$. But we know that in general $C^\infty(U, \mathbb{R})$ is not a free C^∞-algebra, see 4.4 for comparison.

3.9. The basic facts from the theory of Weil functors are completed by the following assertion.

Proposition. *Given two Weil algebras A and B, the composed functor $T_A \circ T_B$ is a Weil functor generated by the tensor product $A \otimes B$.*

Proof. For a convenient vector space E we have $T_A(T_B E) = A \otimes B \otimes E$ and this is compatible with the action of smooth mappings, by 3.3. □

Corollary. *There is a canonical natural equivalence $T_A \circ T_B \cong T_B \circ T_A$ generated by the exchange algebra isomorphism $A \otimes B \cong B \otimes A$.*

3.10. Weil functors and Lie groups. We shall use the notion of a regular infinite dimensional Lie group, modelled on convenient vector spaces, as laid out in [9], following the lead of Omori et. al. [20] and Milnor [15]. We just remark that they have unique smooth exponential mappings, and that no smooth Lie group is known which is not regular. We shall use the notation $\mu : G \times G \to G$ for the multiplication and $\nu : G \to G$ for the inversion. The tangent bundle TG of a regular Lie group G is again a Lie group, the semidirect product $\mathfrak{g} \ltimes G$ of G with its Lie algebra $\mathfrak{g}$.

Now let A be a Weil algebra and let T_A be its Weil functor. Then the space $T_A(G)$ is again a Lie group with multiplication $T_A(\mu)$ and inversion $T_A(\nu)$. By the properties 3.5 of the Weil functor T_A we have a surjective homomorphism $\pi_A : T_A G \to G$ of Lie groups. Following the analogy with the tangent bundle, for $a \in G$ we will denote its fiber over a by $(T_A)_a G \subset T_A G$, likewise for mappings. With this notation we have the following commutative diagram, where we assume that G is a regular Lie group:

$$\begin{tikzcd}
& \mathfrak{g} \otimes N \arrow[r] \arrow[d, equal] & \mathfrak{g} \otimes A \arrow[d, equal] & & \\
0 \arrow[r] & (T_A)_0 \mathfrak{g} \arrow[r] \arrow[d, "(T_A)_0 \exp^G"'] & T_A \mathfrak{g} \arrow[r] \arrow[d, "T_A \exp^G"'] & \mathfrak{g} \arrow[r] \arrow[d, "\exp^G"] & 0 \\
e \arrow[r] & (T_A)_e G \arrow[r] & T_A G \arrow[r, "\pi_A"] & G \arrow[r] & e
\end{tikzcd}$$

The structural mappings (Lie bracket, exponential mapping, evolution operator, adjoint action) are determined by multiplication and inversion. Thus their images under the Weil functor T_A are again the same structural mappings. But note that the canonical flip mappings have to be inserted like follows. So for example

$$\mathfrak{g} \otimes A \cong T_A \mathfrak{g} = T_A(T_e G) \xrightarrow{\kappa} T_e(T_A G)$$

is the Lie algebra of $T_A G$ and the Lie bracket is just $T_A([\ \ ,\ \ ])$. Since the bracket is bilinear, the description of 3.3 implies that $[X \otimes a, Y \otimes b]_{T_A \mathfrak{g}} = [X, Y]_{\mathfrak{g}} \otimes ab$. Also $T_A \exp^G = \exp^{T_A G}$. If $\exp^G$ is a diffeomorphism near 0, $(T_A)_0(\exp^G) : (T_A)_0 \mathfrak{g} \to (T_A)_e G$ is also a diffeomorphism near 0, since T_A is local. The natural transformation $0_G : G \to T_A G$ is a homomorphism which splits the bottom row of the diagram, so $T_A G$ is the semidirect product $(T_A)_0 \mathfrak{g} \ltimes G$ via the mapping $T_A \rho : (u, g) \mapsto T_A(\rho_g)(u)$. So from [9], theorem 5.5, we may conclude that $T_A G$ is again a regular Lie group, if G is regular. If $\omega^G : TG \to T_e G$ is the Maurer Cartan form of G (i.e. the left logarithmic derivative of Id_G) then

$$\kappa_0 \circ T_A \omega^G \circ \kappa : T T_A G \cong T_A T G \to T_A T_e G \cong T_e T_A G$$

is the Maurer Cartan form of $T_A G$.

