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UNIFORMITY AND HOMOGENEITY OF ELASTIC RODS, SHELLS AND COSSERAT THREE-DIMENSIONAL BODIES

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To Ivan Kolář, on the occasion of his 60th birthday.

Abstract. We present a general geometrical theory of uniform bodies which includes three-dimensional Cosserat bodies, rods and shells as particular cases. Criteria of local homogeneity are given in terms on connections.

1. Introduction

An n-dimensional Cosserat medium $\mathcal{B}$ is represented by an n-dimensional manifold B which can be embedded into $\mathbb{R}^3$, the embedded submanifold endowed at each point with a deformable linearly independent basis of 3 vectors. The mechanical response is supposed to depend on the deformations of the underlying n-body as well as on the gradients of the attached deformable basis.

In this paper we present a general geometrical framework for arbitrary Cosserat bodies. The geometrical picture consists of an n-dimensional body $\mathcal{B}$ which is embedded into the Euclidean space R^{n+m}. The geometry of the embedding (the configuration) allows us to construct the principal bundle of linear frames of $\mathbb{R}^{n+m}$ along the embedded submanifold. Thus, a deformation is nothing but a principal bundle isomorphism of two of these configuration bundles. The constitutive equation states that the mechanical response depends on the 1-jet of the deformation. We associate to the body a groupoid of material 1-jets in such a way that the smooth uniformity is equivalent to this groupoid being a Lie groupoid. If the Cosserat body enjoys smooth global uniformity we construct a non-holonomic parallelism and, by prolongating it by means of the material symmetry group, a non-holonomic $\bar{G}$-structure. Its integrability (integrable prolongability, in fact) is equivalent to the local homogeneity of the body. Finally, we consider the case of rods, shells and three-dimensional Cosserat bodies as particular cases.

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Our theory is based on the original works of the Cosserat brothers [5] and is the natural extension of the theory of inhomogeneities developed by Noll and Wang [43, 50, 48, 49] (see also [37, 38, 39, 40, 45]). Our approach generalizes several previous papers on three-dimensional Cosserat bodies (including second grade materials) [46, 47, 6, 7, 14, 8, 15, 18, 10, 11, 12, 13, 41] and shells [19, 20, 21]. A general setting for continua with microstructure [2] was developed in [16, 17]. The homogeneity conditions are obtained as integrability conditions of non-holonomic parallelisms. It seems to us that non-holonomic $\bar{G}$-structures deserve a careful study in order to obtain nice homogeneity conditions for Cosserat bodies. So, the results discussed, for instance, in [32, 33, 34, 35, 36, 44, 52, 25, 29, 42, 4] should be extended to that case.

2. Bundle configurations

Let $\mathcal{B}$ be an n-dimensional manifold. Consider an embedding $\Phi : \mathcal{B} \longrightarrow \mathbb{R}^{n+m}$ of $\mathcal{B}$ into $\mathbb{R}^{n+m}$. Thus, $\Phi(\mathcal{B})$ is an n-dimensional embedded submanifold in $\mathbb{R}^{n+m}$. At every point X in $\Phi(\mathcal{B})$ we consider the set of linear frames $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+m}\}$ of $\mathbb{R}^{n+m}$ at X such that $\{e_1, \cdots, e_n\}$ is a basis of the tangent space $T_X(\Phi(\mathcal{B}))$. Consequently, $\{e_{n+1}, \cdots, e_{n+m}\}$ is a set of linearly independent tangent vectors in $T_X\mathbb{R}^{n+m}$ which are transverse to $\Phi(\mathcal{B})$. We denote by $\widetilde{\mathcal{F}B}_\Phi$ the collection of all these bases at all the points of $\Phi(\mathcal{B})$. We define a canonical projection $\pi_\Phi : \widetilde{\mathcal{F}B}_\Phi \longrightarrow \Phi(\mathcal{B})$ which maps a basis at X onto X.

Proposition 2.1. *$\widetilde{\mathcal{F}B}_\Phi$ is a principal subbundle of the restriction of the linear frame bundle $\mathcal{F}\mathbb{R}^{n+m}$ of $\mathbb{R}^{n+m}$ to $\Phi(\mathcal{B})$, and whose structural group is*

$$G_0 = \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \mid A \in Gl(n, \mathbb{R}), C \in Gl(m, \mathbb{R}), B \in \mathcal{M}(m, n) \right\} \subset Gl(n+m, \mathbb{R}) , \tag{1}$$

where $\mathcal{M}(m, n)$ is the real vector space of matrices of order $m \times n$. Φ will be called a ***configuration*** *and $\widetilde{\mathcal{F}B}_\Phi$ the* ***bundle configuration****.*

Given two configurations $\Phi_1, \Phi_2 : \mathcal{B} \longrightarrow \mathbb{R}^{n+m}$, we put $\kappa = \Phi_2 \circ \Phi_1^{-1}$.

