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OSCILLATION AND NONOSCILLATION OF EMDEN-FOWLER TYPE EQUATION OF SECOND ORDER

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Abstract. Oscillation and nonoscillation criteria are established for the equation

$$u'' + p(t)|u|^{\alpha}|u'|^{1-\alpha} \operatorname{sgn} u = 0,$$

where $\alpha \in]0,1]$, and $p : [0,+\infty[\to [0,+\infty[$ is a locally summable function.

1. Statement of Main Results

Consider the differential equation

$$u'' + p(t)|u|^{\alpha}|u'|^{1-\alpha} \operatorname{sgn} u = 0, \tag{1}$$

where $\alpha \in]0,1]$, and $p : [0,+\infty[\to [0,+\infty[$ is an integrable function. Under the solution of equation (1) is understood a function $u : [0,+\infty[\to]-\infty,+\infty[$ which is locally absolutely continuous together with its first derivative and satisfying equation (1) almost everywhere.

The solution u of equation (1) is said to be proper if there exists $a > 0$ such that $\operatorname{mes}\{t > a : \; u'(t) = 0\} = 0$. The nontrivial solution u of equation (1) is said to be oscillatory if it has at least one zero in any neighborhood of $+\infty$, and it is said to be nonoscillatory, otherwise. As it will be shown below (see Lemma 3), equation (1) cannot have proper oscillatory and proper nonoscillatory solutions simultaneously. This fact is justified by the following

Definition. Equation (1) is said to be nonoscillatory if it has at least one proper nonoscillatory solution, and it is said to be oscillatory, otherwise.

In this paper we intend to concern ourselves with the problem of oscillation and nonoscillation of equation (1). Analogous problem has been considered in [3] when p was in general not of constant signs.

As we shall show below (see Lemma 4), if for some $\lambda < \alpha$ the integral $\int^{+\infty} s^{\lambda} p(s)ds$ diverges, then equation (1) is oscillatory (for a linear equation, i.e., when $\alpha = 1$,

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this assertion goes back to W. B. Fite [1] and E. Hille [2]). Therefore we shall assume that $\int^{+\infty} s^\lambda p(s)ds$ converges for $\lambda < \alpha$.

Introduce the notation:

$$p_*(\lambda) = \alpha \liminf_{t\to+\infty} t^{\alpha-\lambda}\int_t^{+\infty} s^\lambda p(s)ds, \quad p^*(\lambda) = \alpha \limsup_{t\to+\infty} t^{\alpha-\lambda}\int_t^{+\infty} s^\lambda p(s)ds \quad \text{for } \lambda < \alpha,$$

$$p_*(\lambda) = \alpha \liminf_{t\to+\infty} t^{\alpha-\lambda}\int_1^{t} s^\lambda p(s)ds, \quad p^*(\lambda) = \alpha \limsup_{t\to+\infty} t^{\alpha-\lambda}\int_1^{t} s^\lambda p(s)ds \quad \text{for } \lambda > \alpha.$$

Below new oscillation and nonoscillation criteria of equation (1) are given in terms of the numbers $p_*(\lambda)$ and $p^*(\lambda)$. Analogous results for the linear equation (i.e., when $\alpha = 1$) are contained in [4].

Theorem 1. Let either $p_*(0) > \frac{1}{\alpha+1}(\frac{\alpha}{\alpha+1})^\alpha$ or $p_*(\alpha+1) > (\frac{\alpha}{\alpha+1})^{\alpha+1}$. Then equation (1) is oscillatory.

Thus it is natural to suppose that

$$p_*(0) \le \frac{1}{\alpha+1}\Big(\frac{\alpha}{\alpha+1}\Big)^\alpha \quad \text{and} \quad p_*(\alpha+1) \le \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}. \tag{2}$$

Note that in this case the equations

$$x^{\frac{\alpha+1}{\alpha}} - x + p_*(0) = 0, \tag{3}$$

$$\alpha x^{\frac{\alpha+1}{\alpha}} - \alpha x + p_*(\alpha+1) = 0 \tag{4}$$

are solvable. Denote by A the least nonnegative root of equation (3) and by B the largest root of equation (4).

Theorem 2. Let (2) be fulfilled and either

$$p^*(0) > B - A + p_*(0) \tag{5}$$

or

$$p^*(\alpha+1) > B - A + p_*(\alpha+1). \tag{6}$$

Then equation (1) is oscillatory.

Theorem 3. Let (2) be fulfilled and either

$$p^*(\lambda) > \frac{1}{\alpha-\lambda}\Big(\frac{\lambda}{\alpha+1}\Big)^{\alpha+1} + B \tag{7}$$

for some $\lambda < \alpha$ or

$$p^*(\lambda) > \frac{1}{\lambda-\alpha}\Big(\frac{\lambda}{\alpha+1}\Big)^{\alpha+1} - A \tag{8}$$

for some $\lambda > \alpha$. Then equation (1) is oscillatory.