4. Product preserving functors from finite dimensional manifolds to infinite dimensional ones

4.1. Product preserving functors. Let $\mathcal{M}f_{\text{fin}}$ denote the category of all finite dimensional separable Hausdorff smooth manifolds, with smooth mappings as morphisms. Let $F : \mathcal{M}f_{\text{fin}} \to \mathcal{M}f$ be a functor which preserves products in the following sense: The diagram

$$F(M_1) \xleftarrow{F(pr_1)} F(M_1 \times M_2) \xrightarrow{F(pr_2)} F(M_2)$$

is always a product diagram.

Then $F(\text{point}) = \text{point}$, by the following argument:

$$\begin{tikzcd}
F(\text{point}) & F(\text{point}\times\text{point}) \arrow[l, "F(pr_1)"', "\cong"] \arrow[r, "F(pr_2)", "\cong"'] & F(\text{point}) \\
& \text{point} \arrow[ul, "f_1"] \arrow[u, "f"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

Each of f_1, f, and f_2 determines each other uniquely, thus there is only one mapping $f_1 : \text{point} \to F(\text{point})$, so the space $F(\text{point})$ is single pointed.

We also require that F has the following two properties:

(1) The map on morphisms $F : C^\infty(\mathbb{R}^n, \mathbb{R}) \to C^\infty(F(\mathbb{R}^n), F(\mathbb{R}))$ is smooth, where we regard $C^\infty(F(\mathbb{R}^n), F(\mathbb{R}))$ as smooth space, see [2] or [10]. Equivalently the associated mapping $C^\infty(\mathbb{R}^n, \mathbb{R}) \times F(\mathbb{R}^n) \to F(\mathbb{R})$ is smooth.

(2) There is a natural transformation $\pi : F \to \mathrm{Id}$ such that for each M the mapping $\pi_M : F(M) \to M$ is a fiber bundle, and for an open submanifold $U \subset M$ the mapping $F(\text{incl}) : F(U) \to F(M)$ is a pullback.

4.2. C^∞-algebras. An $\mathbb{R}$-algebra is a commutative ring A with unit together with a ring homomorphism $\mathbb{R} \to A$. Then every map $p : \mathbb{R}^n \to \mathbb{R}^m$ which is given by an m-tuple of real polynomials $(p_1, \dots, p_m)$ can be interpreted as a mapping $A(p) : A^n \to A^m$ in such a way that projections, composition, and identity are preserved, by just evaluating each polynomial p_i on an n-tuple $(a_1, \dots, a_n) \in A^n$.

A C^∞-algebra A is a real algebra in which we can moreover interpret all smooth mappings $f : \mathbb{R}^n \to \mathbb{R}^m$. There is a corresponding map $A(f) : A^n \to A^m$, and again projections, composition, and the identity mapping are preserved.

More precisely, a C^∞-algebra A is a product preserving functor from the category C^∞ to the category of sets, where C^∞ has as objects all spaces $\mathbb{R}^n$, $n \geq 0$, and all smooth mappings between them as arrows. Morphisms between C^∞-algebras are then natural transformations: they correspond to those algebra homomorphisms which preserve the interpretation of smooth mappings.

Let us explain how one gets the algebra structure from this interpretation. Since A is product preserving, we have $A(\text{point}) = \text{point}$. All the laws for a commutative ring with unit can be formulated by commutative diagrams of mappings between products of the ring and the point. We do this for the ring $\mathbb{R}$ and apply the product preserving functor A to all these diagrams, so we get the laws for the commutative ring $A(\mathbb{R})$ with unit $A(1)$ with the exception of $A(0) \neq A(1)$ which we will check later for the case $A(\mathbb{R}) \neq \text{point}$. Addition is given by $A(+)$ and multiplication by $A(m)$. For $\lambda \in \mathbb{R}$ the mapping $A(m_\lambda) : A(\mathbb{R}) \to A(\mathbb{R})$ equals multiplication with the element $A(\lambda) \in A(\mathbb{R})$, since the following diagram commutes:

$$\begin{tikzcd}
A(\mathbb{R}) \arrow[d, "\cong"'] \arrow[dr, "A(m_\lambda)"] & & \\
A(\mathbb{R}) \times \text{point} \arrow[r, "\mathrm{Id} \times A(\lambda)"] \arrow[d, "\cong"'] & A(\mathbb{R}) \times A(\mathbb{R}) \arrow[r] \arrow[d, equal] & A(\mathbb{R}) \\
A(\mathbb{R} \times \text{point}) \arrow[r, "A(\mathrm{Id} \times \lambda)"] & A(\mathbb{R} \times \mathbb{R}) \arrow[ur, "A(m)"'] &
\end{tikzcd}$$

