ARCHIVUM MATHEMATICUM (BRNO)
Tomus 32 (1996), 195 – 209

PERIODIC SOLUTIONS FOR NONLINEAR EVOLUTION INCLUSIONS

Dimitrios A. Kandilakis and Nikolaos S. Papageorgiou

Abstract. In this paper we prove the existence of periodic solutions for a class of nonlinear evolution inclusions defined in an evolution triple of spaces (X, H, X^*) and driven by a demicontinuous pseudomonotone coercive operator and an upper semicontinuous multivalued perturbation defined on $T \times X$ with values in H. Our proof is based on a known result about the surjectivity of the sum of two operators of monotone type and on the fact that the property of pseudomonotonicity is lifted to the Nemitsky operator, which we prove in this paper.

1. Introduction

In this paper we study the problem of existence of periodic solutions for evolution inclusions driven by time-dependent, demicontinuous, coercive pseudomonotone operators defined within the framework of an evolution triple of spaces.

A very common approach in dealing with periodic problems of evolution equations, is to impose conditions on the perturbation term, which guarantee the uniqueness of the solution of the corresponding Cauchy problem and thus make it easier to apply on the Poincare map one of the classical fixed point theorems. The first major result in this direction is due to Browder [4] who considers semilinear systems in a Hilbert space driven by a monotone operator with a single-valued perturbation term $f(t, x)$, which is monotone in x. Browder uses the fixed point theorem for nonexpansive maps in a uniformly convex Banach space (in particular in a Hilbert space; see for example Brezis [3] theorem 1.2, p.5 or Goebel-Kirk [9], theorem 4.1, p.40).

The next major result on periodic solutions for evolution equations, can be traced in the work of Prüss [17]. Prüss considers semilinear, time invariant systems and abandons the monotonicity condition on $f(t, .)$ in favor of a Nagumo-type tangential condition. He also assumes that the linear unbounded operator governing the equation generates a compact semigroup or alternatively that the single-valued perturbation term $f(t, x)$ is compact. Subsequently Becker [2] considered

1991 *Mathematics Subject Classification*: 34G20, 34C25, 34G99.
Key words and phrases: evolution triple, compact embedding, pseudomonotone operator, demicontinuity, coercive operator, dominated convergence theorem.
Received October 23, 1995.

semilinear equations driven by a closed densely defined linear operator $A(.)$ which generates a compact semigroup. Using a perturbation term of special form and an extra condition which essentially amounts to saying that $A - \lambda I$ is m-accretive for some $\lambda > 0$, Becker proves the existence of a unique periodic solution.

The first fully nonlinear existence results for the periodic problem, were obtained recently by Vrabie [19] and Hirano [11]. Vrabie's work can be viewed as a nonlinear extension of Becker's result. He assumes that the nonlinear time invariant operator A monitoring the evolution equation is such that $A - \lambda I$ is m-accretive for some $\lambda > 0$ and that $-A$ generates a compact semigroup (via the Crandall-Liggett exponential formula). His perturbation term $f(t, x)$ is single valued, satisfies the Caratheodory conditions (i.e. is measurable in t and continuous in x) and also obeys a rather restrictive assymptotic growth condition. Hirano on the other hand considers an evolution equation defined in a Hilbert space, driven by a time invariant nonlinear operator of the subdifferential type which generates a compact semigroup of contractions and with a single-valued Caratheodory perturbation term of sublinear growth, which satisfies a unilateral condition.

From all the above works none considers multivalued perturbations (i.e. evolution inclusions). The only work in this direction is that of Hu-Papageorgiou [12], who consider evolution inclusions defined in an evolution triple of Hilbert spaces and by using a tangential condition and Galerkin approximations they proved the existence of a periodic solution for a problem with a Caratheodory multivalued perturbation $F(t, x)$ defined on $T \times H$ into H. This result was recently improved by Hu-Papageorgiou[13].

Our work here extends all the above mentioned results. We deal with time-dependent systems having multivalued perturbations and which are defined within the framework of an evolution triple of spaces (X, H, X^*). We only assume that X embeds compactly in H. This hypothesis does not imply that $A(t, .)$, $t \in T$, generates a compact semigroup or that the perturbation term $F(t, x)$ (assumed to be multivalued, defined on $T \times X$ with values in H and demicontinuous in x) is actually compact. Our proof is based on a general surjectivity result for a certain sum of operators of monotone type and on a proposition which shows that the property of pseudomonotonicity can be lifted to the Nemitsky (superposition) operator.

2. Mathematical preliminaries

In this section we fix our notation and briefly recall some basic definitions and facts from the theory of multifunctions and from nonlinear functional analysis, which we will need in the sequel.

So let (Ω, Σ) be a measurable space and Y a separable Banach space. We will be using the following notations:

$P_{f(c)}(Y) = \{A \subseteq Y : A \text{ is nonempty, closed (and convex)}\}$

and $P_{(w)k(c)} = \{A \subseteq Y : A \text{ is nonempty, (weakly-) compact (and convex)}\}$.

A multifunction (set-valued function) $F : \Omega \to P_f(Y)$ is said to be measurable if the R_+−valued function $\omega \to d(x, F(\omega)) = \inf\{\|x - z\| : z \in F(\omega)\}$ is measurable for every $x \in Y$. If there is a σ-finite measure $\mu(.)$ defined on Σ and Σ is μ-complete (or more generally without requiring the presence of $\mu(.)$ when Σ is closed under the Souslin operation), then the above definition of measurability of $F(.)$ is equivalent to saying that $GrF = \{(\omega, x) \in \Omega \times Y : x \in F(\omega)\} \in \Sigma \times B(Y)$ with $B(Y)$ being the Borel field of Y (graph measurability). For further details we refer to the survey paper of Wagner [20].

Let $F : \Omega \to P_f(Y)$ be a measurable multifunction and let $1 \leq p \leq \infty$. By S_F^p we will denote the selectors of $F(.)$ which belong to the Lebesgue-Bochner space $L^p(\Omega, Y)$ i.e. $S_F^p = \{f \in L^p(\Omega, Y) : f(\omega) \in F(\omega) a.e.\}$. This set may be empty. It is easy to check using Aumann's selection theorem (see Wagner [20], theorem 5.10), that S_F^p is nonempty iff $\omega \to \inf\{\|x\| : x \in F(\omega)\} \in L^p(\Omega)$. Moreover S_F^p is closed in $L^p(\Omega, Y)$ and is convex iff $F(\omega)$ is convex for μ-almost all $\omega \in \Omega$.

Let V, W be two Hausdorff topological spaces. A multifunction $G : V \to 2^W \setminus \{\emptyset\}$ is said to be upper semicontinuous (u.s.c.) if for all $C \subseteq W$ closed, the set $G^-(C) = \{v \in V : G(v) \cap C \neq \emptyset\}$ is closed in V. If $G(.)$ is u.s.c. with closed values and W is regular, then $GrG = \{(v, w) \in V \times W : w \in G(v)\}$ is closed in $V \times W$. The converse is true if $\overline{G(V)}$ is compact in W (see DeBlasi-Myjak [7]).

