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LIE ALGEBROIDS AND MECHANICS

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Abstract. We give a formulation of certain types of mechanical systems using the structure of groupoid of the tangent and cotangent bundles to the configuration manifold M; the set of units is the zero section identified with the manifold M. We study the Legendre transformation on Lie algebroids.

0. Introduction

The purpose of this paper is not to prove new results but to investigate certain elementary tools utilized in Analytical Mechanics, mainly in C. Marle's paper [M] and in A. Weinstein's paper [W]. Some ideas are suggested by the papers of P. Dazord [D] and I. Vaisman [V].

The motion of a point in a Euclidian space can be described by the solution of the Newton equation

$$m\frac{d^2x}{dt^2} = F\,,$$

where $\frac{d^2x}{dt^2}$ represents the acceleration and F the force; with suitable unities we may suppose that $m = 1$.

When studying more general mechanical systems (especially constrained mechanical systems) it is more difficult to define the acceleration and the forces. All authors (see for instance [A], [A.M], [G], [H], [S]) do not agree on the definitions. For instance the constraint force is a vector field or a Pfaffian form; there is also an ambiguity about the origin of vectors and covectors.

To unify the terminology we have used the groupoid structure of the bundles TM and T^*M. Each fiber is endowed with an affine structure. The set of units is the zero section identified with the manifold M. The Lie algebroid of these groupoids is the vertical bundle along the zero section. We distinguish vectors and covectors whose origin lies in the zero section from elements whose origin lies in the fibers (called *lifted* elements).

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We point out the importance of semi-basic forms, notion introduced by G. Reeb and developed in [G] and [L.M]. On the cotangent bundle T^*M (or on any vector bundle endowed with a symplectic structure such that the fibers are lagrangian submanifolds, for instance the tangent bundle when there exists a Legendre transformation $TM \to T^*M$), the symplectic duality carries the vertical vector fields onto the semi-basic 1-forms. On the other side the difference of two vector fields on TM defining second order differential equations is a vertical vector field. We remark that a section of the vector bundle $T^*M \times_M T^*M \to T^*M$ can be interpreted either as a vertical vector field or as a semi-basic 1-form, which explains the ambiguity defining the constraint force field.

Following [M], we make use of the Lagrange differential introduced by W. Tulczyjew [T] and generalized by many authors, in particular I. Kolář [Ko].

We investigate the Legendre transformation for Lie algebroids as was initiated by A. Weinstein [W]; we study in particular the Lie algebroid J_1TM and the Lie algebroid of a symplectic groupoid but we do not tackle the theory of Lie bialgebroids.

All manifolds and mapping are supposed to be smooth i.e. C^∞. For any manifold W, the projections of the tangent bundle TW and cotangent bundle T^*W onto W are denoted respectively by p and q.

1. Some facts about Lie algebroids

A vector bundle $\pi : \mathcal{A} \to M$ over a n-dimensional manifold M is a Lie algebroid if it satisfies the following properties:

a) the vector bundle is equipped with a Lie algebra structure [,] on its space of sections.

b) there exists a bundle map $\varrho : \mathcal{A} \to TM$ (called the *anchor map*) which induces a Lie algebra homomorphism (also denoted ϱ) from sections of $\mathcal{A}$ to vector fields on M.

c) for any smooth function f on M and any pair (ξ, η) of sections of $\mathcal{A}$, the following identity is satisfied

$$[f\xi, \eta] = f[\xi, \eta] - (\varrho(\eta)f)\xi \ . \tag{1}$$

We consider adapted local coordinates $(x^1, \dots, x^n, \lambda^1, \dots, \lambda^r)$ on $\mathcal{A}$, where the x^i's are local coordinates on M, the λ^j's are linear coordinates on the fibers, associated with a basis ξ_i of sections of the Lie algebroid. Then the bracket and anchor map may be expressed

$$\begin{aligned} [\xi_i, \xi_j] &= \sum c_{ij}^k \xi_k \\ \varrho(\xi_i) &= \sum a_{ij} \frac{\partial}{\partial x^j} \end{aligned} \tag{2}$$

where the c_{ij}^k and a_{ij} are smooth functions on M.

This notion is due to J. Pradines [P] who has proved that with any Lie groupoid is attached a Lie algebroid as follows (see [L_5]). Let $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ be a Lie groupoid with base Γ_0 (the set of units). An *infinitesimal displacement* in the sense of C. Ehresmann [E] is a vector tangent to a α-fiber, with origin at $u \in \Gamma_0$. The set of all infinitesimal displacements is a vector bundle $\mathcal{A}(\Gamma)$. According to J. Pradines [P], any local section $U \subset \Gamma \to \mathcal{A}(\Gamma)$ extends to a vector field tangent to the α-fibers and invariant under the right translations. It results that the sheaf of germs of section of $\mathcal{A}(\Gamma)$ is a Lie algebra sheaf and hence the vector bundle $\mathcal{A}(\Gamma) \to \Gamma_0$ is a Lie algebroid with anchor map $T\beta : \mathcal{A}(\Gamma) \to T\Gamma_0$. In our previous papers $\mathcal{A}(\Gamma)$ was written $depl\Gamma$. For more details see for instance [L$_5$], [A.D], [W].