Definition 2.1. *A* ***deformation*** *is a principal bundle isomorphism $\tilde{\kappa} : \widetilde{\mathcal{F}B}_{\Phi_1} \longrightarrow \widetilde{\mathcal{F}B}_{\Phi_2}$ between the corresponding bundle configurations which induces the identity map on the structure groups, and it covers κ.*

In other words, $\tilde{\kappa}$ maps a basis $\{Y_1, \cdots, Y_n, Y_{n+1}, \cdots, Y_{n+m}\}$ at $X \in \Phi_1(\mathcal{B})$ such that $\{Y_1, \cdots, Y_n\}$ is a basis of $T_X(\Phi_1(\mathcal{B}))$ and $\{Y_{n+1}, \cdots, Y_{n+m}\}$ are transversal to $\Phi_1(\mathcal{B})$, into a basis $\{\bar{Y}_1, \cdots, \bar{Y}_n, \bar{Y}_{n+1}, \cdots, \bar{Y}_{n+m}\}$ at $\kappa(X)$ of the same type, that is, $\{\bar{Y}_1, \cdots, \bar{Y}_n\}$ is a basis of $T_{\kappa(X)}(\Phi_2(\mathcal{B}))$ and $\{\bar{Y}_{n+1}, \cdots, \bar{Y}_{n+m}\}$ are transversal to $\Phi_2(\mathcal{B})$. With respect to these bases, $\tilde{\kappa}$ is given by a tensor whose associated matrix is as follows:

$$H = \begin{pmatrix} H_1 & 0 \\ H_2 & H_3 \end{pmatrix} . \tag{2}$$

where $H_1 \in Gl(n, \mathbb{R}), H_3 \in Gl(m, \mathbb{R}), H_2 \in \mathcal{M}(m, n)$.

We fix an embedding $\Phi_0 : \mathcal{B} \longrightarrow \mathbb{R}^{n+m}$ once and for all, and the corresponding bundle configuration $\widetilde{\mathcal{F}B}_{\Phi_0}$ will be denoted by $\mathcal{E}_0$, for brevity. We also put $\mathcal{B}_0 = \Phi_0(\mathcal{B})$.

We assume that the elastic response depends on the 1-jet of the deformation so that the constitutive equation reads as

$$W = W_0(j^1_X \tilde{\kappa}) \ , \tag{3}$$

with respect to the reference configuration Φ_0.

Definition 2.2. *$\mathcal{B}_0$ will be called a* ***deformable body****.*

3. Uniformity and material symmetries

Definition 3.1. *Given a deformable body $\mathcal{B}_0$ we say that it is* ***uniform*** *if for any two points X and Y in $\mathcal{B}_0$ there exists a local automorphism $\tilde{\Phi}$ of principal bundles of $\mathcal{E}_0$ from X to Y which induces the identity map between the structure groups and such that*

$$W_0(j^1_Y \tilde{\kappa} \circ j^1_X \tilde{\Phi}) = W_0(j^1_Y \tilde{\kappa}) \ , \tag{4}$$

for all 1-jet of deformation $j^1_Y \tilde{\kappa}$. We will call $j^1_X \tilde{\Phi}$ a ***material*** *1-jet.*

We denote by Φ the local diffeomorphism of $\mathcal{B}_0$ covered by $\tilde{\Phi}$.

Definition 3.2. *A* ***material symmetry*** *at a point $X \in \mathcal{B}_0$ is a 1-jet $j^1_X \tilde{\Phi}$ of a local automorphism $\tilde{\Phi}$ of principal bundles of $\mathcal{E}_0$ at X which induces the identity map between the structure groups and such that*

$$W_0(j^1_X \tilde{\kappa} \circ j^1_X \tilde{\Phi}) = W_0(j^1_X \tilde{\kappa}) \ , \tag{5}$$

for all 1-jet of deformation $j^1_Y \tilde{\kappa}$.

The following result follows immediatly from the above definitions.

Proposition 3.1. *(1) The collection $\Omega(\mathcal{B}_0)$ of all material 1-jets is a groupoid over $\mathcal{B}_0$ with source and target projections given by $\alpha(j^1_X \tilde{\Phi}) = X$ and $\beta(j^1_X \tilde{\Phi}) = \Phi(X)$, respectively.*

(2) The collection $G(X)$ of all material symmetries at a point $X \in \mathcal{B}_0$ has a structure of group. In fact, $G(X) = (\alpha, \beta)^{-1}(X, X)$, where $(\alpha, \beta) : \Omega(\mathcal{B}_0) \longrightarrow \mathcal{B}_0 \times \mathcal{B}_0$ is defined by $(\alpha, \beta)(j^1_X \tilde{\Phi}) = (X, \Phi(X))$.

Definition 3.3. *We say that $\mathcal{B}_0$ enjoys smooth uniformity if $\Omega(\mathcal{B}_0)$ is a Lie groupoid.*

In such a case, there exist local smooth uniformities (i.e., local sections of $(\alpha, \beta) : \Omega(\mathcal{B}_0) \longrightarrow \mathcal{B}_0 \times \mathcal{B}_0$). For the sake of simplicity we will assume, from now on, that $\mathcal{B}_0$ enjoys global smooth uniformity or, in other words, the Lie groupoid $\Omega(\mathcal{B}_0)$ is smoothly transitive.

Proposition 3.2. *Assume that $\mathcal{B}_0$ enjoys smooth uniformity and take a point $X_0 \in \mathcal{B}_0$. Then $\Omega_{X_0}(\mathcal{B}_0) = \alpha^{-1}(X_0)$ is a principal bundle over $\mathcal{B}_0$ with structure group $G(X_0)$ and canonical projection β.*

Proof: It follows the same lines that in Proposition 11.8 in [18]. □

4. Reference crystals and non-holonomic parallelisms

Consider the principal bundle $\mathcal{E}$ over $\mathbb{R}^n$ consisting of all the linear frames $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+m}\}$ at all the points of $\mathbb{R}^n$ such that $\{e_1, \cdots, e_n\}$ is a linear frame of $\mathbb{R}^n$. It is not hard to see that $\mathcal{E}$ is a trivial bundle, say $\mathcal{E} = \mathbb{R}^n \times G_0$.

