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TRANSLATION OF NATURAL OPERATORS ON MANIFOLDS WITH AHS—STRUCTURES

Andreas Čap

To Ivan Kolář, on the occasion of his 60th birthday.

Abstract. We introduce an explicit procedure to generate natural operators on manifolds with almost Hermitian symmetric structures and work out several examples of this procedure in the case of almost Grassmannian structures.

This paper splits into two parts. In the first part, we introduce a procedure to generate invariant operators on manifolds with almost Hermitian symmetric (AHS) structures. This procedure is inspired by the Jantzen–Zuckermann translation principle in representation theory and by curved versions of this principle. A curved translation principle was first applied in [Eastwood–Rice] to four–dimensional conformal geometry, versions for conformal manifolds of arbitrary dimensions and for other structures can be found in [Eastwood] and in [Bailey–Eastwood–Gover].

It should be remarked that there is another approach which leads to powerful curved versions of the translation principle in the conformal case (see [Baston] and [Eastwood–Slovák]) which probably are better suited to prove general existence results than the procedure presented here. The advantage of the latter is the bigger generality, that it immediately leads to explicit formulae for the operators in question, and that it also leads to starting points for translations.

In the second part of the paper, we apply the translation procedure in several simple instances in the case of almost Grassmannian structures, arriving at several examples of invariant operators of low order in this case. Thanks to the notion of jet prolongations of representations developed in [CSS1], these are mainly purely algebraic computations. The final results of these computations can then easily be translated into the language of differential operators.

Our reference for the general theory of AHS–structures is [CSS1]. For the theory of almost Grassmannian structures we refer to [Bailey–Eastwood].

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1. The translation procedure

1.1. The basic ingredients for an AHS–structure are a so called $|1|$–graded Lie algebra, i.e. a semisimple Lie algebra $\mathfrak{g}$ which admits a grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and a connected Lie group G with Lie algebra $\mathfrak{g}$. By $\mathfrak{p}$ we denote the parabolic subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $\mathfrak{g}$ and by P the corresponding subgroup of G.

Then a manifold with an AHS–structure corresponding to G is a manifold whose dimension equals the dimension of $\mathfrak{g}_{-1}$, together with a principal P–bundle and a Cartan connection on this bundle, i.e. a $\mathfrak{g}$–valued one–form which reproduces fundamental vector fields, is P–equivariant and restricts to a linear isomorphism on each tangent space. The construction of this bundle and the canonical Cartan connection from underlying data is the subject of the paper [CSS2].

1.2. For any representation of the group P on a vector space W one can form the associated bundle to the principal P–bundle over a manifold with the corresponding AHS–structure, thus arriving at a natural vector bundle on these manifolds. Particularly, one is interested in representations in which the subgroup P_1 of P which corresponds to the (abelian) Lie subalgebra $\mathfrak{g}_1$ of $\mathfrak{p}$ acts trivial, since these representations correspond to classical geometric objects like tensors or differential forms.

One of the main achievements of [CSS1] is the construction of semi holonomic jet prolongations of representations. Since we will heavily need these, we quickly recall the construction from [CSS1, 5.3–5.6]. For a representation W of P we describe the (semi holonomic) jet prolongations $\mathcal{J}^k(W)$ as representations of the Lie algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$: As a $\mathfrak{g}_0$–module, the first jet prolongation $\mathcal{J}^1(W)$ is defined as $W \oplus (\mathfrak{g}_{-1}^* \otimes W)$, where we identify $\mathfrak{g}_{-1}^* \otimes W$ with the space of linear maps from $\mathfrak{g}_{-1}$ to W. An element $Z \in \mathfrak{g}_1$ acts on a pair (w, φ) by mapping it to $(Z{\cdot}w, X \mapsto (Z{\cdot}\varphi(X) + [Z,X]{\cdot}w))$, where $X \in \mathfrak{g}_{-1}$ and the dot denotes the action of the bracket $[Z,X] \in \mathfrak{g}_0$ on $w \in W$. Note that in the case of trivial $\mathfrak{g}_1$–action on W (which we will mainly deal with) this formula simplifies considerably. It is shown in [CSS1, 5.3] that this actually defines a $\mathfrak{p}$–action which integrates to a P–action.

Next, one makes $\mathcal{J}^1(\_)$ into a functor on the category of P–representations and constructs a natural transformation to the identity (which is just given by projecting out the first component), see [CSS1, 5.4]. From this, one then has two natural homomorphisms $\mathcal{J}^1(\mathcal{J}^1(W)) \to \mathcal{J}^1(W)$, namely the natural transformation just described and the first jet prolongation of the canonical homomorphism $\mathcal{J}^1(W) \to W$, and one defines $\mathcal{J}^2(W)$ as the submodule of $\mathcal{J}^1(\mathcal{J}^1(W))$ on which these two homomorphisms coincide.

Similarly as in the classical construction of semi holonomic jets (see e.g. [Ehresmann]), this can then be iterated to define higher jet prolongations $\mathcal{J}^k(W)$, see [CSS1, 5.5]. For us, the main point is that as a $\mathfrak{g}_0$–module $\mathcal{J}^k(W)$ is isomorphic to

$$W \oplus (\mathfrak{g}_{-1}^* \otimes W) \oplus \cdots \oplus (\otimes^k \mathfrak{g}_{-1}^* \otimes W),$$

while the $\mathfrak{g}_1$ action can be computed inductively by viewing $\mathcal{J}^k(W)$ as a submodule of $\mathcal{J}^1(\mathcal{J}^{k-1}(W))$.

1.3. Using the absolutely invariant derivative introduced in [CSS1, 2.3–2.5], any homomorphism $\tilde{D} : \mathcal{J}^k(W) \to W'$ of P–modules gives rise to a natural operator of order k defined on all manifolds with an AHS–structure corresponding to G, see [CSS1, 5.7].

It can be shown, however, that not all invariant operators are of this form. An example is the square of the Laplacian on conformal manifolds of dimension 4. In fact, one can compute explicitly the corresponding space of module homomorphisms which turns out to be of dimension three, but all these homomorphisms give rise to the zero operator.

The operators which are induced by module homomorphisms as above will be called *strongly invariant* in the sequel. Our aim in this section is to construct a procedure of translating such module homomorphisms, and thus the corresponding strongly invariant operators.

The basic idea of these translations in the form we will construct them is the following: suppose that $\tilde{D} : \mathcal{J}^k(W) \to W'$ is a homomorphism of P–modules and that V is a representation of P which is induced by a representation of the whole group G. The first step is to construct from $\tilde{D}$ a homomorphism $\tilde{D}_V : \mathcal{J}^k(W \otimes V) \to W' \otimes V$, which coincides with $\tilde{D} \otimes \mathrm{id}_V$ in the highest order. The corresponding operator will be called the twisted operator corresponding to D and V.

The second part of the construction is to construct for certain representations V and subrepresentations $\tilde{W}$ of $W \otimes V$ canonical homomorphisms $\mathcal{J}^k(\tilde{W}) \to W \otimes V$. We will only do this in the following special situation: Consider the subgroup P_0 of P corresponding to the Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{p}$, which is always reductive, see [Ochiai, 5.1]. Over this subgroup, we may split V as $V_0 \oplus V_1 \oplus \dots \oplus V_\ell$ in such a way that the action of $\mathfrak{g}_{\pm 1}$ maps each V_i to $V_{i\pm 1}$, and we will restrict to the special case, where this splitting has in fact length two, i.e. $V = V_0 \oplus V_1$. In this situation, we will construct for each P_0–irreducible component $\tilde{W} \subset W \otimes V_0$ (which we view as a representation of P with trivial P_1–action) a homomorphism $\mathcal{J}^1(\tilde{W}) \to W \otimes V$, which generically is a splitting by a differential operator of the projection of natural vector bundles corresponding to the projection $W \otimes V \to (W \otimes V)/(W \otimes V_1) \to \tilde{W}$. A similar construction gives rise to homomorphisms $J^1(W' \otimes V) \to \tilde{W}'$ for certain representations $\tilde{W}'$ with trivial P_1 action.

Having constructed all these homomorphisms one can then simply compose the corresponding strongly invariant operators to obtain a new operator between the bundles corresponding to $\tilde{W}$ and $\tilde{W}'$.

1.4. Twisted operators. Suppose that W and W' are P–representations which are induced from irreducible P_0–representations, i.e. have trivial P_1–action, and suppose that $\tilde{D} : J^k(W) \to W'$ is a P–homomorphism. Recall that as a P_0–module $\mathcal{J}^k(W) \cong W \oplus (\mathfrak{g}_{-1}^* \otimes W) \oplus \dots \oplus (\otimes^k \mathfrak{g}_{-1}^* \otimes W)$. It was observed in [CSS1, 5.8] that $\tilde{D}$ depends in fact only on one of the components $\otimes^i \mathfrak{g}_{-1}^* \otimes W$, and we assume that it depends only on the top component. Thus, $\tilde{D}$ is in fact a P_0–homomorphism $\otimes^k \mathfrak{g}_{-1}^* \otimes W \to W'$, which vanishes on the image of $\otimes^{k-1} \mathfrak{g}_{-1}^* \otimes W$ under the action of P_1.