When Γ is a Lie group, $\mathcal{A}(\Gamma)$ is its Lie algebra and we recover the usual notions.

The problem of integrating Lie algebroids i.e. of finding a Lie groupoid whose Lie algebroid is given is not always possible.

2. Examples of Lie algebroids

1°) The tangent bundle $p : TM \to M$ is a Lie algebroid with ϱ = identity of TM

2°) Any integrable subbundle of TM is a Lie algebroid with the inclusion as anchor map and the induced bracket

3°) Let $P(M, G)$ be a G-principal bundle. C. Ehresmann has associated to P a groupoid called now *gauge groupoid* Φ; this groupoid is the quotient Φ of $P \times P$ by the diagonal action of G. We have proved in [L$_1$], [L$_2$], [L$_4$] that $\mathcal{A}(\Phi)$ is isomorphic to TP/G, the vector bundle over M of all tangent vectors to P mod. the right translations under the action of G. So TP/G is endowed with a Lie algebroid structure whose anchor map is the projection $TP/G \to TM$.

In particular when P is a group (so $M = P/G$ is a homogeneous space), Φ is diffeomorphic to $P \times P/G$ and $\mathcal{A}(\Phi)$ is diffeomorphic to the trivial bundle $T_eP \times P/G$. We recover the "infinitesimal displacements" of the method of moving frames. If P is the affine euclidian group, then $T_eP = SO(p) \oplus \mathbb{R}^p$; an infinitesimal displacement is the sum of a translation and an infinitesimal rotation.

4°) When the principal bundle over M is the frame bundle $H\big(M, GL(n, \mathbb{R})\big)$, we proved in [L$_1$], [L$_2$], [L$_3$] that the vector bundle $TH/GL(n, \mathbb{R}) \to M$ is isomorphic to $J_1TM \to M$, where J_1TM is the set of 1-jets of sections of the tangent bundle $TM \to M$. This isomorphism induces a Lie algebroid structure an $J_1TM \to M$, the anchor map being the natural projection $J_1TM \to TM$.

This property extends to J_qTM, isomorphic to TH^q/L^q_n, when we consider the q-jets.

3. The dual of a Lie algebroid

P. Dazord and D. Sondaz [D.S] have proved the following theorem: Given a vector bundle $\mathcal{A} \to M$ endowed with a Lie algebroid structure, its dual $\mathcal{A}^* \to M$ is endowed with a homogeneous Poisson structure i.e. $\mathcal{L}(Z)\Lambda = -\Lambda$ where Z is the dilations vector field on $\mathcal{A}^*$ and Λ is the Poisson bivector field. Conversely if $\mathcal{A}^*$ is endowed with a homogeneous Poisson structure, then $\mathcal{A}$ is endowed with a Lie algebroid structure.

This theorem generalizes two situations: if M is a point, then $\mathcal{A} = \mathfrak{g}$ (Lie algebra). We recover the Lie-Poisson structure on $\mathfrak{g}^*$. If $\mathcal{A} = TM$, we obtain the natural symplectic structure of T^*M.

We shall explain the first part of the theorem following the paper by A. Weinstein [W]. As the differentials of the functions that are affine on each fiber span the cotangent space at each points of $\mathcal{A}^*$, the Poisson bracket is known if we know the Poisson bracket for any pair of affine functions on the fibers. The functions constant on the fibers are the pullbacks of functions on M while the functions linear on fibers are identified with sections of $\mathcal{A}$. Hence if f and g are functions on M, ξ, η are sections of $\mathcal{A}$, we get

$$\{f,g\} = 0, \quad \{f,\xi\} = \varrho(\xi)\cdot f, \quad \{\xi,\eta\} = [\xi,\eta]\,. \tag{3}$$

Using local coordinates $(x^1,\dots,x^n,\mu^1,\dots,\mu^r)$ on $\mathcal{A}^*$ corresponding to adapted local coordinates $(x^1,\dots,x^n,\lambda^1,\dots,\lambda^r)$ on $\mathcal{A}$, as in section 1, we may write

$$\{x^i,x^j\} = 0 \quad \{\mu^i,\mu^j\} = \sum c_{ij}^k\mu^k \quad \{x^i,\mu^j\} = a_{ji} \tag{4}$$

In the case of an integrable subbundle of TM, the dual $\mathcal{A}^*$ is the cotangent bundle along the leaves of the corresponding foliation. When $\mathcal{A}$ is the gauge Lie algebroid identified to TP/G, then $\mathcal{A}^*$ is T^*P/G; its Poisson structure is induced by the symplectic structure on T^*P.

We have to notice that J_1T^*M is not the dual of J_1TM in a natural way.