We may investigate now the difference between $A(\mathbb{R}) = $ point and $A(\mathbb{R}) \neq$ point. In the latter case for $\lambda \neq 0$ we have $A(\lambda) \neq A(0)$ since multiplication by $A(\lambda)$ equals $A(m_\lambda)$ which is a diffeomorphism for $\lambda \neq 0$ and factors over a one pointed space for $\lambda = 0$. So for $A(\mathbb{R}) \neq$ point which we assume from now on, the group homomorphism $\lambda \mapsto A(\lambda)$ from $\mathbb{R}$ into $A(\mathbb{R})$ is actually injective.

This definition of C^∞-algebras is due to Lawvere [12], for a thorough account see Moerdijk-Reyes [16], for a discussion from the point of view of functional analysis see [3]. In particular there on a C^∞-algebra A the natural topology is defined as the finest locally convex topology on A such that for all $a = (a_1, \dots, a_n) \in A^n$ the evaluation mappings $\varepsilon_a : C^\infty(\mathbb{R}^n, \mathbb{R}) \to A$ are continuous. In [3], 6.6 one finds a method to recognize C^∞-algebras among locally-m-convex algebras. In [14] one finds a characterization of the algebras of smooth functions on finite dimensional algebras among all C^∞-algebras.

4.3. Theorem. *Let $F : \mathcal{M}f_{\text{fin}} \to \mathcal{M}f$ be a product preserving functor. Then either $F(\mathbb{R})$ is a point or $F(\mathbb{R})$ is a C^∞-algebra. If $\varphi : F_1 \to F_2$ is a natural transformation between two such functors, then $\varphi_\mathbb{R} : F_1(\mathbb{R}) \to F_2(\mathbb{R})$ is an algebra homomorphism.*

If F has property (1) then the natural topology on $F(\mathbb{R})$ is finer than the given manifold topology and thus is Hausdorff if the latter is it.

If F has property (2) then $F(\mathbb{R})$ is a local algebra with an algebra homomorphism $\pi = \pi_\mathbb{R} : F(\mathbb{R}) \to \mathbb{R}$ whose kernel is the maximal ideal.

Proof. By definition F restricts to a product preserving functor from the category of all $\mathbb{R}^n$'s and smooth mappings between them, thus it is a C^∞-algebra.

If F has property (1) then for all $a = (a_1, \dots, a_n) \in F(\mathbb{R})^n$ the evaluation mappings are given by

$$\varepsilon_a = \operatorname{ev}_a \circ F : C^\infty(\mathbb{R}^n, \mathbb{R}) \to C^\infty(F(\mathbb{R})^n, F(\mathbb{R})) \to F(\mathbb{R})$$

and thus are even smooth.

If F has property (2) then obviously $\pi_\mathbb{R} = \pi : F(\mathbb{R}) \to \mathbb{R}$ is an algebra homomorphism. It remains to show that the kernel of π is the largest ideal. So if $a \in A$ has $\pi(a) \neq 0 \in \mathbb{R}$ then we have to show that a is invertible in A. Since the following diagram is a pullback,

$$\begin{array}{ccc} F(\mathbb{R} \setminus \{0\}) & \xrightarrow{F(i)} & F(\mathbb{R}) \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi}\downarrow \\ \mathbb{R} \setminus \{0\} & \xrightarrow{i} & \mathbb{R} \end{array}$$

we may assume that $a = F(i)(b)$ for a unique $b \in F(\mathbb{R} \setminus \{0\})$. But then $1/i : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is smooth, and $F(1/i)(b) = a^{-1}$, since $F(1/i)(b).a = F(1/i)(b).F(i)(b) = F(m)F(1/i, i)(b) = F(1)(b) = 1$, compare 4.2. $\square$