The mathematical setting of our problem will be the following: Let $T = [0, b]$ and H be a separable Hilbert space. Let X be a dense subspace of H carrying the structure of a separable, reflexive Banach space, which embeds into H continuously. Identifying H with its dual (pivot space), we have that $X \subseteq H \subseteq X^*$, with all embeddings being continuous and dense. Such a triple of spaces is known in the literature as "evolution triple" (see Zeidler [21]; some times the name Gelfand triple is also used). By $\|.\|$ (resp. $|\,.\,|$, $\|.\|_*$) we will denote the norm of X (resp. of H, X^*). Also by $(.,.)$ we will denote the inner product of H and by $\langle .,. \rangle$ the duality brackets for the pair (X, X^*). The two are compatible in the sense that $\langle .,. \rangle \,|_{H \times X} = (.,.)$. Also for $1 < p < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ define $W_{pq}(T) = \{x \in L^p(T, X) : \dot{x} \in L^q(T, X^*)\}$. In this definition the time derivative of $x(.)$ is taken in the sense of vector-valued distributions. When furnished with the norm $\|x\|_{W_{pq}(T)} = [\|x\|^2_{L^p(T,X)} + \|\dot{x}\|^2_{L^q(T,X^*)}]^{\frac{1}{2}}$, the space $W_{pq}(T)$ becomes a separable, reflexive Banach space. It is well known that $W_{pq}(T)$ embeds continuously into $C(T, H)$ (see Zeidler [21], proposition 23.23, p.422). So every element in $W_{pq}(T)$ after possible modification on a Lebesgue null set is equal to a continuous function from T into H. Furthermore if X embeds compactly into H, then so does $W_{pq}(T)$ in $L^p(T, H)$ (see Zeidler [21], p.450).

We will be studying the following periodic problem defined on T and the evolution triple $(X \subseteq H \subseteq X^*)$:

$$(1) \qquad \left\{ \begin{array}{c} \dot{x}(t) + A(t, x(t)) + F(t, x(t)) \ni h(t) \ a.e. \text{ on } T \\ x(0) = x(b) \end{array} \right\}$$

Here $A : T \times X \to X^*$, $F : T \times X \to 2^H \setminus \{\emptyset\}$ and $h \in L^q(T, X^*)$.

Definition 1. By a solution of (1) we mean a function $x \in W_{pq}(T)$ such that

$$\dot{x}(t) + A(t, x(t)) + f(t) = h(t) \text{ a.e. on } T,\ x(0) = x(b) \text{ and } f \in S^q_{F(.,x(.))}.$$

Remark 1. Since $W_{pq}(T)$ embeds into $C(T, H)$, the pointwise evaluations $x(0)$ and $x(b)$ make sense. Moreover a solution $x(.)$, when viewed as an X^*-valued function is absolutely continuous. Hence it is strongly differentiable almost everywhere and its derivative $\dot{x}(.)$ is of course in $L^q(T, X^*)$ (i.e. $x \in AC^{1,q}(T, X^*)$).

3 Auxiliary results

As we already mentioned, our proof will be based on a general surjectivity result for the sum of two operators of monotone type. Although the result is known (see for example Lions [14], theorem 1.2, p.319 or B-A. Ton [18], corollary 1, p.610), however for the convenience of the reader we state it here. First a definition:

Definition 2. Suppose that Y is a reflexive Banach space, $L : D(L) \subseteq Y \to Y^*$ is a linear densely defined maximal monotone operator and $T : Y \to P_{wkc}(Y^*)$ is a multivalued operator. We will say that $T(.)$ is pseudomonotone with respect to $D(L)$ if for $\{y_n\}_{n\geq 1} \subseteq D(L)$ with $y_n \xrightarrow{w} y$ in Y and $L(y_n) \xrightarrow{w} L(y)$ in Y^* as $n \to \infty$ and for $y_n^* \in T(y_n), n \geq 1$, satisfying $y_n^* \xrightarrow{w} y^*$ in Y^* as $n \to \infty$ and $\overline{\lim}(y_n^*, y_n) \leq (y^*, y)$, we have $y^* \in T(y)$ and $(y_n^*, y_n) \to (y^*, y)$ as $n \to \infty$.

The surjectivity result that we will be using in the sequel is the following:

Proposition 1. *If Y is a reflexive Banach space, $L : D(L) \subseteq Y \to Y^*$ is a linear densely defined maximal monotone operator and $T : Y \to P_{wkc}(Y^*)$ is bounded, pseudomonotone with respect to $D(L)$, u.s.c. from Y into Y_w^* (here Y_w^* denotes the Banach space Y furnished with the weak topology) and coercive (i.e. $inf[\frac{(y^*,y)}{\|y\|_Y} : y^* \in T(y)] \to \infty$ as $\|y\| \to \infty$) then $R(L+T) = Y^*$ (i.e. $L+T$ is surjective).*

Remark 2. Recall that $L : D(L) \subseteq Y \to Y^*$ is a linear, densely defined maximal monotone operator iff $L(.)$ is closed, monotone and $L^*(.)$ is also monotone (see for example Zeidler [21], theorem 32.1, p.897).

The second auxilliary result that we will use in our main existence theorem (see theorem 1), roughly speaking says that the pseudomonotonicity property of $A(t, .)$ lifts to the Nemitsky operator $\widehat{A}(.)$ ($\widehat{A}(x)(.) = A(., x(.))$) as pseudomonotonicity with respect to $D(L)$, with $L(x) = \dot{x}$ for $x \in D(L) = \{x \in W_{pq}(T) : x(0) = x(b)\}$. First recall that an operator $T : X \to X^*$ is said to be demicontinuous (resp. pseudomonotone) if for $x_n \to x$ in X as $n \to \infty$ we have $T(x_n) \xrightarrow{w} T(x)$ in X^* as $n \to \infty$ (resp. if for $x_n \xrightarrow{w} x$ in X as $n \to \infty$ and $\overline{\lim}\ \langle T(x_n), x_n - x\rangle \leq 0$ we have $\langle T(x), x - v\rangle \leq \underline{\lim}\langle T(x_n), x_n - v\rangle$ for all $v \in X$). Our hypothesis on the operator $A(t, x)$ is the following:

H(A): $A : T \times X \to X^*$ is an operator such that

(i) $t \to A(t, x)$ is measurable,

(ii) $A(t, x)$ is demicontinuous and pseudomonotone,

(iii) $||A(t,x)||_* \leq a_1(t) + c_1||x||^{p-1}$ a.e. on T with $a_1 \in L^q(T)$, $c_1 > 0$, $2 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

(iv) $\langle A(t,x), x\rangle \geq c||x||^p$ for almost all $t \in T$, all $x \in X$ and with $c > 0$.

Remark 3. Given $y \in Y$, let $u(y) : T \times X \to R$ be defined by $u(y)(t,x) = \langle A(t,x), y\rangle$. Evidently, because of hypotheses H(A) (i) and (ii), $t \to u(y)(t,x)$ is measurable and $x \to u(y)(t,x)$ is continuous (i.e. $u(y)(.,.)$ is a Caratheodory function). Hence $(t,x) \to u(y)(t,x)$ is jointly measurable (see Wagner [20], lemma 7.5, p.877). Since $y \in Y$ was arbitrary we deduce that $(t,x) \to A(t,x)$ is weakly measurable and since X is separable and reflexive from the Pettis measurability theorem (see Diestel-Uhl [8] theorem 2, p.42) we conclude that $(t,x) \to A(t,x)$ is jointly measurable.