4. The Lie algebroid of a vector bundle

1°) Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be vector bundles over the same manifold M; then the fiber product $E_1 \times_M E_2$ is defined by

$$E_1 \times_M E_2 = \big\{(x_1,x_2) \in E_1 \times E_2; \quad \pi_1(x_1) = \pi_2(x_2)\big\}\,.$$

This fiber product can be considered as a vector bundle in 3 different ways

a) this product is a vector bundle over the base M and $E_1 \times_M E_2$ may be written $E_1 \oplus E_2$.

b) the product is a vector bundle over E_1; then $E_1 \times_M E_2$ may be written $\pi_1^*E_2$.

c) the product is a vector bundle over E_2; then $E_1 \times_M E_2$ may be written $\pi_2^*E_1$.

Let $\pi : E \to M$ be a vector bundle and let p be the projection $TE \to E$. Then there exists a surjective map $\Phi : TE \to E \times_M TM$ defined by

$$\Phi(z) = \big(p(z), T\pi(z)\big)$$

This map Φ may be considered as a morphism of vector bundles over the base E; then its kernel is the *vertical bundle* $VE = \ker T\pi$

The set $\mathcal{B}(E) = \{z \in TE; p(z) = 0\}$ is the restriction of TE to the zero section 0_M of E.

The set $\mathcal{A}(E) = \{z \in TE; \Phi(z) = (0,0)\}$ is the restriction of VE to 0_M.

2°) The zero section 0_M of the vector bundle $\pi : E \to M$ may be identified with M. Each fiber $E_x = \pi^{-1}(x)$ may be considered as an affine space with a distinguished element $0_x \in 0_M$. We remark that the tangent bundle TE_x is the inverse image of E_x by the projection $VE \to E$. In particular $T_{0_x}E_x = VE|_{0_x}$ (the set of all vertical tangent vectors to E with origin 0_x).

For $v \in E_x$, the translation τ_v is the map $E_x \to E_x$ defined by $v' \mapsto v' + v$; in particular $\tau_v(0_x) = v$ and $T\tau_v$ maps $T_{0_x}E_x$ onto T_vE_x; so we get the bijection $\lambda_x : E_x \times T_{0_x}E_x \to TE_x$ defined by $\lambda_x(v,w) = T\tau_v(w)$; the inverse map $(\lambda_x)^{-1}$: $TE_x \to E_x \times T_{0_x}E_x$ is defined by $y \mapsto (v,w)$ with

$$v = p(y)\,, \quad w = T\tau_{-v}(y)\,.$$

We deduce the map

$$\lambda : E \times_M VE|0_M \to VE$$

defined by $\lambda(v,w) = T\tau_v(w)$ for all $x \in M$, $v \in E_x$, $w \in VE|_{0_x}$.

We recover the map λ introduced by C. Marle [M]. The vector $\lambda(v,w)$ is the vertical lift of w with respect to v, in the terminology of Abraham-Marsden [A.M].

3°) The addition in the fibers of $\pi : E \to M$ defines a Lie groupoid structure on E for which $\alpha = \beta = \pi$ and the set of units is 0_M. So the *corresponding Lie algebroid is the vector bundle $VE|0_M$*. It is why this vector bundle has been denoted by $\mathcal{A}(E)$ in the first part of the section.

We could identify $\mathcal{A}(E)$ with E but in the case of the tangent bundle $E = TM$, we shall have to distinguish the two vector bundles.

We remark that the anchor map from $\mathcal{A}(E)$ to TM is the zero map; so the bracket of any pair of sections of $\mathcal{A}(E)$ vanishes and the Poisson tensor Λ on the dual $\mathcal{A}^*(E)$ is null.

5. The case of the tangent bundle TM

Let T^2M be the set of all 2-velocities on M i.e. the set of 2-jets from $\mathbb{R}$ to M, with source $0 \in \mathbb{R}$.

We know that T^2M is a submanifold of TTM (see [L.M]); a tangent vector to TM which is a 2-velocity will be said to be *holonomic*.

The submanifold T^2M is defined by

$$T^2M = \left\{v^2 \in TTM; Tp(v^2) = p(v^2)\right\}. \tag{5}$$

So the projection $\varpi : T^2M \to TM$ which associated a 1-jet to any 2-jet is the restriction to T^2M of both maps Tp and p. Thus $\varpi^{-1}(0_M)$ is the kernel of p and Tp; according to the first part of section 4, it is the vector bundle $VTM|0_M$ i.e. the Lie algebroid $\mathcal{A}(TM)$. In [L.M], the bundle $\varpi^{-1}(0_M)$ was denoted by $\mathfrak{G}^2M$.

Proposition. *The Lie algebroid $\mathcal{A}(TM)$ is a vector bundle isomorphic to TM, with the same transition functions but the action of $\mathbb{R}$ onto $\mathcal{A}(TM)$ considered as a subspace of T^2M is different from the action of $\mathbb{R}$ on TM.*