4.4. Examples. Let A be an augmented local C^∞-algebra with maximal ideal N. Then A is quotient of a free C^∞-algebra $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ of smooth functions on some large product $\mathbb{R}^\Lambda$, which depend globally only on finitely many coordinates, see [16] or [3]. So we have a short exact sequence

$$0 \to I \to C^\infty_{\text{fin}}(\mathbb{R}^\Lambda) \xrightarrow{\varphi} A \to 0.$$

Then I is contained in the codimension 1 maximal ideal $\varphi^{-1}(N)$, which is easily seen to be $\{f \in C^\infty_{\text{fin}}(\mathbb{R}^\lambda) : f(x_0) = 0\}$ for some $x_0 \in \mathbb{R}^\Lambda$. Then clearly φ factors over the quotient of germs at x_0. If A has Hausdorff natural topology, then φ even factors over the Taylor expansion mapping, by the argument in [3], 6.1, as follows. Namely, let $f \in C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ be infinitely flat at x_0. We shall show that f is in the closure of the set of all functions with germ 0 at x_0. Let $x_0 = 0$ without loss. Note first that f factors over some quotient $\mathbb{R}^\Lambda \to \mathbb{R}^N$, and we may replace $\mathbb{R}^\Lambda$ by $\mathbb{R}^N$ without loss. Define $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$,

$$g(x,y) = \begin{cases} 0 & \text{if } |x| \le |y|, \\ (1 - |y|/|x|)x & \text{if } |x| > |y|. \end{cases}$$

Since f is flat at 0, the mapping $y \mapsto (x \mapsto f_y(x) := f(g(x,y))$ is a continuous mapping $\mathbb{R}^N \to C^\infty(\mathbb{R}^N, \mathbb{R})$ with the property that $f_0 = f$ and f_y has germ 0 at 0 for all $y \neq 0$.

Thus the augmented local C^∞-algebras whose natural topology is Hausdorff are exactly the quotients of algebras of Taylor series at 0 of functions in $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$.

It seems that local implies augmented: one has to show that a C^∞-algebra which is a field is 1-dimensional. Is this true?

4.5. Chart description of functors induced by C^∞-algebras. Let $A = \mathbb{R}\cdot 1 \oplus N$ be an augmented local C^∞-algebra which carries a compatible convenient structure, i.e. A is a convenient vector space and each mapping $A : C^\infty(\mathbb{R}^n, \mathbb{R}^m) \to C^\infty(A^n, A^m)$ is a well defined smooth mapping. As in the proof of 4.3 one sees that the natural topology on A is then finer than the given convenient one, thus is Hausdorff. Let us call this an *augmented local convenient C^∞-algebra.*

We want to associate to A a functor $T_A : \mathcal{M}f_{\text{fin}} \to \mathcal{M}f$ from the category $\mathcal{M}f_{\text{fin}}$ of all finite dimensional separable smooth manifolds to the category of smooth manifolds modelled on convenient vector spaces.

Step 1. Let $\pi = \pi_A : A \to A/N = \mathbb{R}$ be the augmentation mapping. This is a surjective homomorphism of C^∞-algebras, so the following diagram commutes for $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$:

$$\begin{tikzcd} A^n \arrow[r, "T_A f"] \arrow[d, "\pi^n"'] & A^m \arrow[d, "\pi^m"] \\ \mathbb{R}^n \arrow[r, "f"] & \mathbb{R}^m \end{tikzcd}$$

If $U \subset \mathbb{R}^n$ is an open subset we put $T_A(U) := (\pi^n)^{-1}(U) = U \times N^n$, which is an open subset in $T_A(\mathbb{R}^n) := A^n$.

Step 2. Now suppose that $f:\mathbb{R}^n\to\mathbb{R}^m$ vanishes on some open set $V\subset\mathbb{R}^n$. We claim that then T_Af vanishes on the open set $T_A(V)=(\pi^n)^{-1}(V)$. To see this let $x\in V$, and choose a smooth function $g\in C^\infty(\mathbb{R}^n,\mathbb{R})$ with $g(x)=1$ and support in V. Then $g.f=0$ thus we have also $0=A(g.f)=A(m)\circ A(g,f)=A(g).A(f)$, where the last multiplication is pointwise diagonal multiplication between A and A^m. For $a\in A^n$ with $(\pi^n)(a)=x$ we get $\pi(A(g)(a))=g(\pi(a))=g(x)=1$, thus $A(g)(a)$ is invertible in the algebra A, and from $A(g)(a).A(f)(a)=0$ we may conclude that $A(f)(a)=0\in A^m$.