Corollary 1. Let

$$\lim_{\lambda\to\alpha} |\alpha-\lambda| p^*(\lambda) > \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}. \tag{9}$$

Then equation (1) is oscillatory.

Corollary 2. Let for some $\lambda \neq \alpha$

$$|\alpha-\lambda| p_*(\lambda) > \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}. \tag{10}$$

Then equation (1) is oscillatory.

Corollary 3. Let

$$\limsup_{t\to+\infty} \frac{1}{\ln t} \int_1^t s^\alpha p(s)ds > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{11}$$

Then equation (1) is oscillatory.

Corollary 4. Let

$$\limsup_{\lambda\to\alpha-} (\alpha-\lambda) \int_1^{+\infty} s^\lambda p(s)ds > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{12}$$

Then equation (1) is oscillatory.

Remark 1. Inequalities (9)–(12) are exact and they cannot be weakened. Indeed, let $p(t) = \frac{1}{\alpha}(\frac{\alpha}{\alpha+1})^{\alpha+1} t^{-\alpha-1}$ for $t \geq 1$. Then $|\alpha-\lambda|p_*(\lambda) = |\alpha-\lambda|p^*(\lambda) = \frac{\alpha}{\ln t}\int_1^t s^\alpha p(s)ds = \alpha(\alpha-\lambda)\int_1^{+\infty} s^\lambda p(s)ds = (\frac{\alpha}{\alpha+1})^{\alpha+1}$, and equation (1) has the nonoscillatory proper solution $u(t) = t^{\frac{\alpha}{\alpha+1}}$ for $t \geq 1$.

Theorem 4. Let either for some $\lambda < \frac{\alpha^2}{\alpha+1}$ or for some $\lambda > \alpha + \frac{\alpha^{\alpha+1}}{(\alpha+1)[(\alpha+1)^\alpha - \alpha^\alpha]}$ the equality

$$|\alpha-\lambda| p^*(\lambda) < \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} \tag{13}$$

be fulfilled. Then equation (1) is nonoscillatory.

2. Some Auxiliary Statements

Lemma 1. Let u be the proper nonoscillatory solution of equation (1). Then there exists $t_0 > 0$ such that

$$u(t)u'(t) > 0 \quad \text{for} \quad t \geq t_0. \tag{14}$$

Proof. For the sake of definiteness we assume that $u(t) > 0$ for $t \geq t_1$. Suppose that $u'(t_2) \leq 0$ for some $t_2 > t_1$. Since $u'(t) \not\equiv 0$ for $t > t_2$ and u' does not increase, there exists $t_3 > t_2$ such that $u'(t_3) < 0$ and $u'(t) \le u'(t_3)$ for $t > t_3$. Owing to this fact, we have

$$u(t) = u(t_3) + \int_{t_3}^{t} u'(s)ds \le u(t_3) + u'(t_3)(t - t_3) \quad \text{for} \quad t > t_3.$$

Let $t_4 = t_3 - \frac{u(t_3)}{u'(t_3)} + 1$. Then from the latter inequality we obtain contradiction that $u(t_4) < 0$. Thus the lemma is proved.

Lemma 2. Let equation (1) have a proper nonoscillatory solution. Then equations (3) and (4) are solvable, and there exists $t_0 > 0$ such that the equation

$$\rho' + \alpha p(t) + \alpha \rho^{\frac{\alpha+1}{\alpha}} = 0 \tag{15}$$

has the solution $\rho : [t_0, +\infty[\to]0, +\infty[$. Moreover,

$$\liminf_{t\to+\infty} t^\alpha \rho(t) \ge A, \quad \liminf_{t\to+\infty} t^\alpha \rho(t) \le B, \tag{16}$$

where A is the least nonnegative root of equation (3) and B is the largest root of equation (4).