We will show that there is a canonical extension of $\tilde{D}\otimes\mathrm{id}_V : \otimes^k\mathfrak{g}_{-1}^*\otimes W\otimes V \to W'\otimes V$ to a homomorphism $\mathcal{J}^k(W\otimes V)\to W'\otimes V$ of P–modules.

The geometrical reason for the existence of the twisted operators is the following: The representation W gives rise to a homogeneous vector bundle on the flat model G/P of our AHS–structure. Its k–th semi holonomic jet prolongation (in the sense of differential geometry) is again a homogeneous vector bundle, so the standard fiber has a canonical P–action, which can be shown to coincide with the action on $\mathcal{J}^k(W)$ introduced above. Now if V is a representation of P which comes from a representation of the whole group G, then one can show that the corresponding bundle over G/P is actually trivial. Using this, it is clear that $\mathcal{J}^k(W\otimes V)$ must be isomorphic to $\mathcal{J}^k(W)\otimes V$, and composing this isomorphism with $\tilde{D}\otimes\mathrm{id}_V$ we get the twisted operator. A purely algebraic proof of this fact can be given as follows:

1.5. Lemma. *Let V be a representation of P which is the restriction of a representation of the whole group G. Then there is a natural isomorphism between the functors $\mathcal{J}^1(\_\otimes V)$ and $\mathcal{J}^1(\_)\otimes V$, which are defined on the category of all P–representations. For any P–representation W the corresponding isomorphism $\mathcal{J}^1(W\otimes V)\to\mathcal{J}^1(W)\otimes V$ is in fact an isomorphism of extensions of $W\otimes V$ with kernel $\mathfrak{g}_{-1}^*\otimes W\otimes V$, that is we have a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{g}_{-1}^*\otimes(W\otimes V) & \longrightarrow & \mathcal{J}^1(W\otimes V) & \longrightarrow & W\otimes V & \longrightarrow & 0\\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & (\mathfrak{g}_{-1}^*\otimes W)\otimes V & \longrightarrow & \mathcal{J}^1(W)\otimes V & \longrightarrow & W\otimes V & \longrightarrow & 0
\end{array}$$

Proof. Let W be any P–representation. To define a linear map $\mathcal{J}^1(W\otimes V)\to\mathcal{J}^1(W)\otimes V$ it suffices to define it on elements of the form $(w_0\otimes v_0, Z_1\otimes w_1\otimes v_1)$ for $Z_1\in\mathfrak{g}_1$, $w_i\in W$ and $v_i\in V$. We map this element to

$$(w_0,0)\otimes v_0+(0,Z_1\otimes w_1)\otimes v_1+\sum_\ell(0,\eta_\ell\otimes w_0)\otimes\xi_\ell\cdot v_0,$$

where $\{\xi_\ell\}$ is a basis of $\mathfrak{g}_{-1}$ and $\{\eta_\ell\}$ is the dual basis of $\mathfrak{g}_1$. This is obviously a $\mathfrak{g}_0$–homomorphism which is independent of the chosen basis and bijective. Then one computes directly, that it also commutes with the action of $\mathfrak{g}_1$, and thus it is a $\mathfrak{p}$–homomorphism and hence also a P–homomorphism. From the definition it is obvious that this is in fact an isomorphism of extensions.

If $\varphi : W\to W'$ is a homomorphism of $\mathfrak{p}$–modules, then the induced homomorphism $\mathcal{J}^1(\varphi\otimes\mathrm{id}_V)$ maps the element $(w_0\otimes v_0, Z_1\otimes w_1\otimes v_1)$ to $(\phi(w_0)\otimes v_0, Z_1\otimes\phi(w_1)\otimes v_1)$ and from this it is clear that the isomorphism constructed above is natural. □

1.6. Proposition. *For each $n\in\mathbb{N}$ there is a natural isomorphism Ψ^n between the functors $\mathcal{J}^n(\_\otimes V)$ and $\mathcal{J}^n(\_)\otimes V$ such that for each W we get a commutative* *diagram*

$$\begin{tikzcd}
\otimes^n \mathfrak{g}_{-1}^* \otimes W \otimes V \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{J}^n(W \otimes V) \arrow[r] \arrow[d, "\Psi^n_W"] & \mathcal{J}^{n-1}(W \otimes V) \arrow[d, "\Psi^{n-1}_W"] \\
\otimes^n \mathfrak{g}_{-1}^* \otimes W \otimes V \arrow[r] & \mathcal{J}^n(W) \otimes V \arrow[r] & \mathcal{J}^{n-1}(W) \otimes V.
\end{tikzcd}$$

Proof. Suppose inductively that Ψ^n has been constructed for all $n \leq k$. For any W the module $\mathcal{J}^{k+1}(W)$ is defined as the submodule of $\mathcal{J}^1(\mathcal{J}^k(W))$, where the footpoint projection $\pi_{\mathcal{J}^k(W)} : \mathcal{J}^1(\mathcal{J}^k(W)) \to \mathcal{J}^k(W)$ coincides with the first jet prolongation $\mathcal{J}^1(\Pi^k_{k-1})$ of the natural projection $\Pi^k_{k-1} : \mathcal{J}^k(W) \to \mathcal{J}^{k-1}(W)$. (Recall that by definition $\mathcal{J}^k(W)$ is a submodule of $\mathcal{J}^1(\mathcal{J}^{k-1}(W))$.)

Consider the isomorphism

$$\mathcal{J}^1(\mathcal{J}^k(W \otimes V)) \xrightarrow{\mathcal{J}^1(\Psi^k_W)} \mathcal{J}^1(\mathcal{J}^k(W) \otimes V) \xrightarrow{\Psi^1_{\mathcal{J}^k(W)}} \mathcal{J}^1(\mathcal{J}^k(W)) \otimes V,$$

which we denote by $\tilde{\Psi}^{k+1}_W$. As a composition of two natural isomorphisms this is a natural isomorphism, too.

The first mapping by definition induces the mapping Ψ^k_W on the footpoints, while the second one gives the identity, since $\Psi^1_{\mathcal{J}^k(W)}$ is an isomorphism of extensions. Thus $(\pi_{\mathcal{J}^k(W)} \otimes \mathrm{id}_V) \circ \tilde{\Psi}^{k+1}_W = \Psi^k_W \circ \pi_{\mathcal{J}^k(W \otimes V)}$. By induction, we can write the right hand side of this equation as $\Psi^1_{\mathcal{J}^{k-1}(W)} \circ \mathcal{J}^1(\Psi^{k-1}_W) \circ \pi_{\mathcal{J}^k(W \otimes V)}$.

On the other hand, applying the functor $\mathcal{J}^1$ to the diagram

$$\begin{tikzcd}
\mathcal{J}^k(W \otimes V) \arrow[r, "\Psi^k_W"] \arrow[d, "\Pi^k_{k-1}"] & \mathcal{J}^k(W) \otimes V \arrow[d, "\Pi^k_{k-1} \otimes \mathrm{id}_V"] \\
\mathcal{J}^{k-1}(W \otimes V) \arrow[r, "\Psi^{k-1}_W"] & \mathcal{J}^{k-1}(W) \otimes V,
\end{tikzcd}$$

which is commutative by induction, and using the naturality of Ψ^1, we see that $\mathcal{J}^1(\Pi^k_{k-1}) \otimes \mathrm{id}_V \circ \tilde{\Psi}^{k+1}_W = \Psi^1_{\mathcal{J}^{k-1}(W)} \circ \mathcal{J}^1(\Psi^{k-1}_W) \circ \mathcal{J}^1(\Pi^k_{k+1})$.

Thus we see that $\tilde{\Psi}^{k+1}_W$ restricts to an isomorphism $\Psi^{k+1}_W : \mathcal{J}^{k+1}(W \otimes V) \to \mathcal{J}^{k+1}(W) \otimes V$. The commutativity of the diagram in the proposition for $n = k+1$ follows immediately from the fact that Ψ^{k+1}_W induces Ψ^k_W on the footpoints and induction. □

1.7. Using Proposition 1.6 it is now obvious how to construct the twisted operators. Suppose that W and W' are P–representations, $\tilde{D} : \mathcal{J}^k(W) \to W'$ is a P–homomorphism and V is a representation of G. Then $(\tilde{D} \otimes \mathrm{id}_V) \circ \Psi^k_W : \mathcal{J}^k(W \otimes V) \to \mathcal{J}^k(W) \otimes V \to W' \otimes V$ is again a P–homomorphism, which we denote by $\tilde{D}_V$, and the highest order component of this homomorphism is just the highest order component of $\tilde{D}$ tensorized with the identity on V.