Step 3. Now let $f:U\to W$ be a smooth mapping between open sets $U\subseteq\mathbb{R}^n$ and $W\subseteq\mathbb{R}^m$. Then we can define $T_A(f):T_A(U)\to T_A(W)$ in the following way. For $x\in U$ let $f_x:\mathbb{R}^n\to\mathbb{R}^m$ be a smooth mapping which coincides with f in a neighborhood V of x in U. Then by step 2 the restriction of $A(f_x)$ to $T_A(V)$ does not depend on the choice of the extension f_x, and by a standard argument one can define uniquely a smooth mapping $T_A(f):T_A(U)\to T_A(V)$. Clearly this gives us an extension of the functor A from the category of all $\mathbb{R}^n$'s and smooth mappings into convenient vector spaces to a functor from open subsets of $\mathbb{R}^n$'s and smooth mappings into the category of c^∞-open (indeed open) subsets of convenient vector spaces.

Step 4. Let M be a smooth finite dimensional manifold, let $(U_\alpha,u_\alpha:U_\alpha\to u_\alpha(U_\alpha)\subset\mathbb{R}^m)$ be a smooth atlas of M with chart changings $u_{\alpha\beta}:=u_\alpha\circ u_\beta^{-1}:u_\beta(U_{\alpha\beta})\to u_\alpha(U_{\alpha\beta})$. Then by step 3 we get smooth mappings between c^∞-open subsets of convenient vector spaces

$$\begin{array}{ccc}
T_A(u_\beta(U_{\alpha\beta})) & \xrightarrow{T_A(u_{\alpha\beta})} & T_A(u_\alpha(U_{\alpha\beta}))\\
\Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi}\\
u_\beta(U_{\alpha\beta}) & \xrightarrow{u_{\alpha\beta}} & u_\alpha(U_{\alpha\beta})
\end{array}$$

form again a cocycle of chart changings and we may use them to glue the c^∞-open sets $T_A(u_\alpha(U_\alpha))=\pi_{\mathbb{R}^m}^{-1}(u_\alpha(U_\alpha))\subset A^m$ in order to obtain a smooth manifold which we denote by T_AM. By the diagram above we see that T_AM will be the total space of a fiber bundle $T(\pi_A,M)=\pi_{A,M}:T_AM\to M$, since the atlas $(T_A(U_\alpha),T_A(u_\alpha))$ constructed just now is already a fiber bundle atlas. So if M is Hausdorff then also T_AM is Hausdorff, since two points x_i can be separated in one chart if they are in the same fiber, or they can be separated by inverse images under $\pi_{A,M}$ of open sets in M separating their projections.

This construction does not depend on the choice of the atlas. For two atlases have a common refinement and one may pass to this.

If $f\in C^\infty(M,M')$ for two manifolds M, M', we apply the functor T_A to the local representatives of f with respect to suitable atlases. This gives local representatives which fit together to form a smooth mapping $T_Af:T_AM\to T_AM'$. Clearly we again have $T_A(f\circ g)=T_Af\circ T_Ag$ and $T_A(\mathrm{Id}_M)=\mathrm{Id}_{T_AM}$, so that $T_A:\mathcal{M}f\to\mathcal{M}f$ is a covariant functor.

4.6. Theorem. Main properties. *Let $A = \mathbb{R} \cdot 1 \oplus N$ be a local augmented convenient C^∞-algebra. Then we have:*

1. The construction of 4.5 defines a covariant functor $T_A : \mathcal{M}f_{\text{fin}} \to \mathcal{M}f$ such that $(T_A M, \pi_{A,M}, M)$ is a smooth fiber bundle with standard fiber N^m if $\dim M = m$. For any $f \in C^\infty(M, M')$ we have a commutative diagram

$$\begin{tikzcd} T_A M \arrow[r, "T_A f"] \arrow[d, "\pi_{A,M}"'] & T_A M' \arrow[d, "\pi_{A,M'}"] \\ M \arrow[r, "f"] & M'. \end{tikzcd}$$

2. The functor $T_A : \mathcal{M}f \to \mathcal{M}f$ is multiplicative: it respects products. It respects immersions, embeddings, etc, similarly as in 3.5. It also respects transversal pullbacks. For fixed manifolds M and M' the mapping $T_A : C^\infty(M, M') \to C^\infty(T_A M, T_A M')$ is smooth.