Consider now the set $\bar{\mathcal{F}}\mathcal{E}_0$ of all 1-jets $j_0^1\tilde{\Psi}$ of local isomorphisms of principal bundles from $\mathcal{E}$ into $\mathcal{E}_0$ with source at the origin in $\mathbb{R}^n$, such that $\tilde{\Psi}$ induces the identity map between the structure groups.

It follows that $\bar{\mathcal{F}}\mathcal{E}_0$ is a principal bundle over $\mathcal{B}_0$ with canonical projection $\bar{\pi} : \bar{\mathcal{F}}\mathcal{E}_0 \longrightarrow \mathcal{B}_0$, $\bar{\pi}(j_0^1\tilde{\Psi}) = \Psi(0)$, where Ψ is the induced local diffeomorphism between the base manifolds covered by $\tilde{\Psi}$. We also have a canonical projection $\bar{\pi}_{1,0} : \bar{\mathcal{F}}\mathcal{E}_0 \longrightarrow \mathcal{E}_0$, given by $\bar{\pi}_{1,0}(j_0^1\tilde{\Psi}) = \tilde{\Psi}(0, 1)$, where $(0, 1)$ is the distinguished element of $\mathcal{E}$, i.e., $0 \in \mathbb{R}^n$ and 1 is the identity matrix in $Gl(n + m, \mathbb{R})$. (It should be noticed that the functor $\mathcal{F}$ coincides with the one previously defined by I. Kolář [29, 31].)

The element $j_0^1\tilde{\Psi}$ will be called a **non-holonomic frame** at the point $\Psi(0) \in \mathcal{B}_0$. The structure group $\bar{G}(n, n + m)$ of $\bar{\pi} : \bar{\mathcal{F}}\mathcal{E}_0 \longrightarrow \mathcal{B}_0$, consists of the 1-jets $j_0^1\tilde{\Psi}$ of local automorphisms of $\mathcal{E}$ which induces the identity map between the structure groups and with source and target at 0.

By a direct application of chain rule, we obtain that the structure group $\bar{G}(n, n+m)$ may be described as follows. A generic element of $\bar{G}(n, n + m)$ is a triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, where

$$\mathcal{A} \in G_0 \ , \ \mathcal{B} \in Gl(n, \mathbb{R}) \ , \ \mathcal{C} \in \ Lin\,(\mathbb{R}^n, \mathfrak{g}_0) \ ,$$

where $\mathfrak{g}_0$ is the Lie algebra of G_0.

We will write

$$\mathcal{A} = (\mathcal{A}_i^j) \ , \ \mathcal{B} = (\mathcal{B}_\alpha^\beta) \ , \ \mathcal{C} = (\mathcal{C}_{i\gamma}^j) \ ,$$

where Latin indices run from 1 to $n + m$, Greek indices run from 1 to n. For simplicity, we introduce new indices $a, b, c, \ldots$ running from 1 to m.

We have

$$\begin{aligned} &\mathcal{A}_\alpha^{n+b} = 0 \ , \text{ if } 1 \le \alpha \le n \ , \ 1 \le b \le m \ , \\ &\mathcal{C}_{\alpha\gamma}^b = 0 \ , \text{ if } 1 \le \alpha \le n \ , \ 1 \le b \le m \ , \end{aligned}$$

Proposition 4.1. *The group $\bar{G}(n, n + m)$ may be identified with the semidirect product $G_0 \times Gl(n, \mathbb{R}) \times Lin\,(\mathbb{R}^n, \mathfrak{g}_0)$, the multiplication group given by*

$$(\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1)(\mathcal{A}_2, \mathcal{B}_2, \mathcal{C}_2) = (\mathcal{A}, \mathcal{B}, \mathcal{C}) \ , \tag{6}$$

where

$$\begin{aligned}\mathcal{A}_i^j &= (\mathcal{A}_1)_i^k(\mathcal{A}_2)_k^j\ ,\\ \mathcal{B}_\alpha^\beta &= (\mathcal{B}_1)_\alpha^\gamma(\mathcal{B}_2)_\gamma^\beta\ ,\\ \mathcal{C}_{i\gamma}^j &= (\mathcal{A}_1)_i^k(\mathcal{B}_1)_\gamma^\beta(\mathcal{C}_2)_{k\beta}^j + (\mathcal{A}_2)_k^j(\mathcal{C}_1)_{i\gamma}^k\ .\end{aligned}$$

Proof: It follows from a direct computation using the chain rule. □

Remark 4.1. It should be noted that $\dim \bar{G}(n, n+m) = (n+1)[n^2+nm+m^2]+n^2$.

Definition 4.1. *The bundle $\bar{\mathcal{F}}\mathcal{E}_0$ will be called the* ***non-holonomic frame bundle*** *of $\mathcal{E}_0$. A global section $\bar{\mathcal{P}}$ of $\bar{\mathcal{F}}\mathcal{E}_0$ will be called a* ***non-holonomic parallelism*** *on $\mathcal{B}_0$. A non-holonomic frame at a point $X_0 \in \mathcal{B}_0$ will be called a* ***reference crystal*** *at that point.*

Suppose now that $\mathcal{B}_0$ enjoys smooth uniformity, and choose a crystal reference $\tilde{Z}_0 = j_0^1\tilde{\Psi}$ at a point X_0. Given a smooth global uniformity on $\mathcal{B}_0$, we can transport the reference crystal at any point X in $\mathcal{B}_0$ by composing the uniformity from X_0 to X with the 1-jet $j_{(0,1)}^1\tilde{\Psi}$. Thus, we get a global section of the bundle $\bar{\mathcal{F}}\mathcal{E}_0$, or, in other words, a **material non-holonomic parallelism** $\bar{\mathcal{P}}$ on $\mathcal{B}_0$.