Hence for every $x : T \to X$ measurable function, $t \to A(t, x(t))$ is measurable from T into X. In particular, because of hypothesis H(A) (iii) if $x(.) \in L^p(T,X)$, then $A(,.x(.)) \in L^q(T,X^*)$. So we can define the Nemitsky operator $\widehat{A} : L^p(t,X) \to L^q(T,X^*)$ corresponding to $A(t,x)$ by $\widehat{A}(x)(.) = A(.,x(.))$. Finally recall that since X is reflexive $L^p(T,X)^* = L(T,X^*)$ (see Diestel-Uhl [8] theorem 1, p.98). By $((.,.))$ we will denote the duality brackets of this pair; i.e. if $x \in L^p(T,X)$ and $v \in L^q(T,X^*)$ then $((v,x)) = \int_0^b \langle v(t), x(t)\rangle dt$.

Let $L : D(L) : L^p(T,X) \to L^q(T,X^*)$ be defined by $L(x) = \dot{x}$ (again the time derivative of $x(.)$ is taken in the sense of vector valued distributions) with $x \in D(L) = \{y \in L^p(T,X) : \dot{y} \in L^q(T,X^*), y(0) = y(b)\} \subseteq W_{pq}(T)$. Evidently this is a linear, densely defined, closed monotone operator. Also from the integration by parts formula for functions in $W_{pq}(T)$ (see Zeidler, prop. 23.23, p.423), we can easily check that $L^* : D(L^*) \subseteq L^p(T,X) \to L^q(T,X^*)$ is given by $L^*(v) = -\dot{v}$ with $v \in D(L) = D(L^*) \subseteq W_{pq}(T)$. So $L^*(.)$ is monotone and thus by virtue of remark 2 we conclude that $L(.)$ is maximal monotone.

Proposition 2. *If $A : T \times X \to X^*$ satisfies hypothesis H(A) then $\widehat{A} : L^p(T,X) \to L^q(T,X^*)$ is demicontinuous and pseudomonotone with respect to $D(L)$.*

Proof. First we will show the demicontinuity of A. So let $x_n \to x$ in $L^p(T,X)$ as $n \to \infty$. We can find a subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $x_{n_k}(t) \to x(t)$ a.e. in X as $k \to \infty$. Then because of hypothesis H(A)(ii) for any given $y \in L^p(T,X)$ we have $\langle A(t, x_{n_k}(t)), y(t)\rangle \to \langle A(t, x(t)), y(t)\rangle$ a.e. on T. Because of hypothesis H(A)(iii) we can apply the generalized dominated convergence theorem (see for example Ash [1], theorem 7.5.2, p.295) and get that $((\widehat{A}(x_{n_k}), y)) = \int_0^b \langle A(t, x_{n_k}(t)), y(t)\rangle dt \to \int_0^b \langle A(t, x(t)), y(t)\rangle dt = ((\widehat{A}(x), y))$ as $k \to \infty$ and since y was arbitrary we get the semicontinuity of $\widehat{A}(.)$.

We will now show the pseudomonotonicity of $\widehat{A}(.)$ with respect to $D(L)$. So let $x_n \to x$ in $W_{pq}(T)$ as $n \to \infty$ and assume that $\overline{lim}((\widehat{A}(x_n), x_n - x)) \leq 0$. Let $\xi_n(t) = \langle A(t, x_n(t)), x_n(t) - x(t)\rangle$. Since $W_{pq}(T)$ embeds continuously in $C(T,H)$, we have that $x_n \xrightarrow{w} x$ in $C(T,H)$ as $n \to \infty$ and so for every $t \in T$ we have $x_n(t) \xrightarrow{w} x(t)$ in H as $n \to \infty$. On the other hand let $N \subseteq T$ be the exceptional Lebesgue-null set outside of which hypotheses H(A)(iii) and (iv) hold. Then for

$t \in T\backslash N$ we have

$$\xi_n(t) \geq \varphi_n(t) = c\|x_n(t)\|^p - (a_1(t) + c_1\|x_n(t)\|^{p-1})\|x(t)\| \tag{2}$$

If $C = \{t \in T : \underline{lim}\xi_n(t) < 0\}$ (which is measurable) and $\mu(C) > 0$ ($\mu(.)$ being the Lebesgue measure on T), then from (2) above for every $t \in C \cap (T\backslash N)$ we will have that $\{x_n(t)\}_{n\geq 1}$ is bounded in X. Since X is reflexive and $x_n(t) \xrightarrow{w} x(t)$ in H as $n \to \infty$, we get that $x_n(t) \xrightarrow{w} x(t)$ in X as $n \to \infty$ for $t \in C \cap (T\backslash N)$. Exploiting the fact that $A(t,.)$ is pseudomonotone we get that $\langle A(t, x_n(t)), x_n(t) - x(t)\rangle \to 0$ as $n \to \infty$ for $t \in C \cap (T\backslash N)$ contradicting the definition of C. So $\xi_n(t) \geq 0$ a.e. on T. Then from Fatou's lemma we have

$$0 \leq \int_0^b \underline{\lim}\xi_n(t)dt \leq \underline{\lim} \int_0^b \xi_n(t)dt \leq \overline{\lim} \int_0^b \xi_n(t)dt \leq 0$$

hence $\int_0^b \xi_n(t)dt \to 0$ as $n \to \infty$.Note that $\mid \xi_n(t) \mid = \xi_n^+(t) - \xi_n^-(t) = \xi_n(t) + 2\xi_n^-(t)$. Since $\underline{\lim}\xi_n(t) \geq 0$ a.e. on T, we have that $\xi_n^-(t) \to 0$ a.e. on T as $n \to \infty$. Also recall that because of hypotheses H(A)(iii) and (iv) $\varphi_n(t) \leq \xi_n(t)$ a.e. on T with $\{\varphi_n\}_{n\geq 1} \subseteq L^1(T)$ being uniformly integrable. So $0 \leq \xi_n^-(t) \leq \varphi_n^-(t)$ a.e on T and of course $\{\varphi_n^-(t)\}$ is uniformly integrable. Hence a new application of the generalized dominated convergence theorem gives us that $\lim\limits_{n\to\infty} \int_0^b \xi_n^-(t)dt = 0$. Thus we finally have that $\lim\limits_{n\to\infty} \int_0^b \mid \xi_n(t) \mid dt = 0$, i.e $\xi_n \to 0$ in $L^1(T)$ as $n \to \infty$. Moreover by passing to a subsequence if necessary, we may assume that $\xi_n(t) \to 0$ a.e. on T as $n \to \infty$. Because $A(t,.)$ is pseudomonotone (see hypothesis H(A)(ii)), we have that $A(t, x_n(t)) \xrightarrow{w} A(t, x(t))$ a.e. in X^* and $\langle A(t, x_n(t)), x_n(t)\rangle \to \langle A(t, x(t)), x(t)\rangle$ a.e. on T as $n \to \infty$. Then from the demicontinuity of $\widehat{A}(.)$ and the generalized dominated convergence theorem we get that $\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x)$ in $L^q(T, X^*)$ and $((\widehat{A}(x_n), x_n)) \xrightarrow{w} ((\widehat{A}(x)\ , x))$. Therefore $\widehat{A}(.)$ is pseudomonotone with respect to $D(L)$.