Proof. Let $(x^1, \ldots, x^n)$, $(u^1, \ldots, u^n)$ be local coordinates associated with two charts and $F = (F^1, \ldots, F^n)$ be the change of coordinates. Then

$$\frac{du^i}{dt}\Big|_{t=0} = \sum \frac{\partial F^i}{\partial x^j}\frac{dx^j}{dt}\Big|_{t=0}$$

$$\frac{d^2u^i}{dt^2}\Big|_{t=0} = \sum \frac{\partial^2 F^i}{\partial x^j \partial x^k}\frac{dx^j}{dt}\Big|_{t=0}\frac{dx^k}{dt}\Big|_{t=0} + \sum \frac{\partial F^i}{\partial x^j}\frac{d^2x^j}{dt^2}\Big|_{t=0}.$$

$$\text{If}\quad \frac{dx^j}{dt}\Big|_{t=0} = 0, \quad \text{then} \quad \frac{d^2u^i}{dt^2}\Big|_{t=0} = \sum \frac{\partial F^i}{\partial x^j}\frac{d^2x^j}{dt^2}\Big|_{t=0}.$$

Moreover by a change of parameter $s = \frac{t}{a}$ (where $a \neq 0$ is a constant), we obtain $\frac{d^2u^i}{ds^2}\big|_{s=0} = a^2\frac{d^2u^i}{dt^2}\big|_{t=0}$.

Corollary. *The bundle $\varpi : T^2M \to TM$ is an affine bundle whose associated vector bundle is $VTM = TM \times_M \mathcal{A}(TM)$.*

Proof. Let v^2 and v'^2 be holonomic vectors tangent to TM at v; we may consider their difference $w^2 = v^2 - v'^2$. As $p(v^2) = p(v'^2)$ we deduce $Tp(v^2) = Tp(v'^2)$; hence $v^2 - v'^2$ is vertical. Conversely given v^2 holonomic and w^2 vertical, $v^2 - w^2$ is holonomic.

A *second order differential equation* is a vector field $X : TM \to TTM$ whose values are holonomic. From the corollary we deduce that if two vector fields on TM define second order differential equations, their difference is a vertical vector field.

Given a second order differential equation $X : TM \to T^2M$, with every 2-velocity $v^2 \in T^2M$ is associated the vertical vector w^2 defined by the relation $w^2 = v^2 - X\varpi(v^2)$; contrary to what is written in [L.M], w^2 does not belong to $\mathcal{A}(TM) = \mathfrak{G}^2M$ but the transform of w^2 by the translation $T\tau_{-v}$ (where $v = \varpi(v^2)$) belongs to $\mathcal{A}(TM)$; this vector $T\tau_{-v}(w^2)$ will be called the *acceleration* of v^2. The vertical vector w^2 could be called the *lifted acceleration*.

This notion of acceleration is not intrinsic; it depends upon the choice of the second order differential equation X. So $T\tau_{-v}(w^2)$ could be called the acceleration of v^2 with respect of X.

For any differentiable curve $\gamma : I \subset \mathbb{R} \to M$, its lift to T^2M defines a curve $\delta : I \to \mathcal{A}(TM)$ called the field of accelerations. A solution of X is a curve with zero acceleration. For instance if X is the spray attached with a linear connection, its solutions are the geodesics of the connection.

In local coordinates $(x^i, \dot{x}^i)$, the components of the acceleration are $\frac{d^2x^i}{dt^2} - \sum \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}$. The acceleration is the covariant derivative of the velocity.

6. The case of the cotangent bundle T^*M

We recall (see [L.M]) that given a vector bundle $\pi : E \to M$ a *semi-basic* form on E is a differential $1-form$ η on E satisfying one of the equivalent properties
a) $\eta(X) = 0$ for any vertical vector field X on E

b) there exists a unique morphism $f : E \to T^*M$ such that for any $y \in E$, then $\eta_y = T_y^*\pi(f(y))$

c) η is a section of the vector bundle $E \times_M T^*M \to E$.

We have proved that $\eta = f^*\theta$, where θ is the Liouville form on T^*M. Conversely for any morphism $g : E \to T^*M$, the form $g^*\theta$ is semi-basic.

As the restriction of π to the zero section 0_M is the identity map, for any $y \in 0_M$, we obtain $\eta_y = f(y)$i.e. the restriction of η to 0_M is a form on M.

By means of adapted local coordinates on E, the semi-basic form η may be written

$$\eta = \sum_{i=1}^{n} a_i(x^1, \ldots, x^n, y^1, \ldots, y^k)\, dx^i\,.$$

If θ is written locally

$$\theta = \sum_{i=1}^{n} p_i\, dx^i\,,$$

the morphism f is expressed by

$$p_i = a_i(x^1, \ldots, x^n, y^1, \ldots, y^k)\,.$$

Along 0_M, η is written $\eta|0_M = \sum_{i=1}^{n} a_i(x^1, \ldots, x^n, 0 \ldots 0)\, dx^i$.

If we consider the bundle $q : T^*M \to M$, the form θ is semi-basic, its associated morphism is the identity map of T^*M. In [L.M], θ was defined utilizing this property.