The Lie group $G(X_0)$ can be transported via $\tilde{Z}_0$ and we obtain a Lie subgroup $\bar{G}$ of $\bar{G}(n, n+m)$:

$$\bar{G} = \tilde{Z}_0^{-1} \circ G(X_0) \circ \tilde{Z}_0\ .$$

If we prolongate $\bar{\mathcal{P}}$ by the action of $\bar{G}$ we obtain a $\bar{G}$-reduction of $\bar{\mathcal{F}}\mathcal{E}_0$. Such a reduction will be called a **non-holonomic $\bar{G}$-structure** on $\mathcal{B}_0$.

Remark 4.2. A classification of the subgroups of $\bar{G}(n, n+m)$ could be obtained in a similar way to that in [6, 18]. The details of this classification as well as the integrability conditions of the corresponding $\bar{G}$-structures are matter of a future research.

Let (x^α) be a coordinate system on $\mathcal{B}_0$ and take local bundle coordinates (x^α, X_i^j) for $\mathcal{E}_0$. We obtain induced coordinates $(x^\alpha, X_i^j, Y_\alpha^\beta, Z_{i\gamma}^j)$ on $\bar{\mathcal{F}}\mathcal{E}_0$. We set

$$\bar{\mathcal{P}}(x^\alpha) = (x^\alpha, \mathcal{P}_i^j, \mathcal{Q}_\alpha^\beta, \mathcal{R}_{i\gamma}^j)\ . \tag{7}$$

From (7) it follows that there are $n+m$ linearly independent vector fields $\{\mathcal{P}_1, \cdots, \mathcal{P}_{n+m}\}$ on $\mathbb{R}^{n+m}$ along $\mathcal{B}_0$ such that the first n vector fields $\{\mathcal{P}_1, \cdots, \mathcal{P}_n\}$ define a linear paralellism on $\mathcal{B}_0$. These vector fields are locally given by

$$\mathcal{P}_i = \mathcal{P}_i^j \frac{\partial}{\partial x^j}\ .$$

The vector fields $\{P_{n+1}, \cdots, P_{n+m}\}$ are transversal to $\mathcal{B}_0$.

There are also n vector fields $\{\mathcal{Q}_1, \cdots, \mathcal{Q}_n\}$ yielding another linear parallelism on $\mathcal{B}_0$, and which come from the induced diffeomorphisms on the base manifolds. Indeed, there is an underlying "uniformity" on $\mathcal{B}_0$ and an induced ordinary reference crystal $j_0^1\Psi$ at X_0 which is transported to any arbitrary point of $\mathcal{B}_0$.

Moreover, there exists a connection Γ in the principal bundle $\pi_0 : \mathcal{E}_0 \longrightarrow \mathcal{B}_0$. In fact, a non-holonomic frame at a point $X \in \mathcal{B}_0$ just defines:

1 a linear frame of $\mathbb{R}^{n+m}$ at X such that its n first vectors are tangent to $\mathcal{B}_0$ and the last m vectors are transversal;
2 a linear frame of $\mathcal{B}_0$ at X;
3 and, a horizontal subspace at X of the principal bundle $\pi_0 : \mathcal{E}_0 \longrightarrow \mathcal{B}_0$, or, in other words, an infinitesimal piece of connection.

We introduce the following notation:

$$\mathcal{N}_1 = \mathcal{P}_{n+1} \ , \ \cdots \ , \ \mathcal{N}_m = \mathcal{P}_{n+m} \ .$$

Next, take local coordinates (x^α, x^a) on $\mathbb{R}^{n+m}$ such that (x^α) are coordinates on $\mathcal{B}_0$ and (x^a) are transversal coordinates.

Thus, we have

$$\mathcal{Q}_\alpha = \mathcal{Q}_\alpha^\beta(x^\gamma)\frac{\partial}{\partial x^\beta} \ , \ \mathcal{P}_\alpha = \sum_{\beta=1}^{n} \mathcal{P}_\alpha^\beta(x^\gamma)\frac{\partial}{\partial x^\beta} \ , \tag{8}$$

$$\mathcal{N}_a = \sum_{\beta=1}^{n} \mathcal{P}_a^\beta(x^\gamma)\frac{\partial}{\partial x^\beta} + \sum_{b=1}^{m} \mathcal{P}_a^b(x^\gamma)\frac{\partial}{\partial x^b} \ ,$$

where, for simplicity, we have written $\mathcal{P}_{n+a}^\alpha = \mathcal{P}_a^\alpha$ and $\mathcal{P}_{n+a}^b = \mathcal{P}_a^b$.

The parallelism $\{\mathcal{P}_1, \dots, \mathcal{P}_n\}$ defines a linear connection Γ_1 on $\mathcal{B}_0$ whose Christoffel components are given by

$$(\Gamma_1)_{\alpha\beta}^\gamma = -(\mathcal{P}^{-1})_\beta^\sigma \frac{\partial \mathcal{P}_\sigma^\gamma}{\partial x^\alpha} \ .$$

That is, the covariant derivative ∇_1 associated with Γ_1 is given by

$$(\nabla_1)_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} = (\Gamma_1)_{\alpha\beta}^\gamma \frac{\partial}{\partial x^\gamma} \ .$$

The parallelism $\{\mathcal{Q}_1, \dots, \mathcal{Q}_n\}$ defines another linear connection Γ_2 on $\mathcal{B}_0$ with Christoffel components given by

$$(\Gamma_2)_{\alpha\beta}^\gamma = -(\mathcal{Q}^{-1})_\beta^\sigma \frac{\partial \mathcal{Q}_\sigma^\gamma}{\partial x^\alpha} \ .$$