The explicit expressions for the isomorphisms Ψ^k_W become quickly fairly complicated, but in the simple situation where V splits as $V_0 \oplus V_1$ over $\mathfrak{g}_0$, such that the action of $\mathfrak{g}_{\pm 1}$ maps each V_i to $V_{i\pm 1}$, we can compute the formulae for twisted operators explicitly:

Proposition. *Let W and W' be irreducible representations of P, and let V be a representation of G such that $X_1\cdot(X_2\cdot v)=0$ for all $X_1,X_2\in\mathfrak{g}_{-1}$ and $v\in V$. Moreover, let $\tilde D:\mathcal{J}^k(W)\to W'$ be a P–homomorphism which does not factor over $\mathcal{J}^{k-1}(W)$. Then the corresponding homomorphism $\tilde D_V:\mathcal{J}^k(W\otimes V)\to W'\otimes V$ maps a jet in $\mathcal{J}^k(W\otimes V)$ with top component $Z_1\otimes\dots\otimes Z_k\otimes w_k\otimes v_k$ and second highest component $Y_1\otimes\dots\otimes Y_{k-1}\otimes w_{k-1}\otimes v_{k-1}$ to*

$$\tilde D(Z_1\otimes\dots\otimes Z_k\otimes w_k)\otimes v_k+ \\ +\sum_{i=0}^{k-1}\sum_\ell \tilde D(Y_1\otimes\dots\otimes Y_i\otimes\eta_\ell\otimes Y_{i+1}\otimes\dots\otimes Y_{k-1}\otimes w_{k-1})\otimes\xi_\ell\cdot v_{k-1},$$

where the ξ_ℓ form a basis of $\mathfrak{g}_{-1}$ and the η_ℓ the dual basis of $\mathfrak{g}_1$.

Proof. To proof the result amounts to computing the top component of the value of Ψ^k_W on the given jet. Obviously, the formula holds for $k=1$, so let us inductively assume that it holds for Ψ^{k-1}_W. To compute the action of Ψ^k_W on the given jet, we first have to interpret it as a one jet with values in $(k-1)$–jets. This one jet has then as foot point a jet with top component $Y_1\otimes\dots\otimes Y_{k-1}\otimes w_{k-1}\otimes v_{k-1}$, while the jet part coincides in the two highest components with

$$Y_1\otimes(0,\dots,0,Y_2\otimes\dots\otimes Y_{k-1}\otimes w_{k-1}\otimes v_{k-1},0)+ \\ +Z_1\otimes(0,\dots,0,Z_2\otimes\dots\otimes Z_k\otimes w_k\otimes v_k).$$

Next we have to apply $\mathcal{J}^1(\Psi^{k-1}_W)$ to this, which means that we have to act with Ψ^{k-1}_W on the footpoint and on the value of the jet part. By the induction hypothesis the foot point of the result coincides in the top component with the sum of $(0,\dots,0,Y_1\otimes\dots\otimes Y_{k-1}\otimes w_{k-1})\otimes v_{k-1}$ with something whose top component vanishes under the action of $\mathfrak{g}_{-1}$.

On the other hand, the top degree of the jet part of the value coincides with

$$Z_1\otimes(0,\dots,0,Z_2\otimes\dots\otimes Z_k\otimes w_k)\otimes v_k+ \\ +\sum_{i=1}^{k-1}\sum_\ell Y_1\otimes(0,\dots,0,Y_2\otimes\dots\otimes Y_i\otimes\eta_\ell\otimes Y_{i+1}\otimes\dots\otimes Y_{k-1}\otimes w_{k-1})\otimes\xi_\ell\cdot v_{k-1}.$$

But applying $\Psi^1_{\mathcal{J}^{k-1}(W)}$, we get in the top degree the sum of this and the term

$$\sum_\ell \eta_\ell\otimes(0,\dots,0,Y_1\otimes\dots\otimes Y_{k-1}\otimes w_{k-1})\otimes\xi_\ell\cdot v_{k-1}.\square$$

1.8 The translation operators. We now start the construction of the second ingredient of the translation procedure, the translation operators. Fix a representation V of G which splits as $V_0\oplus V_1$ over $\mathfrak{g}_0$, such that the action of $\mathfrak{g}_{\pm1}$ maps each V_i to $V_{i\pm1}$, and an irreducible representation W of P. Then $W\otimes V$ splits as $(W\otimes V_0)\oplus(W\otimes V_1)$ over $\mathfrak{g}_0$ and $W\otimes V_1$ is a P–submodule, and thus we can form the quotient $(W\otimes V)/(W\otimes V_1)$, which we denote (with a slight abuse of notation) by $W\otimes V_0$. This again is a representation with trivial P_1–action. Thus we can read off two simple types of translation operators (which are actually of order zero) immediately: If $\tilde W\subset W\otimes V_1$ is an irreducible subrepresentation, then we have a canonical P–homomorphism $\tilde W\hookrightarrow W\otimes V$, and if $\tilde W'$ is a quotient of $W'\otimes V_0$, then there is a natural homomorphism $W'\otimes V\to\tilde W'$.

1.9. Next, let $\tilde W\subset W\otimes V_0$ be an irreducible subrepresentation. Consider the $\mathfrak{g}_0$–module $\mathfrak{g}_{-1}^*\otimes\tilde W$. Since $\mathfrak{g}_{-1}^*\cong\mathfrak{g}_1$ the action of $\mathfrak{g}_1$ defines a $\mathfrak{g}_0$–homomorphism $\partial^*:\mathfrak{g}_{-1}^*\otimes\tilde W\to W\otimes V_1$. (The notation ∂^* for the action comes from the relation to Spencer cohomology, see [CSS2, 1.3]). We denote the image of this map, which is a $\mathfrak{g}_0$–submodule of $W\otimes V_1$ by $\mathfrak{g}_1\cdot\tilde W$.

Now let $\{\xi_\ell\}$ and $\{\eta_\ell\}$ be dual bases of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ as before. Then $Z\otimes e\mapsto\sum_\ell\eta_\ell\otimes[Z,\xi_\ell]\cdot e$ is a well defined $\mathfrak{g}_0$–endomorphism of $\mathfrak{g}_{-1}^*\otimes\tilde W$. Suppose we can split $\mathfrak{g}_{-1}^*\otimes\tilde W=E\oplus F$ into a direct sum of $\mathfrak{g}_0$–submodules, which are invariant under the above endomorphism and such that F lies in the kernel of the action ∂^*. (Such a splitting always exists, taking $F=0$.) Then let Φ be the endomorphism of $\mathfrak{g}_{-1}^*\otimes\tilde W$ which coincides with the above one on E and is the identity on F.

If one has any finite dimensional module over a reductive Lie group or Lie algebra, then one may fix a decomposition into irreducibles, i.e. a way of writing the identity map as a sum of projections onto irreducible subrepresentations. Then any endomorphism f of the module can be represented by a block diagonal matrix (with block sizes corresponding to the multiplicities of the irreducible subrepresentations). We can then form the determinant and the classical adjoint of this matrix, which gives again rise to a module endomorphism. In [Cap] it is shown, that this endomorphism and the determinant are actually independent of the choice of the decomposition into irreducibles, so we denote them by $\Delta(f)$ and $\mathcal{C}_f$, respectively, and call them the determinant and the classical adjoint of f. In particular, if the given module is simply reducible, i.e. all irreducible components are different, then f must act by a scalar on each of these, and $\Delta(f)$ is just the product of these scalars, while $\mathcal{C}_f$ acts on each component with the product of all these scalars except the eigenvalue of f on this component.

Now consider the mapping $\mathcal{J}^1(\tilde W)\to W\otimes V$ which maps $(e_0,Z_1\otimes e_1)$ to $\Delta(\Phi)e_0+\partial^*(\mathcal{C}_\Phi(Z_1\otimes e_1))$. Obviously, this is a $\mathfrak{g}_0$–homomorphism. But for an element $Z_0\in\mathfrak{g}_1$ we have by definition $Z_0\cdot(e_0,Z_1\otimes e_1)=(0,\sum_\ell\eta_\ell\otimes[Z_0,\xi_\ell]\cdot e_0)$. By construction of Φ, we see that this element is mapped to $\partial^*(\mathcal{C}_\Phi(\Phi(Z_0\otimes e_0)))$ which equals $\Delta(\Phi)Z_0\cdot e_0$. But this shows that the above mapping is also $\mathfrak{g}_1$–equivariant and hence a $\mathfrak{p}$– and a P–homomorphism. But this means that we have a corresponding strongly invariant first order operator between the bundles corresponding to $\tilde W$ and to $W\otimes V$ as required.

1.10. The operators constructed above can be used in two ways: First, if $\Delta(\Phi)$ is nonzero, then we may divide by this number, thus obtaining a differential operator which splits the projection of natural vector bundles induced by the canonical projection $W \otimes V \to W \otimes V_0 \to E$.