As the vertical fibration is lagrangian, for any pair (X, Y) of vertical tangent vector fields on T^*M, the relation $d\theta(X, Y) = 0$ holds. Using property (a) of the semi-basic forms, we deduce

Proposition. *On T^*M, the symplectic duality $X \to -i(X)d\theta$ induces a bijection between vertical vector fields and semi-basic forms; in particular this duality transforms elements of $\mathcal{A}(T^*M)$ into elements of T^*M. This natural isomorphism permits to identity $\mathcal{A}(T^*M)$ and T^*M.*

Corollary. *A section of the bundle $T^*M \times_M T^*M \to T^*M$ may be interpreted either as a vertical vector field or as a semi-basic form.*

In local coordinates $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ such a section can be written $\sum_{i=1}^{n} X_i \frac{\partial}{\partial p_i}$ or $\sum_{i=1}^{n} X_i\, dx^i$, where each X_i is a function of $(x^1, \ldots, x^n, p_1, \ldots, p_n)$.

Remarks. 1) Using local coordinates we can show that T^*M is the dual of $\mathcal{A}(TM)$. Let $\gamma : I \subset \mathbb{R} \to M$ and $g : U \subset M \to \mathbb{R}$ (where U is an open subset containing $\gamma(I)$). If $\left.\frac{d\gamma}{dt}\right|_{t=0} = 0$, then $\left.\frac{d^2}{dt^2}(g \circ \gamma)\right|_{t=0} = \sum \left.\frac{\partial g}{\partial x^i} \frac{d^2 x^i}{dt^2}\right|_{t=0}$.

2) Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be vector bundles and $F : E_1 \to E_2$ be a (non necessarly linear) morphism i.e. F is a map such that $\pi_1 = \pi_2 \circ F$. If η_2 is a semi-basic form on E_2, then $\eta_1 = F^*\eta_2$ is semi-basic; moreover at the points y_1 and $y_2 = F(y_1)$, the forms η_1 and η_2 are the pullbacks of the same form $\varphi \in T^*M$. Indeed let $f_2 : E_2 \to T^*M$ such that $\eta_2 = f_2^*\theta$ and $f_1 = f_2 \circ F$. Then $\eta_1 = F^*\eta_2 = f_1^*\theta$. For $y_1 \in E_1$, we get $f_1(y_1) = f_2F(y_1) = \varphi$.

7. The Legendre transformation and the Lagrangian differential

Let $L : TM \to \mathbb{R}$ be a smooth function (the Lagrangian). The Legendre morphism $\mathcal{L} : TM \to T^*M$ associated with L can be defined as a morphism whose restriction $\mathcal{L}_x$ to each fiber T_xM is the differential dL_x of the restriction L_x of L to T_xM. When $\mathcal{L}$ is a diffeomorphism (then $\mathcal{L}$ is called Legendre transformation and L is said to be hyperregular), the tangent bundle TM is endowed with a symplectic form $\Omega = \mathcal{L}^* d\theta$ (where θ is the Liouville form on T^*M); the fibers of TM are Lagrangian submanifolds of TM. As was proved in section 6 for the cotangent bundle, the symplectic duality induces a bijection between vertical vector fields and semi-basic forms.

Let $\mathcal{E} : TM \to \mathbb{R}$ be the energy i.e. the function $\mathcal{E} : i(Z)\, dL - L$, where Z is the Liouville vector field. It is known that the hamiltonian vector field $X_{\mathcal{E}}$ (i.e. such that $i(X_{\mathcal{E}})\Omega = -d\mathcal{E}$) is a second order differential equation. So for any other second order differential equation X, the vector field $X - X_{\mathcal{E}}$ is vertical and the form $\varphi = i(X - X_{\mathcal{E}})\Omega$ is semi-basic; this form φ induces a morphism f from TM to T^*M such that, for every $v \in TM$, we have $\varphi(y) = T_y^*\pi\big(f(y)\big)$. The form φ has been called *force field* by C. Godbillon [G].

By means of local coordinates $(x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n)$ on TM, the integral curves of $X_{\mathcal{E}}$ are locally solutions of the Lagrange equations

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}^i}\Big) - \frac{\partial L}{\partial x^i} = 0\,, \quad (i = 1, \dots, n) \tag{6}$$

so if φ is written $\sum X_i(x^j, \dot{x}^j)\, dx^i$, the integral curves of X are locally solutions of the equations

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}^i}\Big) - \frac{\partial L}{\partial x^i} = X_i(x^j, \dot{x}^j) \tag{7}$$

(see [G]). In terms of local coordinates $(x^1, \dots, x^n, p_1, \dots, p_n)$ on T^*M, the morphism $f : TM \to T^*M$ can be written

$$p_i = X_i(x^j, \dot{x}^j) \qquad (i = 1, \dots, n)$$

These properties can be formulated as follows: the Lagrangian L and hence the vector field $X_{\mathcal{E}}$ being given, we associate with any 2-velocity its acceleration, as indicated in section 5; this acceleration belongs to $\mathcal{A}(TM)$. When L is hyperregular, TM is endowed with a symplectic form and the symplectic duality maps $\mathcal{A}(TM)$ onto T^*M. We deduce a morphism from T^2M to T^*M, which is called the *Lagrangian differential* and denoted $\Delta(L)$ (see [M]). The notion of Lagrangian differential was introduced by W. Tulczyjew [T].