Proof. Let u be the nonoscillatory proper solution of equation (1). By Lemma 1, there exists $t_0 > 0$ such that (14) is fulfilled. Let $\rho(t) = (\frac{u'(t)}{u(t)})^\alpha$ for $t \ge t_0$. It is easily seen that ρ is the solution of equation (15). From (15) we find that

$$-\frac{1}{\alpha}\rho^{-\frac{\alpha+1}{\alpha}}(t)\rho'(t) \ge 1 \quad \text{for} \quad t \ge t_0.$$

Integrating this inequality we obtain

$$(t - t_0)^\alpha \rho(t) \le 1 \quad \text{for} \quad t \ge t_0.$$

Hence

$$\limsup_{t\to+\infty} t^\alpha \rho(t) \le 1. \tag{17}$$

Introduce the notation

$$r = \liminf_{t\to+\infty} t^\alpha \rho(t), \quad R = \limsup_{t\to+\infty} t^\alpha \rho(t).$$

Assume that $p_*(0) \ne 0$ and $p_*(\alpha + 1) \ne 0$ (the solvability of (3) and (4) as well of the estimate (16) is otherwise trivial). Because of (17) we easily find from (15) that

$$t^\alpha \rho(t) = \alpha t^\alpha \int_t^{+\infty} p(s)ds + \alpha t^\alpha \int_t^{+\infty} \rho^{\frac{\alpha+1}{\alpha}}(s)ds \quad \text{for} \ t > t_0 \tag{18}$$

and

$$t^{\alpha}\rho(t) = \frac{\tau^{\alpha+1}\rho(\tau)}{t} + \frac{1}{t}\int_{\tau}^{t}\left(s\rho^{\frac{1}{\alpha}}(s)\right)^{\alpha}\left(\alpha+1-\alpha s\rho^{\frac{1}{\alpha}}(s)\right)ds -$$

$$(19) \qquad -\frac{\alpha}{t}\int_{\tau}^{t} s^{\alpha+1}p(s)ds \quad \text{for} \quad t_0 < t, \tau < +\infty,$$

whence we have

$$r \geq p_*(0) \quad \text{and} \quad R \leq 1 - p_*(\alpha+1).$$

Obviously, for any $0 < \varepsilon < \min\{r, 1-R\}$ there exist $\tau_\varepsilon > t_0$ and $t_\varepsilon > \tau_\varepsilon$ such that

$$r - \varepsilon < t^{\alpha}\rho(t) < R + \varepsilon, \quad \alpha t^{\alpha}\int_{t}^{+\infty} p(s)ds > p_*(0) - \varepsilon \quad \text{for} \;\; t > \tau_\varepsilon,$$

$$\frac{\alpha}{t}\int_{\tau_\varepsilon}^{t} s^{\alpha+1}p(s)ds > p_*(\alpha+1) - \varepsilon \quad \text{for} \;\; t > t_\varepsilon.$$

Due to this fact we have from (18) and (19) that

$$t^{\alpha}\rho(t) > p_*(0) - \varepsilon + (r-\varepsilon)^{\frac{\alpha+1}{\alpha}} \quad \text{for} \quad t > \tau_\varepsilon,$$

$$t^{\alpha}\rho(t) < \frac{\tau_\varepsilon^{\alpha+1}\rho(\tau_\varepsilon)}{t} + (R+\varepsilon)\big(\alpha+1-\alpha(R+\varepsilon)^{\frac{1}{\alpha}}\big) - p_*(\alpha+1) \quad \text{for} \;\; t > t_\varepsilon.$$

From this we find that

$$r^{\frac{\alpha+1}{\alpha}} - r + p_*(0) \leq 0 \quad \text{and} \quad \alpha R^{\frac{\alpha+1}{\alpha}} - \alpha R + p_*(\alpha+1) \leq 0.$$

Consequently, equations (3) and (4) are solvable, and (16) are fulfilled. Thus the lemma is proved.

Lemma 3. Let equation (1) have the proper oscillatory solution v. Then all the proper solutions of equation (1) are oscillatory.

Proof. Assume the contrary. Let u be proper nonoscillatory solution of equation (1). According to Lemma 1, there exists $t_0 > 0$ such that (14) holds. Choose $t_1 > t_0$ and $t_2 > t_1$ so that

$$v(t) > 0, \;\; v'(t) > 0 \;\; \text{for} \;\; t_1 < t < t_2, \;\; v(t_1) = 0, \;\; v'(t_2) = 0.$$

Introduce the notation

$$\rho(t) = \left(\frac{v'(t)}{v(t)}\right)^{\alpha}, \;\; \sigma(t) = \left(\frac{u'(t)}{u(t)}\right)^{\alpha} \quad \text{for} \;\; t_1 < t < t_2.$$

Evidently,

$$(20) \rho'(t) = -\alpha p(t) - \alpha\rho^{\frac{\alpha+1}{\alpha}}(t), \quad \sigma'(t) = -\alpha p(t) - \alpha\sigma^{\frac{\alpha+1}{\alpha}}(t) \quad \text{for} \;\; t_1 < t < t_2.$$

It is not difficult to see that there exist $t_3 \in]t_1, t_2[$ and $\varepsilon \in]0, t_2 - t_3[$ such that

$$\rho(t) < \sigma(t) \quad \text{for} \quad t_3 < t < t_3 + \varepsilon, \quad \rho(t_3) = \sigma(t_3). \tag{21}$$