On the other hand, if $\Delta(\Phi) = 0$ but $\mathcal{C}_\Phi \neq 0$, then we immediately get a homomorphism $\mathcal{J}^1(E) \to W \otimes V_1$ between two representations with trivial P_1 action, and thus a natural operator.

Finally, one should note that the algebra $\mathfrak{g}_0$ always has a one dimensional center, whose action can be changed independently of the action of the rest (this corresponds to changing the conformal weight in the case of conformal structures). It is easy to see, that this changes Φ by adding a multiple of the identity, so one can reach each of the two cases described above by appropriately choosing the action of the center, and the first case is the generic one.

1.11. The remaining translation operators are constructed in a very similar way: Let $\tilde{W}'$ be an irreducible component of $W' \otimes V_1$ and let π be the projection onto $\tilde{W}'$. As in 1.9 we have the action $\partial^* : \mathfrak{g}_{-1}^* \otimes W' \otimes V_0 \to W' \otimes V_1$, and as above we construct a homomorphism Φ', but this time corresponding to a splitting $\mathfrak{g}_{-1}^* \otimes W' \otimes V_0 = E \oplus F$ such that F is contained in the kernel of $\pi \circ \partial^*$.

Now $\mathcal{J}^1(W' \otimes V)$ splits into the four parts $W' \otimes V_0$, $W' \otimes V_1$, $\mathfrak{g}_{-1}^* \otimes W' \otimes V_0$, and $\mathfrak{g}_{-1}^* \otimes W' \otimes V_1$. We define a mapping $\mathcal{J}^1(W' \otimes V) \to \tilde{W}'$, which depends only on the middle two components, and maps a jet such that these are $w_1' \otimes v_1$ and $Z_0 \otimes w_0' \otimes v_0$ to $\pi\big(\Delta(\Phi')w_1' \otimes v_1 - \partial^*(\mathcal{C}_{\Phi'}(Z_0 \otimes w_0' \otimes v_0))\big)$.

Clearly, this is a $\mathfrak{g}_0$–homomorphism. Acting on such a jet with another element $Z \in \mathfrak{g}_1$, the two middle components of the result depend only on the first component of the jet, and if this is $w_0' \otimes v_0$, then the middle components of the result are $w_0' \otimes Z \cdot v_0$ and $\sum_\ell \eta_\ell \otimes [Z, \xi_\ell] \cdot (w_0' \otimes v_0)$, respectively. Using the construction of Φ' one immediately verifies that this element lies in the kernel of the homomorphism defined above, so we have actually constructed a $\mathfrak{p}$–homomorphism (and thus a P–homomorphism) $\mathcal{J}^1(W' \otimes V) \to \tilde{W}'$. Consequently, we also get a strongly invariant operator of order one between the corresponding bundles.

As in 1.10, it is easy to see that for $\Delta(\Phi') \neq 0$ one gets a differential splitting of the inclusion of natural vector bundles corresponding to the inclusion $\tilde{W}' \hookrightarrow W' \otimes V_1 \hookrightarrow W' \otimes V$. In the case where $\Delta(\Phi') = 0$ but $\mathcal{C}_{\Phi'} \neq 0$ we again get a homomorphism between bundles with trivial P_1–action and thus a natural operator. As before, one can switch between these two cases by adjusting the action of the center of $\mathfrak{g}_0$ appropriately.

1.12. Remark. With the constructions carried out in this section, we have four different types of translations at hand. We may go into $W \otimes V$ from a component of $W \otimes V_1$ with a zero order operator or from a component of $W \otimes V_0$ with a first order operator. On the other hand, we can go out of $W' \otimes V$ to a component of $W' \otimes V_0$ with a zero order operator, or to a component of $W' \otimes V_1$ with a first order operator.

It can be easily shown in general, that the simplest of these translations (the one with two operators of order zero) always decreases the order by one (i.e. translating a k–th order operator in this way one ends up with an operator of order $k-1$), while the two "medium" translations (using one operator of first order and one of order zero) preserves the order, and the most complicated translation (with two first order operators) increases the order by one.

It is an interesting problem in representation theory to discuss in general (independently of the structure in question), the properties of the maps Φ and Φ' with respect to their determinants and classical adjoints. Results in this direction will be proved elsewhere.

2. Almost Grassmannian structures

In this section we discuss generalities about almost Grassmannian structures and fix the notations for the computations in the next section. From now on, we will only deal with Lie algebra representations, leaving to the reader the trivial modifications in the group case.

2.1. The $|1|$–graded Lie algebra $\mathfrak{sl}(p+q,\mathbb{K})$. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, let $p,q \geq 2$ be integers and put $\mathfrak{g} := \mathfrak{sl}(p+q,\mathbb{K})$. Then $\mathfrak{g}$ admits a $|1|$–grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{g}_{-1} \cong \mathbb{K}^{p*} \boxtimes \mathbb{K}^q$, $\mathfrak{g}_0 \cong \mathfrak{sl}(p,\mathbb{K}) \oplus \mathfrak{sl}(q,\mathbb{K}) \oplus \mathbb{K}$ and $\mathfrak{g}_1 \cong \mathbb{K}^p \boxtimes \mathbb{K}^{q*}$. This grading is obvious from a block form with blocks of sizes p and q, respectively:

$$\mathfrak{g}_{-1} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \qquad \mathfrak{g}_0 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \qquad \mathfrak{g}_1 = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

We denote by $\mathfrak{p}$ the parabolic subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $\mathfrak{g}$.

We will often compute in a basis of $\mathfrak{g}$ using the following conventions: Indices named $a,b,c,\dots$ run from 1 to q, while indices named $i,j,k,\dots$ run from 1 to p. For the basis elements lower indices indicate elements of $\mathbb{K}^p$ or $\mathbb{K}^q$, while upper indices indicate elements of $\mathbb{K}^{p*}$ or $\mathbb{K}^{q*}$. Moreover, we will use the Einstein sum convention, so if an upper index equals a lower index then one automatically has to sum over these. We will denote by e the basis of $\mathfrak{g}$ consisting of elementary matrices. Thus basis elements of the form e_a^i belong to $\mathfrak{g}_{-1}$, those of the forms e_b^a and e_j^i belong to $\mathfrak{g}_0$, and those of the form e_i^a are in $\mathfrak{g}_1$.

In this language, one now easily computes the basic brackets, c.f. [CSS1, 3.3]:

$$\begin{aligned}
[\ ,\ ] &: \mathfrak{g}_0 \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}, & [(e_j^i, e_b^a), e_c^k] &= \delta_c^a e_b^k - \delta_j^k e_c^i \\
[\ ,\ ] &: \mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1, & [(e_j^i, e_b^a), e_k^c] &= \delta_k^i e_j^c - \delta_b^c e_k^a \\
[\ ,\ ] &: \mathfrak{g}_1 \times \mathfrak{g}_{-1} \to \mathfrak{g}_0, & [e_j^b, e_a^i] &= \delta_a^b e_j^i - \delta_j^i e_a^b
\end{aligned}$$

It is well known that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are dual with respect to the Cartan Killing form on $\mathfrak{g}$. Using the well known relation between the trace form and the Killing form on $\mathfrak{g}$, or by a direct computation one immediately sees, that the bases $\{e_i^a\}$ of $\mathfrak{g}_1$ and $\{e_a^i\}$ of $\mathfrak{g}_{-1}$ are dual with respect to the Killing form up to a factor $\frac{1}{2(p+q)}$. In particular, we may write the identity map as an element of $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}$ as $\frac{1}{2(p+q)} e_i^a \otimes e_a^i$ (using Einstein sum convention).

Next, we need a distinguished generator $\mathbb{I}$ of the center of $\mathfrak{g}_0$. This generator can be fixed by requiring that it acts as the identity on $\mathfrak{g}_1$ and as minus the identity on $\mathfrak{g}_{-1}$. A direct computation immediately shows that $\mathbb{I} = \frac{q}{p+q} e_i^i - \frac{p}{p+q} e_a^a$.

2.2. Representations. We will be concerned with two types of (finite dimensional) representations of the algebra $\mathfrak{p}$. On one hand, we will need representations which are obtained by extending representations of $\mathfrak{g}_0$ trivially to representations of $\mathfrak{p}$. In particular, all irreducible representations of $\mathfrak{p}$ are of this form. Specifically, the one dimensional representations are characterized by the action of the central element $\mathbb{I}$ from above. To avoid ugly numerical factors in our formulae and following the convention introduced in [Bailey–Eastwood, 2.3] we denote by $\mathbb{K}[1]$ the one dimensional representation $\Lambda^p\mathbb{K}^p \cong \Lambda^q\mathbb{K}^{q*}$. Consequently, for $\alpha \in \mathbb{K}$ we denote by $\mathbb{K}[\alpha]$ the one dimensional representation in which the action of the center is fixed by requiring that $\mathbb{I}\cdot x = \frac{\alpha(p+q)}{pq}x$ for all $x \in \mathbb{K}[\alpha]$. For any representation W of $\mathfrak{g}_0$, we will denote by $W[\alpha]$ the tensor product of W with $\mathbb{K}[\alpha]$. This corresponds to the change of conformal weight in the case of conformal structures. Now it is well known what the irreducible representations of $\mathfrak{g}_0$ look like. They can all be written as $(W_1 \boxtimes W_2)[\alpha]$ where W_1 is an irreducible representation of $\mathfrak{sl}(p,\mathbb{K})$ and W_2 is an irreducible representation of $\mathfrak{sl}(q,\mathbb{K})$. For all tensor representations we will use a coordinate notation similar to the one introduced in 2.1 above. Note that on these tensor representations we always take the natural action of the center. Thus the notation $(W_1 \boxtimes W_2)[\alpha]$ does *not* mean that $\mathbb{I}$ acts by multiplication with $\frac{\alpha(p+q)}{pq}$.