8. On constrained mechanical systems. Lagrangian point of view

We shall use Marle's definition for such a system. A *constrained mechanical system* is a triple (M, L, C) where M is a manifold, L a hyperregular Lagrangian on TM, and C a submanifold of the tangent bundle TM, with the following regularity condition: there exists a submanifold M_1 of M such that C is contained in TM_1 and the restriction $p|C$ to C of the projection $p : TM \to M$ is a submersion of C onto M_1. The submanifold C is called the *constraint submanifold*. Moreover we assume that $p|C$ is not a diffeomorphism.

Under these assumptions, for any $x \in M_1$, $C_x = C \cap T_x M_1 = (p|C)^{-1}(x)$ is a submanifold of $T_x M_1$ and hence of $T_x M$. For any $v \in C_x$, the tangent space $T_v C_x$ is the vector space $\ker T_v(p|C)$. This vector space is called space of "admissible infinitesimal virtual displacements" in [M]; in accordance with our terminology, we shall call it the space of "lifted admissible infinitesimal virtual displacements". Then the set of all these displacements is the vertical bundle $VC = \ker T(p|C)$; this bundle VC is a vector subbundle of $V_C TM_1$ (vertical tangent bundle to TM_1 restricted to C) and of $T_C TM_1$, and also of $V_C TM$ and $T_C TM$.

As $p|C$ is a surjective submersion, the vector bundle $(T_C p)^{-1}(TM_1)$ is the (non direct) sum

$$(T_C p)^{-1}(TM_1) = TC + V_C TM\,. \tag{8}$$

As the Lagrangian L is hyperregular, for any $x \in M$ and any $v \in T_x M$, the second order differential $D^2 L_x(v)$ is a symmetric non degenerate bilinear form acting on $T_v T_x M \times T_v T_x M$ and inducing an isomorphism from $T_v T_x M$ onto $T_v^* T_x M$. According to [M], we shall say that L is *normal* on C if moreover the quadratic form associated with $D^2 L_x(v)$ is positive for any $v \in C_x$ with $x \in M_1$. In this case we obtain a scalar product on $V_C TM$ (restriction to C of the vertical bundle VTM).

For any $v \in C_x$, we obtain

$$T_v T_x M = T_v C_x \oplus (T_v C_x)^{\perp} \tag{9}$$

where $(T_v C_x)^{\perp}$ is the orthogonal of $T_v C_x$ with respect of this scalar product. We deduce the splittings

$$V_C TM = VC \oplus (VC)^{\perp} \tag{10}$$

and

$$(T_C p)^{-1}(TM_1) = TC \oplus (VC)^{\perp} \tag{11}$$

If we consider the restriction $X_{\mathcal{E}}|C$ of the hamiltonian vector field $X_{\mathcal{E}}$ to C, this vector field belongs to $(T_C p)^{-1}(TM_1)$; indeed for any $v \in C$, we have $Tp(X_{\mathcal{E}}(v)) = pX_{\mathcal{E}}(v) = v$, as $X_{\mathcal{E}}$ defines a second order differential equation. So we get the splitting

$$X_{\mathcal{E}}|C = X_C + X' \tag{12}$$

where X_C is a vector field tangent to C and X' is vertical. We deduce that X_C is a holonomic vector field tangent to C.

A *motion* of the mechanical system is a smooth curve $\gamma : I \subset \mathbb{R} \to M$ whose lift γ^1 in TM (i.e. $\gamma^1(t) = \frac{d\gamma}{dt}\big|_t$) is an integral curve of the vector field X_C.

In terms of local coordinates $(x^j, \dot{x}^j)$ on TM, we deduce from section 7 that γ is a solution of the second order differential system

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}^i}\Big) - \frac{\partial L}{\partial x^i} = X_i(x^j, \dot{x}^j),$$

where the X_i's are defined by $\varphi = i(X')\Omega = \sum X_i \, dx^i$.

Let $v = \gamma^1(t)$ and τ_{-v} the translation acting on T_xM (with $x = p(v)$) (see section 4). The tangent space T_vC_x is mapped by $T\tau_{-v}$ onto a vector subspace F_v of $\mathcal{A}(TM)|_x$, which may be called in our terminology the space of "*admissible infinitesimal virtual displacements*" at time τ. We get the splitting

$$\mathcal{A}(TM)|_x = F_v \oplus (F_v)^\perp$$

where $(F_v)^\perp$ is the transform of $(T_vC_x)^\perp$ by $T\tau_{-v}$.

According to formula (12) the vertical vector $X'(v)$ is mapped by $T\tau_{-v}$ onto the acceleration (in the sense of section 5) of the 2-velocity $j_t^2\gamma$ with respect of the vector field $X_{\mathcal{E}}$. Thus we obtain the following property which expresses the principle of d'Alembert-Lagrange: for any t, *the acceleration of $j_t^2\gamma$with respect of the vector field $X_{\mathcal{E}}$ is orthogonal to all the admissible infinitesimal virtual displacements at time t.*

The constraint is said to be *perfect*.

The semi-basic form $\varphi = i(X')\Omega$ on TM induces a morphism $f : TM \to T^*M$ such that $\varphi = f^*\theta$ (as seen in section 6) and the Lagrangian differential $\Delta(L)$ of $j_t^2\gamma$ (in the sense of section 7) is an element $f(t)$ of T^*M, called the *constraint force* at time t.