Because of this fact we have from (20) that

$$\rho(t) = \rho(t_3) - \alpha \int_{t_3}^{t} p(s)ds - \alpha \int_{t_3}^{t} \rho^{\frac{\alpha+1}{\alpha}}(s)ds \geq \sigma(t_3) - \alpha \int_{t_3}^{t} p(s)ds -$$
$$-\alpha \int_{t_3}^{t} \sigma^{\frac{\alpha+1}{\alpha}}(s)ds = \sigma(t) \quad \text{for} \quad t_3 < t < t_3 + \varepsilon,$$

but this contradicts (21). Thus the lemma is proved.

Remark 2. In proving Lemma 3 we have in fact proved that the solutions u and v of equation (1) do not exist which for some $0 < a < b < +\infty$ could satisfy the conditions

$$u(t) > 0, \;\; u'(t) > 0, \;\; v(t) > 0, \;\; v'(t) > 0 \quad \text{for} \;\; a < t < b,$$
$$u(a) \geq 0, \;\; u'(b) > 0, \;\; v(a) = 0, \;\; v'(b) = 0.$$

Lemma 4. Let for some $\lambda < \alpha$ the integral $\int^{+\infty} s^\lambda p(s)ds$ be divergent. Then equation (1) is oscillatory.

Proof. Assume the contrary. Let equation (1) have the proper nonoscillatory solution u. By Lemma 1, there exists t_0 such that (14) is fulfilled. Similarly as in proving Lemma 2 we can see that the function $\rho(t) = (\frac{u'(t)}{u(t)})^\alpha$ for $t \geq t_0$ satisfies the relation (15) and the inequality (17). Multiplying the equality (15) by t^λ and integrating from t_0 to t, we see that

$$\int_{t_0}^{t} s^\lambda p(s)ds = -t^\lambda \rho(t) + t_0^\lambda \rho(t_0) + \lambda \int_{t_0}^{t} s^{\lambda-1}\rho(s)ds -$$
$$-\alpha \int_{t_0}^{t} s^\lambda \rho^{\frac{\alpha+1}{\alpha}}(s)ds \quad \text{for} \quad t > t_0.$$

By (17) the right-hand side of the above equality has a finite limit for $t \to +\infty$. Hence the integral $\int^{+\infty} s^\lambda p(s)ds$ converges, but this contradicts the condition of the lemma. Thus the lemma is proved.

Lemma 5. Let there exist the function $v : [t_0, +\infty[\to]0, +\infty[$ which is locally absolutely continuous together with its first derivative and satisfying the inequalities

$$v'(t) > 0, \quad v''(t) + p(t)v^\alpha(t)(v'(t))^{1-\alpha} \leq 0 \quad \text{for} \;\; t > t_0 \tag{22}$$

almost everywhere. Then equation (1) is nonoscillatory.

Proof. Denote by u_0 a solution of equation (1) satisfying the initial conditions

$$u_0(t_0) = 0, \quad u_0'(t_0) = 1$$

and show that $u_0'(t) > 0$ for $t > t_0$.

Assume the contrary. Let there exist $t_1 > t_0$ such that

$$u'(t) > 0 \quad \text{for} \quad t_0 < t < t_1, \quad u'(t_1) = 0.$$

Since the function $w(t) = (t - t_0)\frac{v(t_1)-v(t_0)}{t_1-t_0} + v(t_0)$ satisfies the inequalities

$$w'(t) > 0, \quad w''(t) + p(t)w^\alpha(t)(w'(t))^{1-\alpha} \geq 0 \quad \text{for} \quad t > t_0,$$

equation (1) has the solution u_1 satisfying the conditions (cf., for example [5])

$$u_1(t) > 0, \quad u_1'(t) > 0 \quad \text{for} \quad t_0 \leq t \leq t_1, \quad u_1(t_0) = v(t_0), \quad u_1(t_1) = v(t_1),$$

which is according to Remark 2 impossible. Thus we have proved that $u_0'(t) > 0$ for $t > t_0$. Consequently, u_0 is the proper nonoscillatory solution of equation (1). Thus the lemma is proved.

Lemma 6. Let $p^*(\lambda) < +\infty$ for $\lambda \neq \alpha$. Then the mapping $\lambda \longmapsto (\alpha - \lambda)p^*(\lambda)$ ($\lambda \longmapsto (\alpha - \lambda)p_*(\lambda)$) does not increase (does not decrease) for $\lambda < \alpha$ and does not decrease (does not increase) for $\lambda > \alpha$.