3. Any bounded algebra homomorphism $\varphi : A \to B$ between augmented convenient C^∞-algebras induces a natural transformation $T(\varphi, \quad) = T_\varphi : T_A \to T_B$. If φ is split injective, then $T(\varphi, M) : T_A M \to T_B M$ is a split embedding for each manifold M. If φ is split surjective, then $T(\varphi, M)$ is a fiber bundle projection for each M. So we may view T as a co-covariant bifunctor from the category of augmented convenient C^∞-algebras algebras times $\mathcal{M}f_{\text{fin}}$ to $\mathcal{M}f$.

Proof. 1. The main assertion is clear from 4.5. The fiber bundle $\pi_{A,M} : T_A M \to M$ is a vector bundle if and only if the transition functions $T_A(u_{\alpha\beta})$ are fiber linear $N \otimes E_\alpha \to N \otimes E_\beta$. So only the first derivatives of $u_{\alpha\beta}$ should act on N, so any product of two elements in N must be 0, thus N has to be nilpotent of order 2.

2. The functor T_A respects finite products in the category of c^∞-open subsets of convenient vector spaces by 3.3, step 3 and 5. All the other assertions follow by looking again at the chart structure of $T_A M$ and by taking into account that f is part of $T_A f$ (as the base mapping).

3. We define $T(\varphi, \mathbb{R}^n) := \varphi^n : A^n \to B^n$. By 4.5, step 3, this restricts to a natural transformation $T_A \to T_B$ on the category of open subsets of $\mathbb{R}^n$'s by gluing also on the category $\mathcal{M}f$. Obviously T is a co-covariant bifunctor on the indicated categories. Since $\pi_B \circ \varphi = \pi_A$ (φ respects the identity), we have $T(\pi_B, M) \circ T(\varphi, M) = T(\pi_A, M)$, so $T(\varphi, M) : T_A M \to T_B M$ is fiber respecting for each manifold M. In each fiber chart it is a linear mapping on the typical fiber $N_A^m \to N_B^m$.

So if φ is split injective, $T(\varphi, M)$ is fiberwise split injective and linear in each canonical fiber chart, so it is a split embedding.

If φ is split surjective, let $N_1 := \ker \varphi \subseteq N_A$, and let $V \subset N_A$ be a topological linear complement to N_1. Then for $m = \dim M$ and for the canonical charts we have the commutative diagram:

$$\begin{tikzcd}
T_A M \arrow[r, "T(\varphi, M)"] & T_B M \\
T_A(U_\alpha) \arrow[u] \arrow[r, "T(\varphi, U_\alpha)"] \arrow[d, "T_A(u_\alpha)"'] & T_B(U_\alpha) \arrow[u] \arrow[d, "T_B(u_\alpha)"'] \\
u_\alpha(U_\alpha) \times N_A^m \arrow[r, "\mathrm{Id} \times \varphi | N_A^m"] \arrow[d, equal] & u_\alpha(U_\alpha) \times N_B^m \arrow[d, equal] \\
u_\alpha(U_\alpha) \times N_1^m \times V^m \arrow[r, "\mathrm{Id} \times 0 \times Iso"] & u_\alpha(U_\alpha) \times 0 \times N_B^m
\end{tikzcd}$$

So $T(\varphi, M)$ is a fiber bundle projection with standard fiber $E_\alpha \otimes \ker \varphi$. □

4.7. Theorem. *Let A and B be augmented convenient C^∞-algebras. Then we have:*

(1) *We get the convenient C^∞-algebra A back from the functor T_A by restricting to the subcategory of $\mathbb{R}^n$'s.*
(2) *The natural transformations $T_A \to T_B$ correspond exactly to the bounded C^∞-algebra homomorphisms $A \to B$.*