The second type of representations that we will need is representations of $\mathfrak{p}$ which are restrictions of representations of the whole algebra $\mathfrak{g}$. Specifically, we will need the defining representation V of $\mathfrak{g}$ and its dual V^*. Over $\mathfrak{g}_0$ we have splittings $V = V_0 \oplus V_1 = \mathbb{K}^q \oplus \mathbb{K}^p$ and $V^* = (V^*)_0 \oplus (V^*)_1 = \mathbb{K}^{p*} \oplus \mathbb{K}^{q*}$, and $\mathfrak{g}_{\pm 1}\cdot V_i \subset V_{i\pm 1}$ and similarly for V^*. The geometric objects corresponding to these representations are analogs of the tractors in conformal geometry, which were introduced in [Bailey–Eastwood–Gover].

Finally, we shall also need tensor products of these two types of representations. As above, we can split these (over $\mathfrak{g}_0$) as $W\otimes V = (W\otimes V_0)\oplus(W\otimes V_1)$ and similarly with V^*. We will often write elements in such representations as column vectors with two lines, the lower line corresponding to $W\otimes V_0$ and the upper one to $W\otimes V_1$.

2.3. Jet prolongations and invariant differential operators. As we have noticed in 1.3, for any representations W and W' of $\mathfrak{p}$ a homomorphism $\tilde{D} : \mathcal{J}^k(W) \to W'$ of $\mathfrak{p}$–modules gives rise to an invariant differential operator on manifolds with AHS–structures via the absolutely invariant derivative, see [CSS1, 5.7]. (In fact, one has to restrict to representations which actually integrate to representations of an appropriate group, but this requires only trivial modifications.) Once one has such an operator expressed in these terms, one can compute its formula in terms of usual covariant derivatives with respect to appropriate connections using the methods developed in section 4 of [CSS1] and the explicit formulae for the deformation tensors from [CSS2, 3.7]. Since this procedure is not related to the translations at all, we will not carry it out, but compute only the expressions of the operators in terms of absolutely invariant derivatives. In particular, the symbol ∇ will always denote an invariant derivative with respect to the canonical Cartan connection.

To write down expressions for such invariant operators, we will use an abstract index notation similar to the one developed in [Bailey–Eastwood]. Here we use upper primed indices to indicate $\mathbb{K}^p$–factors, lower primed indices to indicate $\mathbb{K}^{p*}$–factors, upper unprimed indices to indicate $\mathbb{K}^q$–factors, and lower unprimed indices to indicate $\mathbb{K}^{q*}$–factors. Moreover, round brackets indicate symmetrizations and square brackets indicate antisymmetrizations. Finally, lower zeros mean a trace free part. Thus, for example $\varphi_{B_0}^{(A'B')A}$ denotes a section of the natural bundle associated to the representation $S^2\mathbb{K}^p \boxtimes (\mathbb{K}^q \otimes K^{q*})_0$, the exterior tensor product of the symmetric square of $\mathbb{K}^p$ with the trace free part of $L(\mathbb{K}^q, \mathbb{K}^q)$, which is just the adjoint representation of $\mathfrak{sl}(q, \mathbb{K})$. The main purpose of this notation is to indicate appropriate traces by using the same letter in an upper and a lower index. The difference between the notation used here and the one used in [Bailey–Eastwood] is that we identify the tangent bundle with $\mathbb{K}^{p*} \boxtimes \mathbb{K}^q$ and not with $\mathbb{K}^p \boxtimes \mathbb{K}^q$. So in our setting an invariant derivative has the form $\nabla_A^{A'}$, and so on.

3. Examples of translations

3.1. We start with the simplest example, where $W = \mathbb{K}[\alpha]$ and V is the standard representation. In this case the zero component of $W \otimes V$ equals $\mathbb{K}^q[\alpha]$, and the one component is just $\mathbb{K}^p[\alpha]$, so they are both irreducible. The crucial step in the construction of the translation operators is to analyze the endomorphism $\mathfrak{g}_1 \otimes W \otimes V_0$ which maps $Z \otimes w$ to $\sum \eta_\ell \otimes [Z, \xi_\ell]\cdot w$. Computing in coordinates, we see that $e_j^b \otimes e_c$ is mapped to

$$\tfrac{1}{2(p+q)} e_i^a \otimes [e_j^b, e_a^i]\cdot e_c = \tfrac{1}{2(p+q)} e_i^a \otimes (\delta_a^b e_j^i - \delta_j^i e_a^b)\cdot e_c = \tfrac{1}{2(p+q)}(\alpha e_j^b \otimes e_c - \delta_c^b e_j^a \otimes e_a).$$

Over $\mathfrak{g}_0$, the module $\mathfrak{g}_1 \otimes W \otimes V_0 \cong \mathbb{K}^p \boxtimes (\mathbb{K}^q \otimes \mathbb{K}^{q*})$ splits into two irreducible components corresponding to the splitting of $\mathbb{K}^q \otimes \mathbb{K}^{q*}$ into trace part and trace free part, and the whole trace free part lies in the kernel of the action $\partial^* : \mathfrak{g}_1 \otimes W \otimes V_0 \to W \otimes V_1$. On the trace part, the map from above obviously acts by multiplication with $\frac{\alpha - q}{2(p+q)}$. Thus as the map Φ, which occurs in the construction of the translation operators in 1.9, we take the map which is the identity on the trace free part and acts by this scalar on the trace part. Its determinant $\Delta(\Phi)$ (c.f. 1.9) thus equals $\frac{\alpha - q}{2(p+q)}$, while its classical adjoint is just the identity. Thus we get the homomorphism $\mathcal{J}^1(\mathbb{K}^q[\alpha]) \to V[\alpha]$, respectively an operator on the corresponding bundles given by:

$$(e_a, e_i^b \otimes e_c) \mapsto \begin{pmatrix} \delta_c^b e_i \\ \frac{\alpha - q}{2(p+q)} e_a \end{pmatrix} \qquad \varphi^A \mapsto \begin{pmatrix} \nabla_A^{A'} \varphi^A \\ \frac{\alpha - q}{2(p+q)} \varphi^A \end{pmatrix}.$$

In particular, if we choose $\alpha = q$, then the lower component vanishes, and we get a homomorphism $\mathcal{J}^1(\mathbb{K}^q[q]) \to \mathbb{K}^p[q]$, and thus an invariant first order operator between the corresponding bundles defined by $\varphi^A \mapsto \nabla_A^{A'} \varphi^A$.

3.2. If we replace the standard representation V in 3.1 by its dual, then the computations are very similar. In this case we get a homomorphism $\mathcal{J}^1(\mathbb{K}^{p*}[\alpha]) \to$

$V^*[\alpha]$, respectively a first order operator between the corresponding bundles given by:

$$(e^i, e^a_j \otimes e^k) \mapsto \begin{pmatrix} \delta^k_j e^a \\ \frac{p-\alpha}{2(p+q)} e^i \end{pmatrix} \qquad \varphi_{A'} \mapsto \begin{pmatrix} \nabla^{A'}_A \varphi_{A'} \\ \frac{p-\alpha}{2(p+q)} \varphi_{A'} \end{pmatrix}.$$

In particular, for $\alpha = p$ there is a homomorphism $\mathcal{J}^1(\mathbb{K}^{p*}[p]) \to \mathbb{K}^{q*}[p]$, respectively a first order invariant operator between the corresponding bundles.