Proof. We prove this lemma only in the case when $\lambda < \alpha$. For $\lambda > \alpha$ the lemma is proved in a similar way. Let $\varepsilon > 0$. Choose $t_\varepsilon > 0$ so that

$$p_*(\lambda) - \varepsilon < \alpha t^{\alpha-\lambda} \int_t^{+\infty} s^\lambda p(s)ds < p^*(\lambda) + \varepsilon \quad \text{for} \quad t > t_\varepsilon.$$

It is easy to see that whatever $\mu < \alpha$ might be,

$$t^{\alpha-\mu} \int_t^{+\infty} s^\mu p(s)ds = t^{\alpha-\lambda} \int_t^{+\infty} s^\lambda p(s)ds + (\mu-\lambda)t^{\alpha-\mu} \int_t^{+\infty} s^{\mu-\lambda-\alpha}\bigg(\int_s^{+\infty} \tau^\lambda p(\tau)d\tau\bigg)ds.$$

Hence if $\lambda < \mu$, than

$$\begin{aligned}(p_*(\lambda) - \varepsilon)\Big(1 + \frac{\mu - \lambda}{\alpha - \mu}\Big) < \alpha t^{\alpha-\mu} \int_t^{+\infty} s^\mu p(s)ds < \\ < (p^*(\lambda) + \varepsilon)\Big(1 + \frac{\mu - \lambda}{\alpha - \mu}\Big) \quad \text{for} \quad t > t_\varepsilon.\end{aligned}$$

This implies that

$$(\alpha - \mu)p^*(\mu) \leq (\alpha - \lambda)p^*(\lambda) \quad \text{and} \quad (\alpha - \mu)p_*(\mu) \geq (\alpha - \lambda)p_*(\lambda).$$

Thus the lemma is proved.

Lemma 7. Let $p^*(\lambda) < +\infty$ for $\lambda \neq \alpha$. Then

$$\lim_{\lambda\to\alpha-}(\alpha-\lambda)p_*(\lambda) = \lim_{\lambda\to\alpha+}(\lambda-\alpha)p_*(\lambda) \tag{23}$$
$$\Big(\lim_{\lambda\to\alpha-}(\alpha-\lambda)p^*(\lambda) = \lim_{\lambda\to\alpha+}(\lambda-\alpha)p^*(\lambda)\Big).$$

Proof. Let $\lambda < \alpha$, $\mu > \alpha$ and $\varepsilon > 0$. Choose $t_\varepsilon > 1$ so that

$$p_*(\lambda) - \varepsilon < \alpha t^{\alpha-\lambda} \int_t^{+\infty} s^\lambda p(s)ds < p^*(\lambda) + \varepsilon \quad \text{for } t > t_\varepsilon,$$
$$p_*(\mu) - \varepsilon < \alpha t^{\alpha-\mu} \int_1^{t} s^\mu p(s)ds < p^*(\mu) + \varepsilon \quad \text{for } t > t_\varepsilon.$$

It is easily seen that

$$t^{\alpha-\lambda}\int_t^{+\infty} s^\lambda p(s)ds = -t^{\alpha-\mu}\int_1^t s^\mu p(s)ds + (\mu-\lambda)t^{\alpha-\lambda}\int_t^{+\infty} s^{\lambda-\mu-1}\Big(\int_1^s \tau^\mu p(\tau)d\tau\Big)ds,$$
$$t^{\alpha-\mu}\int_1^t s^\mu p(s)ds = -t^{\alpha-\lambda}\int_t^{+\infty} s^\lambda p(s)ds + t^{\alpha-\mu}\int_1^{+\infty} s^\lambda p(s)ds +$$
$$+(\mu-\lambda)t^{\alpha-\mu}\int_1^t s^{\mu-\lambda-1}\Big(\int_s^{+\infty}\tau^\lambda p(\tau)d\tau\Big)ds.$$

From these equalities we have

$$\frac{\mu-\lambda}{\alpha(\alpha-\lambda)}(p_*(\mu)-\varepsilon) - t^{\alpha-\mu}\int_1^t s^\mu p(s)ds < t^{\alpha-\lambda}\int_t^{+\infty} s^\lambda p(s)ds <$$
$$< \frac{\mu-\lambda}{\alpha(\alpha-\lambda)}(p^*(\mu)+\varepsilon) \quad \text{for } t > t_\varepsilon,$$
$$-t^{\alpha-\lambda}\int_t^{+\infty} s^\lambda p(s)ds + \frac{\mu-\lambda}{\alpha(\mu-\alpha)}(p_*(\lambda)-\varepsilon)(1-t^{\alpha-\lambda}) < t^{\alpha-\mu}\int_1^t s^\mu p(s)ds <$$
$$< \int_1^{+\infty} s^\lambda p(s)ds + \frac{\mu-\lambda}{\alpha(\mu-\alpha)}(p^*(\lambda)+\varepsilon) \quad \text{for } t > t_\varepsilon,$$