4. Existence of periodic solutions

In this section we establish the existence of solutions for problem (1). Our hypotheses on the multivalued perturbation term $F(t, x)$ are the following:

H(F): $F : T \times X \to P_{fc}(H)$ is a multifunction such that:

(i) $t \to F(t, x)$ is measurable,

(ii) $GrF(t,.)$ is sequentially closed in $X_w \times H_w$,

(iii) $\mid F(t, x) \mid \leq a_2(t) + c_2\|x\|^{p-1}$ a.e on T with $a_2 \in L^q(T)$, $c > 0$, and

(iv) for almost all $t \in T$, all $x \in X$ and all $v \in F(t, x)$ we have $\langle v, x\rangle \geq -c_3\|x\|^p - c_4(t)$ with $c_3 \geqslant 0$, $c_4(.) \in L^1(T)$ and $c_3 < c$ where $c > 0$ is the constant from H(A)(iv).

Theorem 1. *If hypotheses H(a), H(F) hold, $h \in L^q(T, X^*)$ and X embeds compactly in H then problem (1) has a solution.*

Proof. Recall that $L : D(L) \subseteq L^p(T, X) \to L^q(T, X^*)$ defined by $L(x) = \dot{x}$ for $x \in D(L) = \{y \in L^p(T, X) : \dot{y} \in L^q(T, X^*), y(0) = y(b)\} \subseteq W_{pq}(T)$, is a linear, densely defined maximal monotone operator (see section 3). Next let $T : L^p(T,X) \to 2^{L^q(T,X^*)}$ be defined by $T(x) = \widehat{A}(x) + G(x)$ with $\widehat{A}$ being the Nemitsky operator corresponding to $A(t,x)$ and $G(x) = S^q_{F(.,x(.))}$. First we will show that $T(.)$ is $P_{wkc}(L^q(T,X^*))$ -valued. Since $\widehat{A}(.)$ is single valued and $G(.)$ has closed, convex and bounded values in $L^q(T,H)$ (see hypothesis H(F)), to establish that $T(.)$ is $P_{wkc}(L^q(T,X^*))$ -valued, it suffices to show that $G(.)$ has nonempty values. To this end let $x(.) \in L^p(T,X)$ and let $s_n(.), n \geq 1$, be X-valued simple functions such that $s_n \to x$ a.e. on T in X as $n \to \infty$ and $\|s_n - x\|_{L^p(T,X)} \to 0$ as $n \to \infty$. Because of hypothesis H(F)(i) $t \to F(t, s_n(t)), n \geq 1$, is measurable and so by Aumman's selection theorem there exists a measurable function $v_n : T \to H$ such that $v_n(t) \in F(t, x_n(t))$ for all $t \in T, n \geq 1$. Evidently $\{v_n\}_{n\geq 1}$ is bounded in $L^q(T,H)$ (see hypothesis H(f)(iii)) so we may assume that $v_n \xrightarrow{w} v$ in $L^q(T,H)$. Invoking theorem 3.1 of Papageorgiou [15], we get that

$$v(t) \in \overline{conv} w - \overline{\lim}\{v_n(t)\}_{n\geq 1} \subseteq \overline{conv} w - \overline{\lim} F(t, s_n(t)) \subseteq F(t, x(t)) \text{ a.e. on } T,$$

the last inclusion being a consequence of hypothesis H(F)(iii). So $v \in S^q_{F(.,x(.))}$ hence $G(x) \neq \emptyset$. Therefore $T : L^p(T,X) \to P_{wkc}(L^q(T,X^*))$ and because of hypotheses H(A)(iii) and H(F)(iii) is bounded (i.e. maps bounded sets into bounded sets). Now we will show that $T(.)$ is u.s.c. from $L^p(T,X)$ into $L^q(T,X^*)_w$ (here $L^q(T,X^*)_w$ denotes the Lebesgue-Bochner space $L^q(T,X^*)$ equipped with the weak topology). So let $C \subseteq L^q(T,X^*)$ be weakly closed and let $T^-(C) = \{x \in L^p(T,X) : T(x) \cap C \neq \emptyset\}$. We need to show that $T^-(C)$ is closed. For this, let $x_n \to x$ in $L^p(T,X)$ as $n \to \infty$ with $x_n \in T^-(C)$, $n \geq 1$. Let $g_n \in T(x_n) \cap C$, $n \geq 1$. By passing to a subsequence if necessary, we may assume that $g_n \xrightarrow{w} g$ in $L^q(T,X^*)$ and $x_n(t) \to x(t)$ a.e. on T as $n \to \infty$. By virtue of the last proposition $\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x)$ in $L^q(T,X^*)$ as $n \to \infty$ while if $v_n = g_n - \widehat{A}(x_n), n \geq 1$, and $v = g - \widehat{A}(x)$, then as above via theorem 3.1 of Papageorgiou [15] we have that $v(t) \in F(t,x(t))$ a.e. on T. So $g - \widehat{A}(x) = v$ with $v \in S^q_{F(.,x(.))}$, i.e. $g \in T(x)$. Also $g \in C$ since the latter is weakly closed in $L^q(T,X^*)$. Thus $x \in T^-(C)$ which implies that $T(.)$ is u.s.c. from $L^p(T,X)$ into $L^q(T,X^*)_w$. Next we will show that $T(.)$ is pseudomonotone with respect to $D(L)$. To this end let $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ $g_n \xrightarrow{w} g$ in $L^q(T,X^*)$ as $n \to \infty$ with $g_n \in T(x_n)$, $n \geq 1$, and assume that $\overline{lim}((g_n, x_n - x)) \leq 0$. Note that $g_n = \widehat{A}(x_n) + v_n$ with $v_n \in G(x_n) = S^q_{F(.,x_n(.))}, n \geq 1$. Also observe that we actually have $g_n \in L^q(T,H), n \geq 1$, and because of hypothesis H(F)(iii), by passing to a subsequence if necessary, we may assume that $v_n \xrightarrow{w} v$ in $L^q(T,H)$. Finally note that since $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ and $W_{pq}(T)$ embeds compactly in $L^p(T,H)$ (see section 2), we have that $x_n \to x$ in $L^p(T,H)$. Then:

$$\begin{aligned} &\overline{\lim}((\widehat{A}(x_n), x_n - x)) + \underline{\lim}((v_n, x_n - x)) \\ &= \overline{\lim}((\widehat{A}(x_n), x_n - x)) + \underline{\lim}(v_n, x_n - x)_{L^q(T,H),L^p(T,H)} \\ &\leq \overline{\lim}((\widehat{A}(x_n) + v_n, x_n - x)) = \overline{\lim}((g_n, x_n - x)) \leq 0 \end{aligned} \tag{3}$$