Proof. (1) is obvious. (2) For a natural transformation $\varphi : T_A \to T_B$ (which is smooth) its value $\varphi_{\mathbb{R}} : T_A(\mathbb{R}) = A \to T_B(\mathbb{R}) = B$ is a C^∞-algebra homomorphism which is smooth and thus bounded. The inverse of this mapping is already described in theorem 4.6.4. □

4.8. Proposition. *Let $A = \mathbb{R}\cdot 1 \oplus N$ be a local augmented convenient C^∞-algebra and let M be a smooth finite dimensional manifold.*

Then there exists a bijection

$$\varepsilon : T_A(M) \to \operatorname{Hom}(C^\infty(M, \mathbb{R}), A)$$

onto the space of bounded algebra homomorphisms, which is natural in A and M. Via ε the expression $\operatorname{Hom}(C^\infty(\quad, \mathbb{R}), A)$ describes the functor T_A in a coordinate free manner.

Proof. Step 1. Let $M = \mathbb{R}^n$, so $T_A(\mathbb{R}^n) = A^n$. Then for $a = (a_1, \dots, a_n) \in A^n$ we have $\varepsilon(a)(f) = A(f)(a_1, \dots, a_n)$, which gives a bounded algebra homomorphism $C^\infty(\mathbb{R}^n, \mathbb{R}) \to A$. Conversely, for $\varphi \in \operatorname{Hom}(C^\infty(\mathbb{R}^n, \mathbb{R}), A)$ consider $a = (\varphi(\mathrm{pr}_1), \dots, \varphi(\mathrm{pr}_n)) \in A^n$. Since polynomials are dense in $C^\infty(\mathbb{R}^n, \mathbb{R})$, φ is bounded, and A is Hausdorff, φ is uniquely determined by its values on the coordinate functions pr_i (compare [3], 2.4.(3)), thus $\varphi(f) = A(f)(a)$ and ε is bijective. Obviously ε is natural in A and $\mathbb{R}^n$.

Step 2. Now let $i : U \subset \mathbb{R}^n$ be an embedding of an open subset. Then the image of the mapping

$$\operatorname{Hom}(C^\infty(U, \mathbb{R}), A) \xrightarrow{(i^*)^*} \operatorname{Hom}(C^\infty(\mathbb{R}^n, \mathbb{R}), A) \xrightarrow{\varepsilon^{-1}_{\mathbb{R}^n, A}} A^n$$

is the set $\pi^{-1}_{A,\mathbb{R}^n}(U) = T_A(U) \subset A^n$, and $(i^*)^*$ is injective.

To see this let $\varphi \in \operatorname{Hom}(C^\infty(U,\mathbb{R}), A)$. Then $\varphi^{-1}(N)$ is the maximal ideal in $C^\infty(U,\mathbb{R})$ consisting of all smooth functions vanishing at a point $x \in U$, and $x = \pi(\varepsilon^{-1}(\varphi \circ i^*)) = \pi(\varphi(\mathrm{pr}_1 \circ i), \dots, \varphi(\mathrm{pr}_n \circ i))$, so that $\varepsilon^{-1}((i^*)^*(\varphi)) \in T_A(U) = \pi^{-1}(U) \subset A^n$.

Conversely for $a \in T_A(U)$ the homomorphism $\varepsilon_a : C^\infty(\mathbb{R}^n, \mathbb{R}) \to A$ factors over $i^* : C^\infty(\mathbb{R}^n, \mathbb{R}) \to C^\infty(U, \mathbb{R})$, by steps 2 and 3 of 4.5.

Step 3. The two functors $\operatorname{Hom}(C^\infty(\quad, \mathbb{R}), A)$ and $T_A : \mathcal{M}f \to Set$ coincide on all open subsets of $\mathbb{R}^n$'s, so they have to coincide on all manifolds, since smooth manifolds are exactly the retracts of open subsets of $\mathbb{R}^n$'s see e.g. [6], 1.14.1. Alternatively one may check that the gluing process described in 4.5, step 4, works also for the functor $\operatorname{Hom}(C^\infty(\quad, \mathbb{R}), A)$ and gives a unique manifold structure on it which is compatible to $T_A M$. $\square$

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