whence we find that

$$(\mu-\lambda)p_*(\mu)-(\alpha-\lambda)p^*(\mu)<(\alpha-\lambda)p_*(\lambda),\quad (\alpha-\lambda)p^*(\lambda)<(\mu-\lambda)p^*(\mu),$$
$$-(\mu-\alpha)p^*(\lambda)+(\mu-\lambda)p_*(\lambda)<(\mu-\alpha)p_*(\mu),$$
$$(\mu-\alpha)p^*(\mu)<(\mu-\lambda)p^*(\lambda)+\alpha(\mu-\alpha)\int\limits_1^{+\infty}s^\lambda p(s)ds.$$

Finally by Lemma 6 we obtain

$$\lim_{\lambda\to\alpha-}(\alpha-\lambda)p_*(\lambda)\geq(\mu-\alpha)p_*(\mu),\quad \lim_{\lambda\to\alpha-}(\alpha-\lambda)p^*(\lambda)\leq(\mu-\alpha)p^*(\mu),$$
$$\lim_{\mu\to\alpha+}(\mu-\alpha)p_*(\mu)\geq(\alpha-\lambda)p_*(\lambda),\quad \lim_{\mu\to\alpha+}(\mu-\alpha)p^*(\mu)\leq(\alpha-\lambda)p^*(\lambda).$$

From the last four inequalities we can conclude that the equality (23) is valid. Thus the lemma is proved.

Lemma 8. Let $p^*(\lambda)<+\infty$ for some $\lambda<\alpha$ and $p^*(\mu)<+\infty$ for some $\mu>\alpha$. Then

$$\alpha\limsup_{t\to+\infty}\frac{1}{\ln t}\int\limits_1^t s^\alpha p(s)ds\leq(\alpha-\lambda)p^*(\lambda) \tag{24}$$

and

$$\alpha\limsup_{\lambda\to\alpha-}(\alpha-\lambda)\int\limits_1^{+\infty}s^\lambda p(s)ds\leq(\mu-\alpha)p^*(\mu). \tag{25}$$

Proof. Let $\varepsilon>0$. Choose $t_\varepsilon>1$ so that for $t>t_\varepsilon$

$$\alpha t^{\alpha-\lambda}\int\limits_t^{+\infty}s^\lambda p(s)ds<p^*(\lambda)+\varepsilon,\quad \alpha t^{\alpha-\mu}\int\limits_1^t s^\mu p(s)ds<p^*(\mu)+\varepsilon.$$

We can easily see that

$$\frac{1}{\ln t}\int\limits_1^t s^\alpha p(s)ds=-\frac{t^{\alpha-\lambda}}{\ln t}\int\limits_t^{+\infty}s^\lambda p(s)ds+\frac{\alpha-\lambda}{\ln t}\int\limits_1^t s^{\alpha-\lambda-1}\bigg(\int\limits_s^{+\infty}\tau^\lambda p(\tau)d\tau\bigg)ds,$$
$$\int\limits_1^{+\infty}s^\delta p(s)ds=(\mu-\delta)\int\limits_1^{+\infty}s^{\delta-\mu-1}\bigg(\int\limits_1^s\tau^\mu p(\tau)d\tau\bigg)ds\quad\text{for }\ \delta<\alpha.$$

From these inequalities we have

$$\frac{\alpha}{\ln t}\int_1^t s^\alpha p(s)ds < (\alpha-\lambda)(p^*(\lambda)+\varepsilon) \quad \text{for } t>t_\varepsilon,$$

$$(\alpha-\delta)\int_1^{+\infty} s^\delta p(s)ds < (\mu-\delta)(p^*(\mu)+\varepsilon) \quad \text{for } t>t_\varepsilon.$$

Hence the inequalities (24) and (25) are fulfilled. Thus the lemma is proved.

3. Proof of the Main Results

Proof of Theorem 1. Assume the contrary. Let equation (1) have the proper nonoscillatory solution. Then according to Lemma 2, equations (3) and (4) are solvable. To this end it is necessary that the inequalities (2) be fulfilled. But this contradicts the conditions of the theorem. Thus the theorem is proved.