In other words, the covariant derivative ∇_2 associated with Γ_2 is given by

$$(\nabla_2)_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} = (\Gamma_2)_{\alpha\beta}^\gamma \frac{\partial}{\partial x^\gamma} \ .$$

Finally, let us recall the definition of the induced connection Γ in $\pi_0 : \mathcal{E}_0 \longrightarrow \mathcal{B}_0$. If $\bar{\mathcal{P}}(X) = j_0^1 \tilde{\Psi}$, the horizontal subspace at $\mathcal{P}(X)$ is defined to be

$$H_{\mathcal{P}(X)} = T\varphi(T_0\mathbb{R}^n) \ ,$$

where $\varphi : \mathbb{R}^n \longrightarrow \mathcal{B}_0$ is given by $\varphi(r) = \tilde{\Psi}(r, 1)$. Since the horizontal lift of $\frac{\partial}{\partial x^\alpha}$ is

$$(\frac{\partial}{\partial x^\alpha})^H = \frac{\partial}{\partial x^\alpha} - \Gamma^j_{k\alpha}\mathcal{P}^k_i \frac{\partial}{\partial X^j_i}$$

we deduce that the Christoffel components of Γ are the following [28, 9, 30]:

$$\Gamma^j_{i\beta} = -\mathcal{R}^j_{k\gamma}(\mathcal{P}^{-1})^k_i(\mathcal{Q}^{-1})^\gamma_\beta \ .$$

A direct computation taking into account that $\mathcal{P}^a_\alpha = 0$ and $\mathcal{R}^a_{\alpha\beta} = 0$, shows that

$$\left.\begin{array}{rcl}
\Gamma^\gamma_{\alpha\beta} & = & -\mathcal{R}^\gamma_{\sigma\mu}(\mathcal{P}^{-1})^\sigma_\alpha(\mathcal{Q}^{-1})^\mu_\beta \ , \\
\Gamma^\gamma_{a\beta} & = & -\mathcal{R}^\gamma_{c\mu}(\mathcal{P}^{-1})^c_a(\mathcal{Q}^{-1})^\mu_\beta - \mathcal{R}^\gamma_{\sigma\mu}(\mathcal{P}^{-1})^\sigma_a(\mathcal{Q}^{-1})^\mu_\beta \ , \\
\Gamma^c_{\alpha\beta} & = & -\mathcal{R}^c_{\gamma\mu}(\mathcal{P}^{-1})^\gamma_\alpha(\mathcal{Q}^{-1})^\mu_\beta \ , \\
\Gamma^c_{a\beta} & = & -\mathcal{R}^c_{\gamma\mu}(\mathcal{P}^{-1})^\gamma_a(\mathcal{Q}^{-1})^\mu_\beta - \mathcal{R}^c_{d\mu}(\mathcal{P}^{-1})^d_a(\mathcal{Q}^{-1})^\mu_\beta \ .
\end{array}\right\}$$

Since there exists a left action of G_0 on $\mathbb{R}^{n+m}$ we can construct an associated vector bundle with $\mathcal{E}_0$ which becomes the Whitney sum $T\mathcal{B}_0 \oplus \mathcal{N}$, where $\mathcal{N}$ is the normal bundle generated by the vector fields $\{\mathcal{N}_1, \cdots, \mathcal{N}_m\}$. The connection Γ induces a connection in $T\mathcal{B}_0 \oplus \mathcal{N}$ whose tangent component defines a linear connection Γ_3 with covariant derivative ∇_3 given by

$$(\nabla_3)_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} = \Gamma^\gamma_{\beta\alpha}\frac{\partial}{\partial x^\gamma} \ .$$

Taking into account that

$$\mathcal{P}^\beta_b(\mathcal{P}^{-1})^\mu_\beta + \mathcal{P}^c_b(\mathcal{P}^{-1})^\mu_c = 0 \ ,$$

and putting

$$\begin{array}{rcl}
\nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} & = & \Gamma^\gamma_{\beta\alpha}\frac{\partial}{\partial x^\gamma} + \Gamma^c_{\beta\alpha}\frac{\partial}{\partial x^c} \ , \\
\nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^d} & = & \Gamma^\gamma_{d\alpha}\frac{\partial}{\partial x^\gamma} + \Gamma^c_{d\alpha}\frac{\partial}{\partial x^c} \ ,
\end{array}$$

we compute the covariant derivative of $\mathcal{N}_a$ with respect to Γ:

$$\nabla_{\frac{\partial}{\partial x^\alpha}}\mathcal{N}_b = \left(\frac{\partial \mathcal{P}^\gamma_b}{\partial x^\alpha} - \mathcal{R}^\gamma_{b\epsilon}(\mathcal{Q}^{-1})^\epsilon_\alpha\right)\frac{\partial}{\partial x^\gamma} + \left(\frac{\partial \mathcal{P}^d_b}{\partial x^\alpha} - \mathcal{R}^d_{b\epsilon}(\mathcal{Q}^{-1})^\epsilon_\alpha\right)\frac{\partial}{\partial x^d} \ . \tag{9}$$