3.3. Next, we will discuss translations of the invariant operator D between the bundles corresponding to $W := \mathbb{K}^q[q]$ and $W' := \mathbb{K}^p[q]$ constructed in 3.1. Let us first translate it with the standard representation V. Let $\tilde{D}$ be the homomorphism on one–jets corresponding to the operator. Then by 1.7, the twisted homomorphism $\tilde{D}_V$ differs from $\tilde{D} \otimes \mathrm{id}_V$ by $\sum_\ell \tilde{D}(\eta_\ell \otimes w) \otimes \xi_\ell \cdot v$. Clearly, this depends only on the component $W \otimes V_1$ of $W \otimes V$, which is isomorphic to $\mathbb{K}^q \boxtimes \mathbb{K}^p[\alpha]$ and thus irreducible. This correction maps $e_b \otimes e_j$ to

$$\tfrac{1}{2(p+q)} \tilde{D}(e^a_i \otimes e_b) \otimes e^i_a \cdot e_j = \tfrac{1}{2(p+q)} \delta^a_b e_i \otimes \delta^i_j e_a = \tfrac{1}{2(p+q)} e_j \otimes e_b,$$

which is an element in $W' \otimes V_0 \cong \mathbb{K}^p \boxtimes \mathbb{K}^q[\alpha]$. Thus, the twisted operator D_V can be written as

$$\begin{pmatrix} \psi^{A'A} \\ \varphi^{BC} \end{pmatrix} \mapsto \begin{pmatrix} \nabla^{A'}_A \psi^{B'A} \\ \nabla^{C'}_B \varphi^{BC} + \frac{1}{2(p+q)} \psi^{C'C} \end{pmatrix}.$$

From this we can immediately read off, that the simplest translation, which uses two translation operators of order zero, just gives $\frac{1}{2(p+q)}$ times the identity on $\mathbb{K}^p \boxtimes \mathbb{K}^q[q]$.

3.4. To start the construction of the nontrivial translation operators, we have to consider the mapping $Z \otimes w \otimes v \mapsto \sum_\ell \eta_\ell \otimes [Z, \xi_\ell] \cdot (w \otimes v)$ on $\mathfrak{g}_1 \otimes W \otimes V_0 \cong \mathbb{K}^p \boxtimes (\mathbb{K}^{q*} \otimes \mathbb{K}^q \otimes \mathbb{K}^q)$. This endomorphism maps $e^b_j \otimes e_c \otimes e_d$ to

$$\begin{aligned} &\tfrac{1}{2(p+q)} e^a_i \otimes (\delta^b_a e^i_j - \delta^i_j e^b_a) \cdot (e_c \otimes e_d) = \\ &\qquad = \tfrac{1}{2(p+q)} (\alpha e^b_j \otimes e_c \otimes e_d - \delta^b_c e^a_j \otimes e_a \otimes e_d - \delta^b_d e^a_j \otimes e_c \otimes e_a). \end{aligned}$$

In this case, the situation is more complicated than before, since $W \otimes V_0$ splits into the two irreducible representations $\tilde{W}_0 := S^2\mathbb{K}^q$ and $\tilde{W}_1 := \Lambda^2\mathbb{K}^q$. First, we construct a homomorphism $\mathcal{J}^1(\tilde{W}_0) \to W \otimes V$: The module $\mathfrak{g}_1 \otimes \tilde{W}_0$ is the exterior tensor product of $\mathbb{K}^p$ with $\mathbb{K}^{q*} \otimes S^2\mathbb{K}^q$, which splits as a direct sum of $\mathbb{K}^q$ (the trace part) and another module (the trace free part). Since $W \otimes V_1 \cong \mathbb{K}^p \boxtimes \mathbb{K}^q[\alpha]$, the whole trace free part must be contained in the kernel of the action $\partial^* : \mathfrak{g}_1 \otimes W \otimes V_0 \to W \otimes V_1$. To compute the action of the above mapping on the trace part we have to evaluate it on elements of the form $e^b_j \otimes e_b \otimes e_c + e^b_j \otimes e_c \otimes e_b$, and one immediately verifies, that on such an element it acts by multiplication with $\frac{\alpha - q - 1}{2(p+q)}$.

Thus, the appropriate map Φ in the construction of the translation operators acts by this scalar on the trace part and as the identity on the trace free part. So its determinant is just the above scalar, while its classical adjoint is the identity. Thus for any α we get a homomorphism $\mathcal{J}^1(S^2\mathbb{K}^q[\alpha]) \to \mathbb{K}^q \otimes V[\alpha]$, respectively an operator between the corresponding bundles of the form:

$$(e_a \odot e_b, e^c_j \otimes (e_d \odot e_f)) \mapsto \begin{pmatrix} \delta^c_f e_d \otimes e_j + \delta^c_d e_f \otimes e_j \\ (\frac{\alpha-q-1}{2(p+q)})(e_a \otimes e_b + e_b \otimes e_a) \end{pmatrix}$$
$$\varphi^{(AB)} \mapsto \begin{pmatrix} \nabla^{A'}_A \varphi^{(AB)} \\ (\frac{\alpha-q-1}{2(p+q)})\varphi^{(AB)} \end{pmatrix}.$$

In particular, we get a homomorphism $\mathcal{J}^1(S^2\mathbb{K}^q[q+1]) \to \mathbb{K}^p \boxtimes \mathbb{K}^q[q+1]$, and thus a first order invariant operator between the corresponding bundles.

On the other hand, to use this for translation of the operator D from 3.1, we have to put $\alpha = q$, which reduces the factor in the above formula to $\frac{-1}{2(p+q)}$. Composing the operator D_V from 3.3 with our operator we get a second order operator between the bundles corresponding to $S^2\mathbb{K}^q[q]$ and $\mathbb{K}^q \otimes V[q]$. Computing this composition, one gets $\varphi^{(AB)} \mapsto \begin{pmatrix} \nabla^{B'}_B \nabla^{A'}_A \varphi^{(AB)} \\ 0 \end{pmatrix}$. Thus we see that the first type of "medium" translations gives only the zero operator, but the top component immediately gives us a second order operator between the bundles corresponding to $S^2\mathbb{K}^q[q]$ and $\mathbb{K}^p \otimes \mathbb{K}^p[q]$. To come to an irreducible target, we have to either symmetrize or antisymmetrize. Symmetrizing, we get a true second order operator, given by $\varphi^{(AB)} \mapsto \nabla^{(B'}_B \nabla^{A')}_A \varphi^{(AB)}$. Antisymmetrizing on the other hand, one gets a commutator of absolutely invariant derivatives and thus an operator of order zero involving the curvature of the canonical Cartan connection (see [CSS1, 2.5]). In fact, the operators obtained here equal the result of the most complicated translation, since the other translation operator equals a scalar multiple of the identity on elements having a zero in the lower row.

3.5. Starting with the component $\tilde{W}_1 = \Lambda^2\mathbb{K}^q[\alpha]$, everything looks quite similar. As above, the module $\mathfrak{g}_1 \otimes \tilde{W}_1$ splits into a trace part and a trace free part, which lies in the kernel of the action. The eigenvalue on the trace part is easily seen to be $\frac{\alpha-q+1}{2(p+q)}$. The homomorphism $\mathcal{J}^1(\Lambda^2\mathbb{K}^q[\alpha]) \to (\mathbb{K}^q \otimes V)[\alpha]$ respectively the corresponding operator is given by

$$(e_a \wedge e_b, e^c_j \otimes (e_d \wedge e_f)) \mapsto \begin{pmatrix} \delta^c_f e_d \otimes e_j - \delta^c_d e_f \otimes e_j \\ (\frac{\alpha-q+1}{2(p+q)})(e_a \otimes e_b - e_b \otimes e_a) \end{pmatrix}$$
$$\varphi^{[AB]} \mapsto \begin{pmatrix} \nabla^{A'}_A \varphi^{[BA]} \\ (\frac{\alpha-q+1}{2(p+q)})\varphi^{[AB]} \end{pmatrix}.$$

In particular, we get a homomorphism $\mathcal{J}^1(\Lambda^2\mathbb{K}^q[q-1]) \to \mathbb{K}^p \boxtimes \mathbb{K}^q[q-1]$, and thus a first order invariant operator between the corresponding bundles.

In the case $\alpha = q$, which is relevant for translation, the scalar in the formula becomes $\frac{1}{2(p+q)}$. Composing D_V from 3.3 with the resulting operator again gives zero in the lower row and immediately an invariant operator in the top row. This time, we get by antisymmetrizing a true second order operator between the bundles corresponding to $\Lambda^2\mathbb{K}^q[q]$ and $\Lambda^2\mathbb{K}^p[q]$, which is given by $\varphi^{[AB]}\mapsto\nabla_A^{[A'}\nabla_B^{B']}\varphi^{[AB]}$, while symmetrizing gives an operator of order zero which involves curvatures.

3.6. We finish the discussion of translation of the operator from 3.1 with a few remarks:
(1) If one considers in this situation the translation operators as constructed in 1.11, one has to deal with homomorphisms $\mathcal{J}^1(\mathbb{K}^p\otimes V[\alpha])\to\mathbb{K}^p\otimes\mathbb{K}^p[\alpha]$, and to obtain an irreducible target one has to symmetrize or antisymmetrize in the end, and we only consider the case of symmetrization. Following the general scheme described in 1.11, one gets the operator:

$$\begin{pmatrix}\psi^{A'B'}\\ \varphi^{C'B}\end{pmatrix}\mapsto\left(\tfrac{\alpha-q+1}{2(p+q)}\right)\psi^{(A'B')}-\nabla_A^{(A'}\varphi^{B')A}.$$

In particular, we get a first order invariant operator between the bundles corresponding to $\mathbb{K}^p\boxtimes\mathbb{K}^q[q-1]$ and $S^2\mathbb{K}^p[q-1]$.