Since $(v_n, x_n - x)_{L^q(T,H),L^p(T,H)} \to 0$ as $n \to \infty$, from (3) above we get that $\overline{\lim}((\widehat{A}(x_n), x_n - x)) \le 0$. But from Proposition 2 we know that $\widehat{A}(.)$ is pseudomonotone with respect to $D(L)$. So $\widehat{A}(x_n) \to \widehat{A}(x)$ in $L^q(T, X^*)$ and $((\widehat{A}(x_n), x_n)) \to ((\widehat{A}(x_n), x))$ as $n \to \infty$. As in the proof of proposition 2 we set

$$\xi_n(t) = \langle A(t, x_n(t)), x_n(t) - x(t)\rangle, \quad n \ge 1 .$$

From that proof we know that $\xi_n \to 0$ in $L^1(T)$ as $n \to \infty$ and by passing to a subsequence if necessary we may also assume that $\xi_n(t) \to 0$ a.e. in T as $n \to \infty$. Also from hypotheses H(A)(iii) and (iv) we have:

$$\xi_n(t) \ge c\|x_n(t)\|^p - (a_1(t) + c_1\|x_n(t)\|^{p-1})\|x(t)\| \text{ a.e. on } T. \tag{4}$$

Our claim is that (4) above implies that for almost all $t \in T$ we have $\sup_{n\ge1}\|x_n(t)\| < \infty$. Suppose not. Then there is a $C \subseteq T$ with $\mu(C) > 0$ such that $\sup_{n\ge1}\|x_n(t)\| = \infty$ for all $t \in C$. Let $\varepsilon = \frac{\mu(C)}{2}$. By Egoroff's theorem there exists $C_1 \subseteq T$ with $\mu(C_1) \le \varepsilon$ such that $\xi_n(t) \to 0$ as $n \to \infty$ uniformly on $T\backslash C_1$. Let $C_2 = (T\backslash C_1) \cap C$. We remark that $\mu(C_2) > 0$. Indeed if this is not the case, then

$$\begin{aligned} b &\ge \mu((T\backslash C_1) \cup C) = \mu(T\backslash C_1) \cup C) + \mu(C_2) \\ &= \mu(T\backslash C_1) + \mu(C) \ge b - \frac{\mu(C)}{2} + \mu(C) > b \end{aligned}$$

a contradiction. Then for $t \in (T\backslash C_1) \cap C \cap (T\backslash N_1) = C \cap (T\backslash N_1)$ (with N_1 being the Lebesgue-null subset of T outside of which we have $\xi_n(t) \to 0$ as $n \to \infty$), we have $| \xi_n(t) | \le M_1$ and so from (4) we deduce that $\sup_{n\ge1}\|x_n(t)\| < \infty$ for all $t \in (T\backslash C_1) \cap C \cap (T\backslash N_1)$, a contradiction to the choice of C. Therefore $\mu(C) = 0$ and so $\sup_{n\ge1}\|x_n(t)\| < \infty$ for all $t \in T\backslash N_2$ with $\mu(N_2) = 0$. Hence if we fix $t \in T\backslash N_2$ and pass to an appropriate subsequence (depending in general on t), we will have $x_n(t) \xrightarrow{w} x(t)$ in X as $n \to \infty$. As before from theorem 3.1 of Papageorgiou [15] we get that $v \in G(x) = S^q_{F(.,x(.))}$ and so $g_n = \widehat{A}(x_n) + v_n \xrightarrow{w} g = \widehat{A}(x) + v$ in $L^q(T, X^*)$ as $n \to \infty$ and $v \in G(x)$. Also $((g_n, x_n)) = ((\widehat{A}(x_n), x_n)) + (v_n, x_n)_{L^q(T,H),L^q(T,H)} \to ((\widehat{A}(x), x)) + (v, x)_{L^q(T,H),L^q(T,H)} = ((g, x))$ as $n \to \infty$. Therefore $T(.)$ is pseudomonotone with respect to $D(L)$.

Finally from hypotheses H(A)(iv) and H(F)(iv) we have for $g = \widehat{A}(x) + v$ and $v \in G(x)$:

$$\begin{aligned} ((v, x)) &= ((\widehat{A}(x), x)) + (g, x)_{L^q(T,H),L^q(T,H)} \\ &\ge c\|x\|^p_{L^p(T,X)} - c_3\|x\|^p_{L^p(T,X)} - \|c_4\|_1 \\ &= (c - c_3)\|x\|^p_{L^p(T,X)} - \|c_4\|_1 \end{aligned}$$

Since $c_3 < c$ (see hypothesis H(F)(iv)) we conclude that $T(.)$ is coercive. Now rewrite problem (1) as the following equivalent operator inclusion:

$$L(x) + T(x) \ni h.$$

Since $(L+T)(.)$ is surjective (see proposition 1) we deduce that the above operator inclusion has a solution $x \in D(L)$. Therefore there exists $x \in W_{pq}(T)$ such that

$$\left\{ \begin{array}{c} \dot{x}(t) + A(t, x(t) + F(t, x(t)) \ni h(t) \ a.e. \ on \ T \\ x(0) = x(b) \end{array} \right\}$$

□

Remark 4. We should point out that hypotheses H(F)(i) and (ii) do not generally imply that the multifunction $(t, x) \to F(t, x)$ is measurable or even graph measurable (see Papageorgiou[16] for counterexamples to this effect). Hence for a measurable $x : T \to X$, we cannot say that $t \to F(t, x(t))$ is measurable which means that we cannot a priori guarantee the nonemptiness of $G(x) = S^q_{F(.,x(.))}$.

5. Examples

In this section we work out two examples of parabolic PDE's, illustrating the applicability of theorem 1.

In both examples T=[0,b] and $Z \subseteq R^N$ is a bounded domain with a C^1 boundary Γ. Let $D_k = \frac{\partial}{\partial z_k}, k \in \{1, 2, ..., N\}$ and $D = grad = (D_k)_{k=1}^N$.

(A) In the first example we consider the following multivalued periodic PDE:

$$(5) \qquad \left\{ \begin{array}{c} \dfrac{dx}{dt} - \Delta x + r \displaystyle\sum_{k=1}^{N} (\sin x) D_k x + u(t, z) \text{ in } T \times Z \\ u(t, z) \in [f_1(t, z, x(t, z)), f_2(t, z, x(t, z))] \text{ a.e. on } T \times Z \\ x(0, z) = x(b, z) \text{ a.e. on } Z, \ x \mid_{T \times \Gamma} = 0 \end{array} \right\}$$

Our hypotheses on the functions $f_1(t, z, x)$ and $f_2(t, z, x)$ are the following:
H(f): $f_1, f_2 : T \times Z \to R$ are functions such that
(i) for every measurable function $x : Z \to R$ the functions
$(t, z) \to f_1(t, z, x(z)), f_2(t, z, x(z))$ are both measurable,
(ii) $x \to f_1(t, z, x), -f_2(t, z, x)$ are both lower semicontinuous,
(iii) $\mid f_1(t, z, x) \mid, \mid f_2(t, z, x) \mid \leq a_2(t, z) + c_2 \mid x \mid^{p-1}$ a.e. on $T \times Z$ with $a_2 \in L^q(T, L^2(Z))$ and $c_2 > 0$, and
(iv) for almost all $(t, z) \in T \times Z$ and all $x \in R$, $f_1(t, z, x)x, f_2(t, z, x)x \geq -c_3$ for some $c_3 > 0$ (sign condition).