Thus the relation between the acceleration and the constraint force (which generalizes Newton equation) is given by the symplectic duality.

9. On constrained mechanical systems. Hamiltonian point of view

We consider the constrained mechanical system (M, L, C) where L is hyperregular on TM and normal on C. Let D be the image of C by the corresponding Legendre transformation $\mathcal{L}$. The linear isomorphism $T\mathcal{L}$ maps VTM onto VT^*M and V_CTM onto V_DT^*M; in particular VC is mapped onto VD. We set $W = T\mathcal{L}\Big((VC)^\perp\Big)$.

We deduce from section 8 the splittings

$$V_DT^*M = VD \oplus W \tag{14}$$

$$(T_Dq)^{-1}(TM_1) = TD \oplus W \tag{15}$$

We shall show that W is the "projection bundle" of [M]; thus we recover C. Marle's results. For this purpose we shall study the links between the bilinear form $D^2L_x(v)$ (with $v \in C$, $x = p(v)$) and $T\mathcal{L}|T_vT_xM$.

By the translation τ_{-v} along the fiber T_xM, the form $D^2L_x(v)$ is transformed into a bilinear form Φ_x on $\mathcal{A}(TM)|_x \times \mathcal{A}(TM)|_x$ inducing an isomorphism Φ_x^b of $\mathcal{A}(TM)|_x$ onto its dual T_x^*M. The vector subspace F_v of $\mathcal{A}(TM)|_x$ (image of T_vC_x by $T\tau_{-v}$) is mapped by Φ_x^b onto a vector subspace $\widetilde{F_v}$ of T_x^*M and $(F_v)^\perp$ is mapped onto the annihilator $(F_v)^\circ$ of F_v into T_x^*M.

Now let us consider the translation $\tau_{\mathcal{L}(v)}$ along the fiber T_x^*M. The image by $T\tau_{\mathcal{L}(v)}$ of $\widetilde{F_v}$ is $T\mathcal{L}(T_vC_x)$; the image of $(F_v)^\circ$ is $T\mathcal{L}\Big((T_vC_x)^\perp\Big) = W_{\mathcal{L}(v)}$.

This property of W corresponds to the definition of W in [M], according to the second part of section 4. Remark that W can be defined when L is not normal; $W_{\mathcal{L}(v)}$ is the image of $(F_v)^\circ$ by $T\tau_{\mathcal{L}(v)}$.

The transform by $\mathcal{L}$ of the vector field $X_{\mathcal{E}}|C$ is the restriction $Y_H|D$ of the hamiltonian vector field, where $H = \mathcal{E}\cdot\mathcal{L}^{-1}$. As $X_{\mathcal{E}}|C$ belongs to $(T_Cp)^{-1}(TM_1)$, the *vector field* $Y_H|D$ *belongs to* $(T_Dq)^{-1}(TM_1)$.

From formula (15), we deduce the splitting

$$Y_H|D = Y_D + Y_W\,,$$

where Y_D is tangent to D and Y_W is a section of W; Y_W is the vertical vector field, transform of X' by $\mathcal{L}$.

The semi-basic form $\Psi = i(Y_W)\,d\theta$ is the form $(\mathcal{L}^{-1})^*\varphi$.

According to remark 2 of section 6, at the points v and $\mathcal{L}(v)$ the forms φ and ψ are the pullbacks of the same element of T_x^*M.

This explains why the constrained Lagrange equations (7) and the constrained Hamilton's equations

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} + X_i(x^i(t), p_i(t))$$

are equivalent. Here we have identified Y_W and Ψ.

10. The Legendre transformation on Lie algebroids

1°) Let $\pi : E \to M$ be a vector bundle. Every vector field Y on E induces a map $g = T\pi\cdot Y$ from E to TM.

For any $x \in M$ and any $y \in \pi^{-1}(x)$, the vector $Y(y)$ which belongs to T_yE is mapped by $T_y\pi$ onto a vector v belonging to T_xM; hence $p\circ g = \pi$ and g is a morphism.

Consider now the map $Tg : TE \to TTM$; the linear map T_yg carries $Y(y)$ onto $w \in T_vTM$; so $p(T_yg\circ Y)(y) = v$. On the other side $Tp(T_yg\circ Y)(y) = T_y(p\circ g)Y(y) = T_y\pi(Y(y)) = v$. So $Tg(Y(y))$ belongs to T^2M. We deduce

Proposition. *Let* $\pi : E \to M$ *be a vector bundle and* Y *be a vector field on* E. *Then the mapping* $g = T\pi\cdot Y$ *is a morphism from* E *to* TM *such that, for any* $y \in E$, *the image of* $Y(y)$ *by* Tg *is a holonomic tangent vector to* TM *i.e. a 2-velocity.*

Corollary. *If moreover there exists on $g(E)$ a vector field X such that the pair (X, Y) is g-related i.e. $T_y g(Y(y)) = X(g(y))$ for any $y \in E$, then the vector field X defines a second order differential equation.*

Definition. If there is given a morphism $\rho : E \to TM$, any vector field Y on E such that $g = T\pi \cdot Y$ coincides with ρ, will be called *admissible*.