Concerning translations, the situation is then somehow dual to the one we met in 3.4: Composing this operator with the operator D_V from 3.3 one immediately sees that this depends only on the φ–component. Thus, the "medium" translation is once more trivial and we directly get the most complicated translation, which just gives the operator from 3.4.
(2) Sticking to the symmetric case, the construction of 3.3 and 3.4 can be iterated. Inductively, one gets for each $k\geq 1$ a homomorphism $\mathcal{J}^k(S^k\mathbb{K}^q[q])\to S^k\mathbb{K}^p[q]$, and thus a k–th order operator between the corresponding bundles. The formula for the corresponding operator is always $\varphi^{(A\dots B)}\mapsto\nabla_A^{(A'}\dots\nabla_B^{B')}\varphi^{(A\dots B)}$.

In fact, the situation is always as in 3.4, i.e. the "medium" translations are automatically trivial.
(3) It is easy to verify, that the translation carried out in 3.3 and 3.4 can actually be reversed. If one computes the simplest translation of the second order operator from 3.4 with the dual V^* of the standard representation, then up to a nonzero factor one gets the first order operator from 3.1.
(4) If one tries to translate the operator D from 3.1 with the dual of the standard representation, then the result is less nice. The simplest translation gives only a nonzero multiple of the identity on $\mathbb{K}[\alpha]$, which can be viewed as reversing the translation from 3.1. The two "medium" translations lead to some nonzero first order operators, which can easily be obtained directly similar to the one in 3.1. The most complicated translation gives a prospective second order operator between $\mathbb{K}^{p*}\boxtimes\mathbb{K}^q[\alpha]$ and $\mathbb{K}^p\otimes\mathbb{K}^{q*}[\alpha]$, which would be highly interesting since these are (up to the central factor) just vector fields and one–forms, respectively. Unfortunately, the homomorphism one obtains by translation gives an operator of order zero involving the curvature of the canonical Cartan connection. It can be verified by a direct computation that there is no second order operator with nonzero symbol between these bundles.

3.7. We now turn to an example in which we translate a standard operator to a nonstandard operator. To get the starting point for this translation, we consider the representation $W\colon = \Lambda^2\mathbb{K}^{q*}[\alpha]$. Tensorizing this with the standard representation V of $\mathfrak{sl}(p+q,\mathbb{K})$ we get the zero component $\Lambda^2\mathbb{K}^{q*}\otimes\mathbb{K}^q[\alpha]$ which contains $\tilde{W}_0\colon = \mathbb{K}^{q*}[\alpha]$ as an irreducible component (the trace part). To construct the translation operator $\mathcal{J}^1(\tilde{W}_0)\to W\otimes V$ we have to analyze the mapping $\mathfrak{g}_1\otimes\tilde{W}_0\to\mathfrak{g}_1\otimes\tilde{W}_0$ defined by $Z\otimes w\otimes v\mapsto\sum_\ell\eta_\ell\otimes[Z,\xi_\ell]\cdot(w\otimes v)$. Viewed as an endomorphism of $\mathfrak{g}_1\otimes\otimes^2\mathbb{K}^{q*}\otimes\mathbb{K}^q$ this maps $e^b_j\otimes e^c\otimes e^d\otimes e_f$ to

$$\begin{aligned}&\tfrac{1}{2(p+q)}e^a_i\otimes(\delta^b_a e^i_j-\delta^i_j e^b_a)\cdot(e^c\otimes e^d\otimes e_f)=\\ &=\tfrac{1}{2(p+q)}\big(\alpha e^b_j\otimes e^c\otimes e^d\otimes e_f+e^c_j\otimes e^b\otimes e^d\otimes e_f+e^d_j\otimes e^c\otimes e^b\otimes e_f-\delta^b_f e^a_j\otimes e^c\otimes e^d\otimes e_a\big).\end{aligned}$$

The submodule $\mathfrak{g}_1\otimes\tilde{W}_0$ is generated by all elements of the form $e^b_j\otimes e^c\otimes e^d\otimes e_d-e^b_j\otimes e^d\otimes e^c\otimes e_d$, and this element is mapped to

$$\tfrac{1}{2(p+q)}\big(\alpha e^b_j\otimes e^c\otimes e^d\otimes e_d-\alpha e^b_j\otimes e^d\otimes e^c\otimes e_d+e^c_j\otimes e^b\otimes e^d\otimes e_d-e^c_j\otimes e^d\otimes e^b\otimes e_d\big).$$

Finally, the module $\mathfrak{g}_1\otimes\tilde{W}_0$ is isomorphic to $\mathbb{K}^p\boxtimes\mathbb{K}^{q*}\otimes\mathbb{K}^{q*}$, so it splits into two irreducible components corresponding to the symmetric and antisymmetric parts in the second component. Since $W\otimes V_1\cong\mathbb{K}^p\boxtimes\Lambda^2\mathbb{K}^{q*}$, the symmetric part must lie in the kernel of the action $\partial^*:\mathfrak{g}_1\otimes\tilde{W}_0\to W\otimes V_1$, so we just have to compute the eigenvalue of the above endomorphism on the skew symmetric part. From the last formula it is easy to see that this eigenvalue is just $\frac{\alpha-1}{2(p+q)}$.

Putting $\alpha=1$ we get a homomorphism $\mathcal{J}^1(\mathbb{K}^{q*}[1])\to\mathbb{K}^p\boxtimes\Lambda^2\mathbb{K}^{q*}[1]$ and an operator between the corresponding bundles, which is given by $\varphi_A\mapsto\nabla^{A'}_{[A}\varphi_{B]}$.

3.8. We want to translate the operator D between the bundles corresponding to $W\colon = \mathbb{K}^{q*}[1]$ and $W'\colon = \mathbb{K}^p\boxtimes\Lambda^2\mathbb{K}^{q*}[1]$ obtained in the end of 3.7 with the standard representation V. First, we compute the twisted operator D_V. Similarly as in 3.3, the difference between the corresponding homomorphism $\tilde{D}_V$ and $\tilde{D}\otimes\mathrm{id}_V$ depends only on the component $W\otimes V_1\cong\mathbb{K}^p\boxtimes\mathbb{K}^{q*}$, and one computes that it maps $e^b\otimes e_j$ to $\frac{1}{2(p+q)}\tilde{D}(e^a_i\otimes e^b)\otimes e^i_a\cdot e_j=\frac{1}{2(p+q)}(e^a\otimes e^b\otimes e_j\otimes e_a-e^b\otimes e^a\otimes e_j\otimes e_a)$. Thus, the twisted operator can be written as

$$\begin{pmatrix}\psi^{A'}_A\\ \varphi^B_C\end{pmatrix}\mapsto\begin{pmatrix}\nabla^{A'}_{[A}\psi^{B'}_{B]}\\ \nabla^{C'}_{[C}\varphi^D_{E]}-\frac{1}{2(p+q)}\psi^{C'}_{[C}\,\mathrm{id}^D_{E]}\end{pmatrix}.$$

Note that the last term in the lower line corresponds just to the inclusion of $\mathbb{K}^p\boxtimes\mathbb{K}^{q*}$ into $\mathbb{K}^p\boxtimes\Lambda^2\mathbb{K}^{q*}\otimes\mathbb{K}^q$ as the trace part. Thus, from the formula for the twisted operator one reads off, that the result of the simplest translation (using two operators of order zero) is the identity map on $\mathbb{K}^p\boxtimes\mathbb{K}^{q*}[1]$ plus the zero map to the other irreducible component of $\mathbb{K}^p\boxtimes\Lambda^2\mathbb{K}^{q*}\otimes\mathbb{K}^q$.