Remark 5. Problems like (5) arise when we deal with partial differential equations involving nonmonotone discontinuities; i.e. the perturbation term $f(t, z, x)$

is nonmonotone and discontinuous in x. Then in order to guarantee the existence of solutions, we need to pass to a multivalued problem which is derived from the original one by filling the gaps at the discontinuity points. Namely we introduce the functions $f_l(t,z,x) = \underline{\lim}_{y \to x} f(t,z,y)$ and $f_u(t,z,x) = \overline{\lim}_{y \to x} f(t,z,y)$ and replace $f(t,z,x)$ with the interval $[f_l(t,z,x), f_u(t,z,x)]$. Note that $f_l(t,z,.)$ and $-f_u(t,z,.)$ are both lower semicontinuous. If in addition we assume that for $x : Z \to R$ measurable, the functions $(t,z) \to f_l(t,z,x(z))$ and $(t,z) \to f_u(t,z,x(z))$ are measurable and if $f(t,z,.)$ satisfies a growth condition like H(f)(iii), then we fall within the framework of problem (5), with H(f) valid (see Chang [6]).

Theorem 2. *If hypothesis H(F) holds, $N \leq 3$, $| r | < \frac{\lambda_1}{1+\lambda_1}$ with λ_1 being the first eigenvalue of $(-\Delta, H_0^1(Z))$ and $h \in L^2(T \times Z)$, then problem (5) has a solution $x \in L^p(T, H_0^1(Z)) \cap C(T, L^2(Z))$ with $\frac{\partial x}{\partial t} \in L^q(T, H^{-1}(Z))$.*

Proof. In this case the evolution triple is $X = H_0^1(Z), H = L^2(Z)$ and $X^* = H^{-1}(Z)$. From the Sobolev embedding theorem we know that X embeds compactly in H. In addition, since $N \leq 3$ we can see that X embeds compactly in $L^4(Z)$.

Let $A_1 : X \to X^*$ be defined by

$$\langle A_1(x), y \rangle = \int_Z (Dx(z), Dy(z))_{R^N} dz, \ \ x, y \in X$$

Evidently $A_1(.)$ is linear, continuous, monotone and $\langle A_1(x), x \rangle = \|Dx\|^2_{L^2(Z,R^N)} = \frac{\lambda_1}{1+\lambda_1}\|x\|^2$. Next let $v : X \to X^*$ be defined by $v(x)(.) = r\sum_{k=1}^N (\sin x(.))D_k x(.) \in L^2(Z)$. We claim that $v(.)$ is a compact map. So let $x_n \xrightarrow{w} x$ in X as $n \to \infty$. We will show that $v(x_n) \to v(x)$ in X^* as $n \to \infty$. Suppose not. Then by passing to a suitable subsequence if necessary, we may assume that there exist $\varepsilon > 0$ and $\{y_n\}_{n \geq 1} \subseteq X$ with $\|y_n\| \leq 1$ such that $0 < \varepsilon \leq \langle v(x_n) - v(x), y_n \rangle$, $n \geq 1$. We may assume that $y_n \xrightarrow{w} y$ in X and so $y_n \to y$ in $L^4(Z)$ as $n \to \infty$. Then we have:

$$\begin{aligned}
\langle v(x_n) - v(x), y_n \rangle &= \int_Z ((\sin x_n)(D_k x_n) y_n(z) - (\sin x)(D_k x) y_n(z)) dz \\
&= \int_Z (\sin x_n - \sin x)(D_k x_n) y_n(z) dz + \int_Z (\sin x)(D_k x_n)(y_n - y)(z) dz \\
&\quad + \int_Z (\sin x)(D_k(x_n - x) y(z) dz + \int_Z (\sin x)(D_k x)(y - y_n)(z) dz
\end{aligned}$$

Recall that $| \sin x_n(z) - \sin x(z) | \leq | x_n(z) - x(z) |$. So applying Holder's inequality with three factors we have (in what follows $M_i, i \in \{1,2,3,4,5\}$, are positive constants):

$$\begin{aligned}
| \int_Z (\sin x_n - \sin x)(D_k x_n) y_n(z) dz | &\leq \|x_n - x\|_{L^4(Z)} \|x_n\| \|y_n\| \\
&\leq M_1 \|x_n - x\|_{L^4(Z)} \to 0 \ ,
\end{aligned}$$

$$\begin{aligned}| \int_Z (\sin x)(D_k x_n)(y_n - y)(z)dz | &\leq M_2 \|x_n\| \|y_n - y\|_{L^4(Z)} \\ &\leq M_3 \|y_n - y\|_{L^4(Z)} \to 0,\end{aligned}$$

$$| \int_Z (\sin x)(D_k(x_n - x)y(z))dz | \to 0 \text{ as } n \to \infty \text{ since } D_k x_n \xrightarrow{w} D_k x \text{ in } L^2(Z)$$

and

$$| \int_Z (\sin x)(D_k x)(y - y_n)(z)dz | \to 0 \text{ as } n \to \infty \text{ since } y_n \xrightarrow{w} y \text{ in } L^2(Z)$$

as $n \to \infty$. Thus finally we have $\langle v(x_n) - v(x), y_n \rangle \to 0$ as $n \to \infty$, a contradiction to the choice of the $y_n' s$. So $v(.)$ is compact as claimed. Also note that

$$| v(x) | = | r \sum_{k=1}^{N} (\sin x(.)) D_k x(.) | \leq M_4 \|Dx\|_{L^2(Z,R^N)} \leq M_5 \|x\|$$

and

$$\begin{aligned}| \langle v(x), x \rangle | &\leq | r | \sum_{k=1}^{N} | \int_Z (\sin x(z))(D_k x(z))x(z)dz | \\ &\leq | r | \sum_{k=1}^{N} (\int_Z | D_k x(z) |^2 dz)^{\frac{1}{2}} | x | \leq | r | \|x\|^2\end{aligned}$$

hence $\langle v(x), x \rangle \geq - | r | \|x\|^2$. Let $A = A_1 + v$. Since the sum of a maximal monotone operator with a compact operator is pseudomonotone, we conclude that A satisfies hypothesis H(A) (recall that $| r | < \frac{\lambda_1}{1+\lambda_1}$). Next let $F : T \times X \to P_{fc}(H)$ be defined by

$$\begin{aligned}&F(t,x) = \{v \in H : f_1(t,z,x(z)) \leq v(z) \leq f_2(t,z,x(z)) \text{ a.e. on } Z\} \\ &= \{v \in H : \int_C f_1(t,z,x(z))dz \leq \int_C v(z)dz \leq \int_C f_2(t,z,x(z))dz \text{ for every Borel } C\end{aligned}$$