Next, we will introduce the notion of prolongability of non-holonomic parallelisms. As we have seen, a material non-holonomic parallelism $\bar{\mathcal{P}}$ induces a global field of frames $\mathcal{P}$ along $\mathcal{B}_0$, a linear parallelism $\mathcal{Q}$ on $\mathcal{B}_0$, and a connection on the principal bundle $\pi_0 : \mathcal{E}_0 \longrightarrow \mathcal{B}_0$. The global section $\mathcal{P}$ of π_0 gives a new flat connection $\bar{\Gamma}$ by defining the horizontal lift of a tangent vector $U \in T_X\mathcal{B}_0$ as follows:

$$U^{\bar{H}} = T\mathcal{P}(X)(U) \in T_{\mathcal{P}(X)}\mathcal{E}_0 \ .$$

Thus, we have

$$(\frac{\partial}{\partial x^\alpha})^{\bar{H}} = \frac{\partial}{\partial x^\alpha} + \frac{\partial \mathcal{P}_i^j}{\partial x^\alpha} \frac{\partial}{\partial X_i^j} .$$

Definition 4.2. *We say that $\bar{\mathcal{P}}$ is a prolongation if both connections, Γ and $\bar{\Gamma}$, coincide. If, moreover, $\mathcal{Q}$ is integrable, $\bar{\mathcal{P}}$ is said to be an integrable prolongation.*

The reason for the above terminology is that an integrable prolongation is a non-holonomic parallelism which is obtained from $\mathcal{P}$ and $\mathcal{Q}$. In fact, note that a non-holonomic frame $j_0^1 \tilde{\Psi}$ at a point $X = \Psi(0) \in \mathcal{B}_0$ is a linear frame of $\mathcal{E}_0$ at the point $\tilde{\Psi}(0)$. Thus, given a global section $\mathcal{P}$ of $\pi_0 : \mathcal{E}_0 \longrightarrow \mathcal{B}_0$ and a linear parallelism $\mathcal{Q}$ of $\mathcal{B}_0$, we can construct a non-holonomic parallelism denoted by $\mathcal{P}^1(\mathcal{Q})$ as follows: $\mathcal{P}^1(\mathcal{Q})(X)$ is defined to be the linear frame at $\mathcal{P}(X)$ which consists of the tangent vectors $\{T\mathcal{P}(X)(\mathcal{Q}_1), \cdots, T\mathcal{P}(X)(\mathcal{Q}_1)\}$, completed with a suitable family of vertical tangent vectors. Of course, $\mathcal{P}^1(\mathcal{Q})$ defines $\mathcal{P}$, $\mathcal{Q}$, and the connection $\bar{\Gamma}$.

Proposition 4.2. *A non-holonomic parallelism $\bar{\mathcal{P}}$ is an integrable prolongation if and only if the torsion tensor T_2 of Γ_2, the difference tensor $D_{13} = \nabla_1 - \nabla_3$, and the m 1-forms $\nabla \mathcal{N}_a$, $1 \leq a \leq m$, simultaneously vanish.*

Proof: If $T_2 = 0$, there exist local coordinates (x^α) on $\mathcal{B}_0$ such that

$$\mathcal{Q}_\alpha^\beta = \delta_\alpha^\beta ,$$

or, equivalently,

$$\mathcal{Q}_\alpha = \frac{\partial}{\partial x^\alpha} .$$

Thus, the non-holonomic parallelism $\bar{\mathcal{P}}$ can be locally written as follows:

$$\bar{\mathcal{P}}(x^\alpha) = (x^\alpha, \mathcal{P}_i^j, 1, \mathcal{R}_{i\beta}^j) .$$

Moreover, the difference tensor D_{13} also vanishes. This implies that

$$\mathcal{R}_{\alpha\beta}^\gamma = \frac{\partial \mathcal{P}_\alpha^\gamma}{\partial x^\beta} .$$

Now, we will use that the transversal vector fields $\mathcal{N}_a$ are parallel, and we deduce that

$$\mathcal{R}_{b\beta}^\gamma = \frac{\partial \mathcal{P}_b^\gamma}{\partial x^\beta} , \mathcal{R}_{b\beta}^c = \frac{\partial \mathcal{P}_b^c}{\partial x^\beta} .$$

Finally, we know that

$$\mathcal{R}_{\alpha\beta}^c = 0 , \mathcal{P}_\alpha^c = 0 .$$

Thus, the result follows.

The converse is trivial. □

The tensors T_2, D_{13} and $\nabla \mathcal{N}_a$ will be called the **inhomogeneity tensors** of the given material non-holonomic parallelism $\bar{\mathcal{P}}$.

Definition 4.3. *A non-holonomic $\bar{G}$-structure on $\mathcal{B}_0$ is said to be an integrable prolongation if around each point of $\mathcal{B}_0$ there exists a local section which is an integrable prolongation.*

From Proposition 4.2 it follows the following

Proposition 4.3. *A non-holonomic $\bar{G}$-structure on $\mathcal{B}_0$ is an integrable prolongation if and only if it admits local sections whose inhomogeneity tensors vanish.*

5. Homogeneity

Definition 5.1. *$\mathcal{B}$ is said to be* ***homogeneous*** *if there exists a uniform configuration $\Phi : \mathcal{B} \longrightarrow \mathbb{R}^{n+m}$ such that:*

(i) $\Phi(\mathcal{B})$ is an open subset of $\mathbb{R}^n$, where $\mathbb{R}^n$ is considered as a natural subspace of $\mathbb{R}^{n+m}$ defined by the vanishing of the coordinates x^{n+1}, x^{n+2} and x^{n+m}. Here $(x^1, \cdots, x^n, x^{n+1}, \cdots, x^{n+m})$ denote the standard coordinates in $\mathbb{R}^{n+m}$;

(ii) There exists a global deformation $\tilde{\kappa}$ from $\widetilde{\mathcal{FB}_\Phi}$ into $\mathcal{E}$ covering a global diffeomorphism $\kappa : \Phi(\mathcal{B}) \longrightarrow \mathbb{R}^n$ such that $\bar{\mathcal{P}} = \tilde{\kappa}^{-1}$ defines a material non-holonomic parallelism, i.e.,

$$\bar{\mathcal{P}}(X) = j_0^1(\tilde{\kappa}^{-1} \circ F\tau_{\kappa(X)}) \ , \ \forall X \in \Phi(\mathcal{B}) \ ,$$

where $\tau_{\kappa(X)} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ denotes the translation on $\mathbb{R}^n$ by the vector $\kappa(X)$, and $F\tau_{\kappa(X)}$ is the induced mapping between frame bundles.