3.9. To start the construction of the more complicated translations we have to analyze the endomorphism $Z\otimes w\otimes v\mapsto\sum_\ell\eta_\ell\otimes[Z,\xi_\ell]\cdot(w\otimes v)$ on $\mathfrak{g}_1\otimes W\otimes V_0\cong\mathbb{K}^p\boxtimes(\mathbb{K}^{q*}\otimes\mathbb{K}^{q*}\otimes\mathbb{K}^q)$. This maps $e^b_j\otimes e^c\otimes e_d$ to

$$\begin{aligned}&\tfrac{1}{2(p+q)}e^a_i\otimes(\delta^b_a e^i_j-\delta^i_j e^b_a)\cdot(e^c\otimes e_d)=\\ &\qquad=\tfrac{1}{2(p+q)}\big(\alpha e^b_j\otimes e^c\otimes e_d+e^c_j\otimes e^b\otimes e_d-\delta^b_d e^a_j\otimes e^c\otimes e_a\big).\end{aligned}$$

Now $W \otimes V_0 \cong \mathbb{K}^{q*} \otimes \mathbb{K}^q$ splits into two components, one isomorphic to $\mathbb{K}[\alpha]$ (the trace part), the other one isomorphic to the adjoint representation of $\mathfrak{sl}(q, \mathbb{K})$ (the trace free part). We are interested in the part $\tilde{W} = \mathbb{K}[\alpha]$. Thus, we have to evaluate the above mapping on the element $e^b_j \otimes e^c \otimes e_c$ and clearly it acts just by multiplication by $\frac{\alpha}{2(p+q)}$. Hence, similarly as in 3.1, we get a homomorphism $\mathcal{J}^1(\mathbb{K}[\alpha]) \to \mathbb{K}^{q*} \otimes V[\alpha]$, respectively an operator between the corresponding bundles given by

$$(1, e^b_j) \mapsto \begin{pmatrix} e^b \otimes e_j \\ \frac{\alpha}{2(p+q)} e^a \otimes e_a \end{pmatrix} \qquad \varphi \mapsto \begin{pmatrix} \nabla^{A'}_A \varphi \\ \frac{\alpha}{2(p+q)} \varphi \, \mathrm{id}^B_C \end{pmatrix}.$$

In particular, this gives an invariant operator for $\alpha = 0$, which is just the exterior derivative from functions to one forms.

To translate the operator D from 3.7, we have to insert $\alpha = 1$. Composing our translation operator with the twisted operator from 3.8 we get a second order operator between the bundles corresponding to $\mathbb{K}[1]$ and $\mathbb{K}^p \boxtimes \Lambda^2\mathbb{K}^{q*} \otimes V[1]$, which is given by $\varphi \mapsto \begin{pmatrix} \nabla^{A'}_{[A} \nabla^{B'}_{B]} \varphi \\ 0 \end{pmatrix}$. As before, the "middle" translation is automatically trivial, and we immediately get a second order operator between the bundles corresponding to $\mathbb{K}[1]$ and $\mathbb{K}^p \otimes \mathbb{K}^p \boxtimes \Lambda^2\mathbb{K}^{q*}[1]$. To get an irreducible target, we have to either symmetrize or antisymmetrize in the $\mathbb{K}^p$–factor, and symmetrizing we get an operator of order zero involving the curvature of the canonical Cartan connection. The remaining interesting part is a second order operator between the bundles corresponding to $\mathbb{K}[1]$ and $\Lambda^2\mathbb{K}^p \boxtimes \Lambda^2\mathbb{K}^{q*}[1]$ given by $\varphi \mapsto \nabla^{[A'}_{[A} \nabla^{B']}_{B]} \varphi$.

3.10. Remarks. (1) The relevance of the example given in 3.9 shows up, if one looks at the classification of invariant operators in the flat case in terms of Bernstein–Gelfand–Gelfand resolutions (see [Baston–Eastwood, chapter 8]). In this classification, the operator constructed in the end of 3.7 occurs as a standard operator in a pattern corresponding to a singular infinitesimal character, while the operator in 3.9 to which it is translated, occurs as a non–standard operator in a pattern corresponding to a different (but also singular) infinitesimal character. This shows, that our method allows translations between various infinitesimal characters, and from standard to nonstandard operators.

(2) It is well known that almost Grassmannian structures in the case $p = q = 2$ coincide with conformal structures in dimension 4. In this case, $\Lambda^2\mathbb{K}^{q*} \cong \Lambda^2\mathbb{K}^p \cong \mathbb{K}[1]$, so the operator from 3.7 actually goes between the bundles corresponding to $\mathbb{K}^{q*}[1]$ and $\mathbb{K}^p[2]$. It can be verified, that this is just a Dirac operator. The operator from 3.9 goes between the bundles corresponding to $\mathbb{K}[1]$ and $\mathbb{K}[3]$ and is actually the conformally invariant Laplacian.

(3) It should be remarked that in the case $q = 2$ the translation carried out in 3.9 coincides with the one from 3.5.

3.11. We finish with a more complicated example by carrying out the most complicated translation of the operator obtained in 3.9 with the standard representation. For simplicity, we will carry out the last step only in the case $p = q = 2$.

First, we construct in general the twisted operator corresponding to the one constructed in 3.9 and the standard representation. The correction of the corresponding homomorphism compared to $\tilde{D}\otimes \mathrm{id}_V$ is of first order and depends only on the component $\mathfrak{g}_{-1}\otimes \mathbb{K}^p[1]$ of $\mathfrak{g}_{-1}\otimes V[1]$. Using the formula from proposition 1.7 we see that this correction maps $e^b_j\otimes e_k$ to

$$\tfrac{1}{2(p+q)}\tilde{D}(e^a_i\otimes e^b_j+e^b_j\otimes e^a_i)\otimes e^i_a{\cdot}e_k=\tfrac{1}{p+q}e^a\wedge e^b\otimes e_k\wedge e_j\otimes e_a.$$

Expressing this in terms of operators, we see that the twisted operator, which maps sections of the bundle corresponding to $V[1]$ to sections of the bundle corresponding to $\Lambda^2\mathbb{K}^p\boxtimes\Lambda^2\mathbb{K}^q\otimes V[1]$, is given by

$$\begin{pmatrix}\varphi^{A'}\\ \psi^A\end{pmatrix}\mapsto\begin{pmatrix}\nabla^{[A'}_{[A}\nabla^{B']}_{B]}\varphi^{C'}\\ \nabla^{[D'}_{[C}\nabla^{E']}_{D]}\psi^F+\frac{1}{p+q}\nabla^{[D'}_{[C}\varphi^{E']}\mathrm{id}^E_{D]}\end{pmatrix}.$$

The zero component $\mathbb{K}^q[1]$ of $V[1]$ is irreducible, and we can read off the corresponding translation operator $\mathcal{J}^1(\mathbb{K}^q[1])\to V[1]$ directly from 3.1. Putting $\alpha=1$ in the last formula of 3.1 we see that the corresponding operator is given by $\varphi^A\mapsto\begin{pmatrix}\nabla^{A'}_A\varphi^A\\ \frac{1-q}{2(p+q)}\varphi^A\end{pmatrix}$. Composing this with the twisted operator we get a third order operator between the bundles corresponding to $\mathbb{K}^q[1]$ and $\Lambda^2\mathbb{K}^p\boxtimes\Lambda^2\mathbb{K}^q\otimes V[1]$, which is given by

$$\varphi^A\mapsto\begin{pmatrix}\nabla^{[A'}_{[A}\nabla^{B']}_{B]}\nabla^{C'}_C\varphi^C\\ \frac{1-q}{2(p+q)}\nabla^{[D'}_{[D}\nabla^{E']}_{E]}\varphi^F-\frac{1}{p+q}\mathrm{id}^F_{[D}\nabla^{[D'}_{E]}\nabla^{E']}_G\varphi^G\end{pmatrix}.$$

3.12. To finish the translation, we have to construct the second translation operator. We will only do this in the case $p=q=2$, in which $\Lambda^2\mathbb{K}^p\boxtimes\Lambda^2\mathbb{K}^q\otimes V[1]\cong V[3]$. Thus, in this case there is only one relevant translation operator, which corresponds to a homomorphism $\mathcal{J}^1(V[3])\to\mathbb{K}^p[3]$. To construct this operator following the procedure from 1.11, we need the mapping Φ', which we can immediately read off from 3.1. In the case $\alpha=3$ which is relevant here, we see from 3.1 that $\Delta(\Phi')=\frac{1}{2(p+q)}$, while the classical adjoint of Φ' is just the identity. Thus, we see from 1.11 that the homomorphism $\mathcal{J}^1(V[3])\to\mathbb{K}^p[3]$, respectively the corresponding translation operator is given by

$$\begin{pmatrix}\dots\\ e^b_j\otimes e_c\\ e_k\\ \dots\end{pmatrix}\mapsto\tfrac{1}{2(p+q)}e_k-\delta^b_c e_j\qquad\begin{pmatrix}\varphi^{A'}\\ \psi^A\end{pmatrix}\mapsto\tfrac{1}{2(p+q)}\varphi^{A'}-\nabla^{A'}_A\psi^A.$$

Composing this with the operator obtained in the end of 3.11 we get (in the case $p=q=2$) a third order operator between the bundles corresponding to $\mathbb{K}^q[1]$ and $\mathbb{K}^p[3]$, which is given by (omitting a factor $\frac{1}{2(p+q)}$)

$$\varphi^A\mapsto\nabla^{[A'}_{[A}\nabla^{B']}_{B]}\nabla^{C'}_C\varphi^C+\nabla^{C'}_C\nabla^{[A'}_{[A}\nabla^{B']}_{B]}\varphi^C+2\nabla^{C'}_{[A}\nabla^{[A'}_{B]}\nabla^{B']}_C\varphi^C.$$

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Institut für Mathematik, Universität Wien
Strudlhofgasse 4, 1090 Wien, AUSTRIA
E-mail: Andreas.Cap@esi.ac.at