Proof of Theorem 2. Assume the contrary. Let equation (1) have the proper nonoscillatory solution. Then according to Lemma 2, there exists $t_0>0$ such that equation (15) has the solution $\rho:[t_0,+\infty[\to]0,+\infty[$ satisfying the conditions (16). From (15) we easily obtain that (18) and (19) are fulfilled. Assume $p_*(\alpha+1)\neq 0$ ($p_*(\alpha+1)=0$ i.e., $B=1$). Clearly, for any $0<\varepsilon<1-B$ ($0<\varepsilon<1$) there exist t_ε such that

$$t^\alpha\rho(t)<B+\varepsilon,\ \ t^\alpha\rho(t)>A-\varepsilon \quad \text{for} \quad t>t_\varepsilon,$$
$$t^\alpha\rho(t)\big(\alpha+1-\alpha t\rho^{\frac{1}{\alpha}}(t)\big)<c(\varepsilon) \quad \text{for} \quad t>t_\varepsilon,$$

where $c(\varepsilon)=(B+\varepsilon)(\alpha+1-\alpha(B+\varepsilon)^{\frac{1}{\alpha}})$ ($c(\varepsilon)=1$). Owing to this fact we find from (18) and (19) that

$$\alpha t^\alpha\int_t^{+\infty}p(s)ds<(B+\varepsilon)^{\frac{\alpha+1}{\alpha}}-A+\varepsilon \quad \text{for} \quad t>t_\varepsilon$$

and

$$\frac{\alpha}{t}\int_{t_\varepsilon}^t s^{\alpha+1}p(s)ds<\frac{t_\varepsilon^{\alpha+1}\rho(t_\varepsilon)}{t}+c(\varepsilon)-A+\varepsilon \quad \text{for} \quad t>t_\varepsilon.$$

Consequently,

$$p^*(0)\le p_*(0)+B-A \quad \text{and} \quad p^*(\alpha+1)\le p_*(\alpha+1)+B-A,$$

which contradicts (5) and (6). Thus the theorem is proved.

Proof of Theorem 3. Assume the contrary. Let equation (1) have the proper nonoscillatory solution. Then according to Lemma 2, there exists $t_0>0$ such that equation (15) has the solution $\rho : [t_0, +\infty[\to]0, +\infty[$ satisfying the estimates (16). Clearly, for any $\varepsilon > 0$ there exists $t_\varepsilon > t_0$ such that

$$t^\alpha \rho(t) < B + \varepsilon, \quad t^\alpha \rho(t) > A - \varepsilon \quad \text{for} \quad t > t_\varepsilon.$$

From (15) we easily find that

$$\alpha t^{\alpha-\lambda} \int_t^{+\infty} s^\lambda p(s) ds = t^\alpha \rho(t) + t^{\alpha-\lambda} \int_t^{+\infty} s^{\lambda-\alpha-1} \big(s\rho^{\frac{1}{\alpha}}(s)\big)^\alpha \big(\lambda - \alpha s \rho^{\frac{1}{\alpha}}(s)\big) ds$$
$$\text{for} \quad t > t_\varepsilon, \quad \lambda < \alpha,$$

$$\alpha t^{\alpha-\lambda} \int_{t_\varepsilon}^{t} s^\lambda p(s) ds = -t^\alpha \rho(t) + \frac{t_\varepsilon{}^\lambda \rho(t_\varepsilon)}{t} + t^{\alpha-\lambda} \int_{t_\varepsilon}^{t} s^{\lambda-\alpha-1} \big(s\rho^{\frac{1}{\alpha}}(s)\big)^\alpha \big(\lambda - \alpha s \rho^{\frac{1}{\alpha}}(s)\big) ds$$
$$\text{for} \quad t > t_\varepsilon, \quad \lambda > \alpha.$$

Since $\max\big\{x^\alpha(\lambda - \alpha x) \ : 0 \le x < +\infty\big\} = (\frac{\lambda}{\alpha+1})^{\alpha+1}$, from the last two inequalities we have

$$\alpha t^{\alpha-1} \int_t^{+\infty} s^\lambda p(s) ds < B + \varepsilon + \frac{1}{\alpha - \lambda} \Big(\frac{\lambda}{\alpha+1}\Big)^{\alpha+1} \quad \text{for } t > t_\varepsilon, \ \lambda < \alpha$$

and

$$\alpha t^{\alpha-1} \int_{t_\varepsilon}^{t} s^\lambda p(s) ds < \frac{t_\varepsilon^\lambda \rho(t_\varepsilon)}{t} + \frac{1}{\lambda - \alpha} \Big(\frac{\lambda}{\alpha+1}\Big)^{\alpha+1} - A + \varepsilon \quad \text{for } t > t_\varepsilon, \ \lambda > \alpha.$$

Consequently,

$$p^*(\lambda) \le \frac{1}{\alpha - \lambda} \Big(\frac{\lambda}{\alpha+1}\Big)^{\alpha+1} + B \quad \text{for} \quad \lambda < \alpha,$$
$$p^*(\lambda) \le \frac{1}{\lambda - \alpha} \Big(\frac{\lambda}{\alpha+1}\Big)^{\alpha+1} - A \quad \text{for} \quad \lambda > \alpha,$$

but this contradicts (7) and (8). Thus the theorem is proved.