Recall that the Borel σ-field of T is countably generated. Let $\{C_n\}_{n \geq 1}$ be a field of such generators. We have:

$$\begin{aligned}GrF(.,x) = \cap_{n \geq 1} \{(t,v) \in T \times H : &\int_{C_n} f_1(t,z,x(z))dz \leq \int_{C_n} v(z)dz \\ &\leq \int_{C_n} f_2(t,z,x(z))dz \}\end{aligned}$$

Thanks to Fubini's theorem (see hypothesis H(f)(i)), we have that $GrF(.,x) \in \ell(T) \times B(H)$, where $\ell(T)$ is the Lebesgue $\sigma-$field of T and $B(H)$ is the Borel $\sigma-$field of H. Thus the multifunction $t \to F(t,x)$ is Lebesgue measurable. Also because X embeds compactly in H, we can easily see that $GrF(t,.)$ is sequentially closed in $X_w \times X_w$. Moreover since $p \geq 2$ and $N \leq 3$ from the Sobolev embedding theorem we have that X embeds continuously in $L^{2(p-1)}(Z)$ so from H(f)(iii) we get:

$$| F(t,x) | \leq \widehat{a}_2(t) + \widehat{c}_2 ||x||^{p-1} \text{ a.e. on } T \text{ with } \widehat{a}_2 \in L^q(T),\ \widehat{c}_2 > 0.$$

Finally, hypothesis H(f)(iv) implies that there exists $\widehat{c}_3 > 0$ such that $\langle v, x\rangle \geq -\widehat{c}_3$ for almost all $t \in T$ all $x \in X$ and all $v \in F(t,x)$.

We can now rewrite problem (5) as the following equivalent abstract evolution inclusion:

$$\left\{ \begin{matrix} \dot{x}(t) + A(t,x(t)) + F(t,x(t)) \ni \widehat{h}(t) \text{ a.e. on } T \\ x(0) = x(b) \end{matrix} \right\}$$

Here $\widehat{h}(t) = h(t,.) \in H$. Invoking theorem 1 we conclude that (5) has at least one solution $x(.,.) \in L^p(T, H_0^1(Z)) \cap C(T, L^2(Z))$ with $\frac{\partial x}{\partial t} \in L^q(T, H^{-1}(Z))$. $\square$

(B) For the second example we consider the following nonlinear distributed parameter control system with a priori feedback:

$$(6) \qquad \left\{ \begin{matrix} \dfrac{\partial x}{\partial t} - \displaystyle\sum_{k=1}^{N} D_k a_k(t,z,x,Dx) + u(t,z) = h(t,z) \text{ in } T \times Z \\ x(0,z) = x(b,z) \text{ a.e. on } Z,\ x\, |_{T\times\Gamma} = 0 \\ ||u(t,.)||_{L^2(Z)} \leq \gamma(t, ||x(t,.)||_{L^2(Z)}) \end{matrix} \right\}$$

We make the following hypotheses for the data of (6):

H(a): $a_k : T \times Z \times R \times R^N \to R$, $k \in \{1,2,...,N\}$, are functions such that:
(i) $(t,z) \to a_k(t,z,x,\eta)$ is measurable,
(ii) $(x,\eta) \to a_k(t,z,x,\eta)$ is continuous,
(iii) $\sum_{k=1}^N (a_k(t,z,x,\eta) - a_k(t,z,x,\eta'))(\eta_k - \eta'_k) \geq 0$ a.e. on $T \times Z$, for all $x \in R$ and all $\eta, \eta' \in R^N$,
(iv) $| a_k(t,z,x,\eta) | \leq \beta_1(t,z) + c_1(| x |^{p-1} + ||\eta||^{p-1})$ a.e. on $T \times Z$ with $\beta_1 \in L^q(T,Z)$ and $c_1 > 0$, and
(v) $\sum_{k=1}^N a_k(t,z,x,\eta)\eta_k \geq c||\eta||^p$ a.e. on $T \times Z$.

H(γ)$\gamma : T \times R_+ \to R_+$ is a function such that
(i) $t \to \gamma(t,r)$ is measurable,
(ii) $r \to \gamma(t,r)$ is continuous, and
(iii) $\gamma(t,r) \leq \beta_2(t)$ a.e. on T with $\beta_2 \in L^q(T)$.

Theorem 3. *If hypotheses H(a), H(γ) hold and $h \in L^2(T \times Z)$ then problem (6) has a solution $x \in L^p(T, W_0^{1,p}(Z)) \cap C(T, L^2(Z))$ with $\frac{\partial x}{\partial t} \in L^q(T, W^{-1,q}(Z))$.*

Proof. In this case $X = W_0^{1,p}(Z)$, $H = L^2(Z)$ and $X^* = W^{-1,q}(Z)$. From the Sobolev embedding theorem, we know that X embeds compactly in H. Let $A : T \times X \to X^*$ be defined by:

$$\langle A(t,x), y \rangle = \sum_{k=1}^{N} \int_Z a_k(t,z,x,Dx) D_k y(z) dz, \;\; x, y \in W_0^{1,p}(Z).$$

From Fubini's theorem, we see that $t \to \langle A(t,x), y \rangle$ is measurable. Since $y \in W_0^{1,p}(Z)$ is arbitrary, we deduce that $t \to A(t,x)$ is weakly measurable and so by the Pettis measurability theorem $t \to A(t,x)$ is measurable. Also by using the dominated convergence theorem we can readily check that $t \to A(t,x)$ is demicontinuous, while from theorem 3.1 of Gossez-Mustonen [10], we know that $x \to A(t,x)$ is pseudomonotone. In addition from hypotheses H(a)(iv) and (v) we have:

$$\|A(t,x)\|_* \le \widehat{\beta}_1(t) + \widehat{c}_1 \|x\|^{p-1} \text{ a.e. on } T \text{ with } \widehat{\beta}_1 \in L^q(T), \; \widehat{c}_1 > 0,$$
$$\text{and } \;\; \langle A(t,x), x \rangle \ge \widehat{c} \|x\|^p \text{ a.e. on } \; T \text{ with } \; \widehat{c} > 0.$$

Next define $F : T \times X \to P_{wkc}(H)$ by $F(t,x) = \{u \in H : \mid u \mid \le \gamma(t, \mid x \mid)$. Evidently $t \to F(t,x)$ is measurable and $GrF(t,.)$ is sequentially closed in $X_w \times X_w$. Recall thay X embeds compactly in H). Also $\mid F(t,x) \mid \le \widehat{\beta}_2(t)$ a.e. on T. Hence for almost all $t \in T$, all $x \in X$ and all$v \in F(t,x)$ we have $\langle v, x \rangle \ge \widehat{\beta}_2(t) \mid x \mid$.