$\mathcal{B}$ is said to be ***locally homogeneous*** *if for every point $X \in \mathcal{B}$ there exists an open neighborhood which is homogeneous.*

This definition is referred to a particular chosen reference crystal. More generally, we will say that $\mathcal{B}$ is homogeneous if it is homogeneous with respect to at least one reference crystal.

We will obtain a geometrical characterization of the local homogeneity.

For the sake of simplicity, we first assume that the group of material symmetries is trivial. So, we have the following

Theorem 5.1. *$\mathcal{B}$ is locally homogeneous (with respect to a chosen reference crystal) if and only if there exists a uniform configuration Φ such that the associated material non-holonomic parallelism $\bar{\mathcal{P}}$ is an integrable prolongation.*

Proof: If $\mathcal{B}$ is locally homogeneous, and $\tilde{\kappa}$ is as in the above definition, we obtain

$$\bar{\mathcal{P}}(x^\alpha) = (x^\alpha, \mathcal{P}_i^j, 1, \frac{\partial \mathcal{P}_i^j}{\partial x^\alpha}) \ .$$

Therefore, $\bar{\mathcal{P}}$ is an integrable prolongation.

Assume now that the inhomogeneity tensors associated with a material non-holonomic parallelism $\bar{\mathcal{P}}$ identically vanish. We assume that $\bar{\mathcal{P}}$ was obtained from a configuration $\Phi : \mathcal{B} \longrightarrow \mathbb{R}^{n+m}$. Then, from Proposition 4.2, it is an integrable prolongation. This means that there exist local coordinates (x^α) on $\Phi(\mathcal{B})$ such that

$$\bar{\mathcal{P}}(x^\alpha) = (x^\alpha, \mathcal{P}_i^j, 1, \frac{\partial \mathcal{P}_i^j}{\partial x^\alpha}) .$$

Next, we define a principal bundle automorphism

$$\tilde{\kappa} : \widetilde{\mathcal{F}B}_\Phi \longrightarrow \mathcal{E}$$

as follows:

$$\tilde{\kappa}(x^\alpha, X_i^j) = (x^\alpha, \mathcal{P}_i^k X_k^j) .$$

$\tilde{\kappa}$ is the required deformation. □

To end this section, we will investigate what happens if a change of reference crystal is performed. Notice that a change of reference crystal consists of composing the material non-holonomic parallelism $\bar{\mathcal{P}} = (\mathcal{P}, \mathcal{Q}, \mathcal{R})$ with an element $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in the Lie group $\bar{G}(n, n+m)$. The new material non-holonomic parallelism is then given by $\bar{\mathcal{P}}' = (\mathcal{P}', \mathcal{Q}', \mathcal{R}')$, where

$$(\mathcal{P}')_i^j = \mathcal{A}_i^k \mathcal{P}_k^j \ , \ (\mathcal{Q}')_\alpha^\beta = \mathcal{B}_\alpha^\gamma \mathcal{Q}_\gamma^\beta \ , \ (\mathcal{R}')_{i\gamma}^j = \mathcal{A}_i^k \mathcal{B}_\gamma^\beta \mathcal{R}_{k\beta}^j + \mathcal{P}_k^j \mathcal{C}_{i\gamma}^k \ .$$

So, the new connections Γ'_1 and Γ'_2 coincide with the former ones, Γ_1 and Γ_2. This fact implies that, if the torsion tensor T_2 of $\bar{\mathcal{P}}$ vanishes, the same is true for $\bar{\mathcal{P}}'$. Therefore, the first test in order to know if a material non-holonomic parallelism is an integrable prolongation is to check the torsion tensor T_2. If T_2 does not vanish, we can conclude that any $\bar{\mathcal{P}}$ would be not an integrable prolongation. If T_2 vanishes, but the other tensors do not so, we can try for a change of reference crystal. Consider the vector fields

$$D_{\alpha\beta} = (\nabla_1)_{\mathcal{Q}_\alpha} \mathcal{P}_\beta - (\nabla_3)_{\mathcal{Q}_\alpha} \mathcal{P}_\beta \ , \ D_{\alpha b} = \nabla_{\mathcal{Q}_\alpha} \mathcal{N}_b \ .$$

By the same argument that in [16], we conclude the following.

Theorem 5.2. *$\mathcal{B}$ is locally homogeneous if and only if there exists a uniform configuration Φ such that the associated material non-holonomic parallelism $\bar{\mathcal{P}}$ have $T_2 = 0$ and $D_{\alpha\beta} = D_{\alpha b} = 0$.*

6. Particular cases

6.1. Elastic rods. (see [1, 5])

In this case, $n = 1$, $m = 2$. That is, $\mathcal{B}_0$ is a curve in $\mathbb{R}^3$. Since $n = 1$, we allways have that the linear parallelism $\{\mathcal{Q}\}$ is integrable, so that T_2 identically vanishes. Proposition 4.2 becomes as follows.