Proof of Corollary 1. We shall assume that $p^*(\lambda) < +\infty$ (according to Theorem 3, the equation is otherwise oscillatory). Then by Lemma 7, the limit in the right-hand side of the inequality (9) exists. Obviously,

$$\lim_{\lambda \to \alpha-} \Big[|\alpha - \lambda| p^*(\lambda) - \Big(\frac{\lambda}{\alpha+1}\Big)^{\alpha+1} - (\alpha - \lambda) B\Big] > 0.$$

This implies that (7) is fulfilled for some $\lambda < \alpha$. Therefore by Theorem 3 equation (1) is oscillatory. Thus the corollary is proved.

Proof of Corollary 2. If for some $\lambda \ne \alpha$, $p^*(\lambda) = +\infty$, then according to Theorem 3, equation (1) is oscillatory. We shall assume that $p^*(\lambda) < +\infty$ for

$\lambda \neq \alpha$. By Lemma 6, if (10) holds for some $\lambda \neq \alpha$, then the condition (9) is also fulfilled. Hence according to Corollary 1, equation (1) is oscillatory.

To convince ourselves that Corollary 3 (Corollary 4) is valid, it should be noted that according to Lemma 8, it follows from (11) ((12)) that (9) is fulfilled. Hence by Corollary 1, equation (1) is oscillatory.

Proof of Theorem 4. Introduce the notation

$$f(t) = \alpha \int_t^{+\infty} s^\lambda p(s)ds \quad \text{for} \quad t > 1, \quad \lambda < \frac{\alpha^2}{\alpha+1},$$

$$f(t) = -\alpha \int_1^t s^\lambda p(s)ds \quad \text{for} \quad t > 1, \quad \lambda > \alpha + \frac{\alpha^{\alpha+1}}{(\alpha+1)[(\alpha+1)^\alpha - \alpha^\alpha]},$$

$$k = \Big(\frac{\alpha}{\alpha+1}\Big)^\alpha - \frac{1}{\alpha-\lambda}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}.$$

From (13) for some $t_0 > 1$ we have

$$0 \le k + t^{\alpha-\lambda} f(t) < \Big(\frac{\alpha}{\alpha+1}\Big)^\alpha \quad \text{for} \quad t > t_0, \quad \lambda < \frac{\alpha^2}{\alpha+1},$$

$$\Big(\frac{\alpha}{\alpha+1}\Big)^\alpha < k + t^{\alpha-\lambda} f(t) \le k \tag{26}$$

$$\text{for} \quad t > t_0, \quad \lambda > \alpha + \frac{\alpha^{\alpha+1}}{(\alpha+1)[(\alpha+1)^\alpha - \alpha^\alpha]}.$$

We can easily see that if $\lambda < \frac{\alpha^2}{\alpha+1}$ and $0 \le x < (\frac{\alpha}{\alpha+1})^\alpha$ or $\lambda > \alpha + \frac{\alpha^{\alpha+1}}{(\alpha+1)[(\alpha+1)^\alpha - \alpha^\alpha]}$ and $\frac{\alpha^2}{\alpha+1} < x \le k$, then

$$\alpha x^{\frac{\alpha+1}{\alpha}} - \lambda x + k(\lambda - \alpha) \le 0,$$

whence according to (26), we have

$$\alpha\big(k + t^{\alpha-\lambda} f(t)\big)^{\frac{\alpha+1}{\alpha}} - \lambda\big(k + t^{\alpha-\lambda} f(t)\big) + k(\lambda-\alpha) \le 0 \quad \text{for} \quad t > t_0.$$

The latter inequality is equivalent to

$$\rho'(t) \le \frac{\lambda}{t}(\rho(t) + f(t)) - \frac{\alpha}{t^{\frac{\lambda}{\alpha}}}(\rho(t) + f(t))^{\frac{\alpha+1}{\alpha}} \quad \text{for} \quad t > t_0,$$

where $\rho(t) = kt^{\lambda-\alpha}$. Then the function

$$v(t) = \exp\Big[\int_{t_0}^t \Big(\frac{\rho(s) + f(s)}{s^\lambda}\Big)^{\frac{1}{\alpha}} ds\Big] \quad \text{for} \quad t > t_0$$

satisfies the inequality (22). Hence by Lemma 5, equation (1) is nonoscillatory. Thus the theorem is proved.

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