Then we can equivalently rewrite (6) in the form of the evolution inclusion (1) with $h(t)(.) = h(t,.) \in H$. Apply theorem 1 to get the desired solution. ☐

Along the same lines we can also consider higher order distributed parameter control systems with a priori feedback. In what follows $\alpha = (\alpha_k)_{k=1}^N$ is a multi-index of positive integers. The number $\mid \alpha \mid = \sum_{k=1}^N \alpha_k$ is the length of the multi-index. Also let $D^\alpha = D_1^{\alpha_1} ... D_N^{\alpha_N}$, $\eta(x) = \{D^\beta x : \mid \beta \mid \le m-1\}$ and $\xi(x) = \{D^\delta x : \mid \delta \mid = m\}$. The number of all multi-indices of length $\le m$ is given by $N_1 = \frac{(N+m)!}{N!m!}$ and the number of all multi-indices of length $\le m-1$ is given by $N_2 = \frac{(N+m-1)!}{N!(m-1)!}$ so $N_3 = N_1 - N_2$ is the number of multi-indices of length exactly m. We consider the following system:

$$\left\{ \begin{array}{c} \dfrac{\partial x}{\partial t} + \displaystyle\sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(t,z,\eta(x(t,z)), \xi(x(t,z))) \\ +u(t,z) = h(t,z) \text{ in } T \times Z \\ x(0,z) = x(b,z) \text{ a.e. on } Z, \; D^\beta x \mid_{T \times \Gamma} = 0 \text{ for } \mid \beta \mid \le m-1 \\ \|u(t,.)\|_{L^2(Z)} \le \gamma(t, \|\eta(x,(t,.))\|_{L^p(Z,R^{N_2})}) \text{ a.e. on } T. \end{array} \right\} \tag{7}$$

Our hypotheses on the data of (7) are the following:
$\mathbf{H(A)}_1 : A_\alpha : T \times Z \times R^{N_2} \rightarrow R^{N_3}$ are functions such that
(i) $(t, z) \rightarrow A_\alpha(t, z, \eta, \xi)$ is measurable,
(ii) $(\eta, \xi) \rightarrow A_\alpha(t, z, \eta, \xi)$ is continuous,
(iii) $| A_\alpha(t, z, \eta, \xi) | \leq \beta_1(t, z) + c_1(\|\eta\|^{p-1} + \|\xi\|^{p-1})$ a.e. on $T \times Z$ with $\beta_1 \in L^q(T \times Z)$, $c_1 > 0$,
(iv) $\sum_{|\alpha|=m}(A_\alpha(t, z, \eta, \xi) - A_\alpha(t, z, \eta, \xi')(\xi_\alpha - \xi'_\alpha) > 0$ a.e. on $T \times H$ for all $\eta \in R^{N_2}, \xi, \xi' \in R^{N_3}$, and
(v) $\sum_{|\alpha|\leq m} A_\alpha(t, z, \eta, \xi)\xi_\alpha \geq c_2\|\xi\|^p = c_3(t, z)$ a.e. on $T \times Z$ with $c_2 > 0$ and $c_3 \in L^1(T \times Z)$.

Theorem 4. *If hypotheses H(A)$_1$, H(γ) hold and $h \in L^2(T \times Z)$ then problem (7) has a solution $x \in L^p(T, W_0^{m,p}(Z)) \cap C(T, L^2(Z))$ with $\frac{\partial x}{\partial t} \in L^q(T, W^{-m,p}(Z))$.*

Proof. In this case the evolution triple consists of $X = W_0^{m,p}(Z)$, $H = L^2(Z)$ and $X^* = W^{-m,p}(Z)$. Again X embeds compactly in H. The operator $A : T \times X \rightarrow X^*$ is given by

$$\langle A(t, x), y \rangle = \sum_{|\alpha|\leq m} \int_Z A_\alpha(t, z, \eta(x(z)), \xi(x(z)))D^\alpha y(z)dz$$

From theorem 1 of Browder [4] we know that $A(t, .)$ is pseudomonotone. Also let $F : T \times X \rightarrow P_{wkc}(H)$ be defined by $F(t, x) = \{v \in H : | v | \leq \gamma(t, \|x\|_{W_0^{m-1,p}(Z)})$. Then the sequential closedness of $GrF(t, .)$ in $X_w \times H_w$ follows from the compact embedding of $W_0^{m,p}(Z)$ into $W_0^{m-1,p}(Z)$.

References

[1] Ash, R., *Real Analysis and Probability*, Academic Press, New York (1972).
[2] Becker, R. I., *Periodic solutions of semilinear equations of evolution of compact type*, J. Math. Anal. Appl. **82** (1981), 33-48.
[3] Brezis, H., *Operateurs Maximaux Monotones*, North Holland, Amsterdam (1973).
[4] Browder, F., *Existence of periodic solutions for nonlinear equations of evolution*, Proc. Nat. Acad. Sci. USA **53** (1965), 1100-1103.
[5] Browder, F., *Pseudomonotone operators and nonlinear elliptic boundary value problems on unbounded domains*, Proc. Nat. Acad. Sci. USA **74** (1977), 2659-2661.
[6] Chang, K.-C., *Variational methods for nondifferentiable functions and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), 102-129.
[7] Blasi, F. S., Myjak, J., *On continuous approximations for multifunctions*, Pacific J. Math. **123** (1986), 9-31.
[8] Diestel, J., Uhl, J. J., *Vector Measures*, Math. Surveys, **15**, AMS Providence, Rhode Island (1977).
[9] Goebel, K., Kirk, W., *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge (1990).

[10] Gossez, J. P., Mustonen, V., *Pseudomonotonicity and the Leray-Lions condition*, Differential and Integral Equations **6** (1993), 37-45.

[11] Hirano, N., *Existence of periodic solutions for nonlinear evolution equations in Hilbert spaces*, Proc. AMS 120 (1994), 185-192.

[12] Hu, S., Papageorgiou, N. S., *On the existence of periodic solutions for a class of nonlinear evolution inclusions*, Bolletino UMI 7-B (1993), 591-605.

[13] Hu, S., Papageorgiou, N. S., *Galerkin approximations for nonlinear evolution inclusions*, Comm. Math. Univ. Carolinae **35** (1994), 705-720.

[14] Lions, J. L., *Quelques Methods de Resolution des Problemes aux Limites Nonlineaires*, Dunod, Paris (1969).

[15] Papageorgiou, N. S., *Convergence theorems for Banach space valued integrable multifunctions*, Inter. J. Math. and Math. Sci. **10** (1987), 433-464.

[16] Papageorgiou, N. S., *On measurable multifunctions with applications to random multivalued equations*, Math. Japonica **32** (1987), 437-464.

[17] Prüss, J., *Periodic solutions for semilinear evolution equations*, Nonl. Anal. TMA **3** (1979), 221-235.

[18] Ton, B.-A., *Nonlinear evolution equations in Banach spaces*, Proc. AMS 109 (1990), 653-661.

[19] Vrabie, I., *Periodic solutions for nonlinear evolution equations in a Banach space*, Proc. AMS 109 (1990), 653-661.

[20] Wagner, D., *Survey of measurable selection theorems*, SIAM J. Control Opt. **15** (1977), 859-903.

[21] Zeidler, E., *Nonlinear Functinal Analysis and its Applications*, Springer-Verlag, New York (1990).

Dimitrios A. Kandilakis
Department of Mathematics
University of the Aegean
83200 Karlovassi, Samos
GREECE

Nikolaos S. Papageorgiou
Department of Mathematics
National Technical University
Zografou Campus
15780 Athens, GREECE