Proposition 6.1. *$\bar{\mathcal{P}}$ is an integrable prolongation if and only if the difference tensor $D_{13} = \nabla_1 - \nabla_3$, and the 1-forms $\nabla\mathcal{N}_1$ and $\nabla\mathcal{N}_2$ simultaneously vanish.*

If the group of material symmetries is continuous, we obtain a $\bar{G}$-structure on the curve $\mathcal{B}_0$, where $\bar{G}$ is a Lie subgroup of $\bar{G}(1,3)$.

A particular case is obtained when we consider principal bundle isomorphisms $\tilde{\kappa} : \mathcal{F}(\Phi_1(\mathcal{B})) \longrightarrow \mathcal{F}((\Phi_2(\mathcal{B}))$ such that the tangent part is precisely given by the tangent map of the induced diffeomorphisms $\kappa : \Phi_1(\mathcal{B}) \longrightarrow \Phi_2(\mathcal{B})$. In this case, $\mathcal{P}_1 = \mathcal{Q}_1$, and, then, $\Gamma_1 = \Gamma_2$.

6.2. Elastic shells. (see [1, 3, 5, 19, 20, 21, 22, 26, 27, 51])

In this case, $n = 2$, $m = 1$. That is, $\mathcal{B}_0$ is a surface in $\mathbb{R}^3$. Thus, the non-holonomic parallelism $\bar{\mathcal{P}}$ defines two linear parallelisms $\{\mathcal{P}_1, \mathcal{P}_2\}$ and $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ on the surface $\mathcal{B}_0$, and a normal vector field $\mathcal{N}$.

Proposition 4.2 becomes as follows.

Proposition 6.2. *$\bar{\mathcal{P}}$ is an integrable prolongation if and only if the tensor torsion T_2, the difference tensor $D_{13} = \nabla_1 - \nabla_3$, and the 1-form $\nabla\mathcal{N}$ simultaneously vanish.*

If the group of material symmetries is continuous, we obtain a $\bar{G}$-structure on the surface $\mathcal{B}_0$, where $\bar{G}$ is a Lie subgroup of $\bar{G}(2,3)$.

A particular case is obtained when we consider principal bundle isomorphisms $\tilde{\kappa} : \mathcal{F}(\Phi_1(\mathcal{B})) \longrightarrow \mathcal{F}(\Phi_2(\mathcal{B}))$ such that the tangent part is precisely given by the tangent map of the induced diffeomorphisms $\kappa : \Phi_1(\mathcal{B}) \longrightarrow \Phi_2(\mathcal{B})$. In this case, $\mathcal{P}_\alpha = \mathcal{Q}_\alpha$, $\alpha = 1, 2$, and, then, $\Gamma_1 = \Gamma_2$.

6.3. Cosserat media. (see [5, 14, 15, 17, 24])

Assume that $n = 3$, $m = 0$. In this case, a bundle configuration $\widetilde{\mathcal{F}\mathcal{B}_\Phi}$ is just the linear frame bundle $\mathcal{F}(\Phi(\mathcal{B}))$ of $\Phi(\mathcal{B})$, that is, the collection of all bases at all the points of $\Phi(\mathcal{B})$. Thus, the Lie group G_0 is $Gl(n, \mathbb{R})$. A deformation is a principal bundle isomorphism $\tilde{\kappa} : \mathcal{F}(\Phi_1(\mathcal{B})) \longrightarrow \mathcal{F}(\Phi_2(\mathcal{B}))$ covering a diffeomorphism $\kappa : \Phi_1(\mathcal{B}) \longrightarrow \Phi_2(\mathcal{B})$. Chosen an uniform configuration $\Phi_0 : \mathcal{B} \longrightarrow \mathbb{R}^n$, we obtain a non-holonomic parallelism $\bar{\mathcal{P}} : \mathcal{B}_0 \longrightarrow \bar{\mathcal{F}}\mathcal{E}_0$ (we follow the notations introduced in the precedent sections). It should be noted that $\bar{\mathcal{F}}\mathcal{E}_0$ is just the so-called non-holonomic second order frame bundle of $\mathcal{B}_0$, and, hence, $\bar{\mathcal{P}}$ is a non-holonomic second order parallelism. Thus, we have two linear parallelisms $\mathcal{P}$ and $\mathcal{Q}$, and a linear connection Γ on $\mathcal{B}_0$. There are no transversal vector fields, and Proposition 4.2 becomes as follows.

Proposition 6.3. *$\bar{\mathcal{P}}$ is an integrable prolongation if and only if the torsion tensor T_2 of Γ_2 and the difference tensor $D_{13} = \nabla_1 - \nabla_3$ simultaneously vanish.*

If the group of material symmetries is continuous, we obtain a material non-holonomic second order $\bar{G}$-structure, where $\bar{G}$ is a Lie subgroup of the second order non-holonomic group $\bar{G}(n) = \bar{G}(3,3)$.

Particular cases are obtained if we only consider deformations such that they are the natural prolongation of the diffeomorphisms between the bases, that is, $\tilde{\Phi} = \mathcal{F}\Phi$. This occurs for second grade material bodies [6, 7, 8, 10, 11, 12, 13]. In this case, $\mathcal{P}_\alpha = \mathcal{Q}_\alpha$ and, hence, $\Gamma_1 = \Gamma_2$. So, we have the following.

Proposition 6.4. *The following statements are equivalent:*

(1) $\bar{\mathcal{P}}$ *is an integrable prolongation;*
(2) *it is an integrable parallelism of second order;*
(3) *the torsion tensor* T_2 *of* Γ_2 *and the difference tensor* $D_{13} = \nabla_1 - \nabla_3$ *simultaneously vanish.*

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