ARCHIVUM MATHEMATICUM (BRNO)
Tomus 32 (1996), 163 – 165

SEMIMODULARITY IN LOWER CONTINUOUS STRONGLY DUALLY ATOMIC LATTICES

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Abstract. For lattices of finite length there are many characterizations of semimodularity (see, for instance, Grätzer [3] and Stern [6]–[8]). The present paper deals with some conditions characterizing semimodularity in lower continuous strongly dually atomic lattices. We give here a generalization of results of paper [7].

1. Preliminaries

Let L be a lattice. By $[a,b]$ $(a \le b, a, b \in L)$ we denote an interval, that is the set of all $c \in L$ for which $a \le c \le b$. For $a, b \in L$ we say that a is a lower cover of b and we write $a \prec b$ if and only if $a < b$ and $[a,b] = \{a, b\}$.

A lattice L is called strongly dually atomic (see [4]), if for any $a, b \in L$ with $a < b$ there is $p \in [a,b]$ such that $p \prec b$. A complete lattice L is lower continuous, if $a \vee \bigwedge C = \bigwedge\{a \vee c : c \in C\}$ for all $a \in L$ and for all chains C in L.

Semimodularity is usually defined as follows:

Definition. A lattice L is called (upper) semimodular, if for all $a, b \in L$, $a \wedge b \prec a$ implies $b \prec a \vee b$.

It is immediate that modular lattices and geometric lattices are semimodular. There are many semimodular lattices being neither modular nor geometric (see Birkhoff [1], Crawley-Dilworth [2], Grätzer [3] and Stern [8]).

2. Results

First we put $J(L) := \{u \in L : u = a \vee b$ implies $u = a$ or $u = b\}$. The elements of $J(L)$ are called the join-irreducibles of L. In a strongly dually atomic lattice L the unique lower cover of a join-irreducible $(0 \ne) u \in J(L)$ is denoted by u'. As a preparation we need.

1991 *Mathematics Subject Classification*: 06C10.
Key words and phrases: lower continuous lattices, strongly dually atomic lattices, semimodular and atomic lattices.
Received April 11, 1995.

Lemma. *Let L be a lower continuous strongly dually atomic lattice. If $p \prec q$ $(p, q \in L)$, then there exists a join-irreducible $u \in J(L)$ such that $p \vee u = q$ and $p \wedge u = u'$.*

Proof. Consider the set $T := \{t \in L : p \vee t = q\}$. T is nonempty, since $q \in T$. Let C be a chain in T. The lower continuity yields

$$p \vee \bigwedge C = \bigwedge \{p \vee c : c \in C\} = q\,.$$

Thus $\bigwedge C \in T$, and T contains a minimal element u, by Zorn's lemma. Clearly, $u \in J(L)$, $p \vee u = q$ and from $u \nleq p$ it follows that $p \wedge u \leq u'$.
Observe that $u' \leq p$. Indeed, if $u' \nleq p$, then $p \vee u' = q$, that is $u' \in T$ and $u' < u$, contradicting the minimality of u. Thus we have $u' \leq p \wedge u$. Hence we obtain $p \wedge u = u'$ which completes the proof. □

Remark 1. For lattices of finite length this lemma was proved in Stern [5] (see also [8], p. 25).

Our main result is the following

Theorem 1. *Let L be a lower continuous strongly dually atomic lattice. Then the following conditions are equivalent:*

(i) *L is semimodular,*
(ii) *L satisfies the exchange property for join-irreducibles, i.e., for all $u, v \in J(L)$ and arbitrary $b \in L$, $v \leq b \vee u$ and $v \nleq b \vee u'$ imply $u \leq b \vee v \vee u'$,*
(iii) *$b \wedge u \prec u$ implies $b \prec b \vee u$ for all $u \in J(L)$ and $b \in L$.*

Proof. Implication (i) ⇒ (ii) follows from the proof of Theorem of [7].

(ii) ⇒ (iii). Suppose that (iii) does not hold. Let $u \in J(L)$, $b, q \in L$ be elements such that $u \wedge b = u' \prec u$ and $b < q < b \vee u$. Since L is strongly dually atomic there is an element $p \in L$ such that $b \leq p \prec q < b \vee u$.
By Lemma we get the existence of a join-irreducible element $v \in J(L)$ with $p \vee v = q$. It follows that $v \leq b \vee u$ and $v \nleq b = b \vee u'$. Applying (ii) we obtain that $u \leq b \vee v \vee u' = b \vee v$. Then $b \vee u \leq b \vee v \leq q$. This contradiction shows that (iii) holds.

(iii) ⇒ (i). Let $a, b \in L$ be elements for which $a \wedge b \prec a$. Without loss of generality we may assume that a, b are incomparable elements. By Lemma, there exists a join-irreducible element $u \in J(L)$ such that $(a \wedge b) \vee u = a$ and $b \wedge u = u'$. Applying (iii) we get that $b \prec b \vee u$. Since $a \vee b = (a \wedge b) \vee u \vee b = b \vee u$, we obtain $b \prec a \vee b$, which shows that L is semimodular. □

Remark 2. Since every lattice of finite length is lower continuous and strongly dually atomic, from this theorem it follows Theorem of [7].

We recall that a lattice L is atomistic if every non-zero element of L is a join of atoms. In an atomistic lattice each join-irreducible element is an atom. Then, as the consequence of Theorem 1 we get the following result which is a generalization of Corollary of [7].

Theorem 2. *Let L be an atomistic lower continuous strongly dually atomic lattice. Then the following conditions are equivalent:*

(i) *L is semimodular,*

(ii) *L satisfies the Steinitz-MacLane exchange property, that is, for all atoms $p, q \in L$ and for arbitrary $b \in L$, the relations $p \leq b \vee q$ and $p \not\leq b$ imply $q \leq b \vee p$,*

(iii) *L has the covering property, i.e., $b \wedge p = 0$ implies $b \prec b \vee p$ for any atom $p \in L$ and for arbitrary $b \in L$.*

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