The notion of admissible vector field has been introduced by A.Weistein [W] but, contrary to this author, we keep the notion of second order differential equation for holonomic vector fields on TM.

In the special case of the Legendre transformation $\mathcal{L} : TM \to T^*M$ induced by a hyperregular Lagrangian function $L : TM \to \mathbb{R}$, we have proved in [L.M], that the hamiltonian vector filed Y_H on T^*M is admissible for the mapping $\mathcal{L}^{-1}$ and the image of Y_H by $\mathcal{L}^{-1}$ is the holonomic vector field $X_\mathcal{E}$, where $\mathcal{E} = i(Z)\, dL - L$ is the energy and $H = \mathcal{E} \circ \mathcal{L}^{-1}$.

2°) As for the tangent bundle, a smooth function L on the Lie algebroid $\pi : \mathcal{A} \to M$ defines a morphism $\mathcal{L} : \mathcal{A} \to \mathcal{A}^*$ (see [L.M]); the restriction $\mathcal{L}_x$ of $\mathcal{L}$ to the fiber $\mathcal{A}_x = \pi^{-1}(x)$ is defined by the differential of L_x (restriction of L to $\mathcal{A}_x$). If $\mathcal{L}$ is a diffeomorphism, then the Poisson structure on $\mathcal{A}^*$ is carried by $\mathcal{L}^{-1}$ onto a Poisson structure on $\mathcal{A}$ called the *Lagrange Poisson structure* by A. Weinstein [W].

With the notation of section 3, the Legendre transformation may be written

$$\mu^j = \frac{\partial L}{\partial \lambda^j} \qquad j = 1, \ldots, r\,.$$

Hence the brackets relations for the Lagrange Poisson structure are

$$\{x^i, x^k\} = 0, \quad \left\{\frac{\partial L}{\partial \lambda^j}, \frac{\partial L}{\partial \lambda^s}\right\} = \sum C^m_{js} \frac{\partial L}{\partial \lambda^m}, \quad \left\{x^i, \frac{\partial L}{\partial \lambda^j}\right\} = a_{ji}\,.$$

Weinstein's result may be formulated as follows: *the hamiltonian vector field $X_\mathcal{E}$ associated with the energy $\mathcal{E} = i(Z)\, dL - L$ by means of the Lagrange Poisson structure is admissible with respect of the anchor map $\rho : \mathcal{A} \to TM$ i.e. the map $T\pi \circ X_\mathcal{E}$ is the anchor map.* The vector field $X_\mathcal{A}$ is called the *lagrangian vector field.*

Using a trivialization $U \times E$ of $\pi^{-1}(U)$ (where U is an open set of M and E a vector space), we write $\mathcal{E} = \sum \lambda^j \frac{\partial L}{\partial \lambda^j} - L$ and $X_\mathcal{E} = \sum \alpha_i \frac{\partial}{\partial x^i} + \sum \beta_j \frac{\partial}{\partial \lambda^j}$; then $T\pi(X_\mathcal{E}) = \sum \alpha_i \frac{\partial}{\partial x^i}$ with $\alpha_i = X_\mathcal{E} \cdot x^i = \{x^i, \mathcal{E}\}$. Following Weinstein's computations, we obtain $\{x^i, \mathcal{E}\} = \sum \{x^i, \frac{\partial L}{\partial \lambda^j}\}\lambda^j + \sum\{x^i, \lambda^j\}\frac{\partial L}{\partial \lambda^j} - \{x^i, L\}$. As $\{x^i, x^k\} = 0$, we get $\{x^i, \mathcal{E}\} = \sum\{x^i, \frac{\partial L}{\partial \lambda^j}\}\lambda^j = \sum a_{ji}\lambda_j = \rho(\sum \lambda_j \mathcal{E}_j) \cdot x^i$.

From this property, it can be deduced that the projection onto M of any integral curve of $X_\mathcal{E}$ lies in a leaf of the foliation of M determined by the image of ρ in TM.

Remark. The hamiltonian vector field Y_H on $\mathcal{A}^*$ (with $H = \mathcal{E} \circ \mathcal{L}^{-1}$) is admissible for the map $\rho' = \rho \circ \mathcal{L}^{-1}$ from $\mathcal{A}^*$ to TM. Indeed let $\tilde{\pi}$ be the projection $\mathcal{A}^* \to M$;

then $\widetilde{\pi} \circ \mathcal{L} = \pi$. As Y_H is the transform of $X_{\mathcal{E}}$ by $\mathcal{L}$, it can be checked that $T\widetilde{\pi} \circ Y_H = T\pi \circ X_{\mathcal{E}}$.

The integral curves of $X_{\mathcal{E}}$ and Y_H have the same projection on M. So these projections lie in the projections of the symplectic leaves of the Poisson structure on $\mathcal{A}^*$.

Examples a) If $\mathcal{A}$ is a completely integrable subbundle of TM, we are in the case of a *mechanical system with holonomic constraints*. In this case the motion $t \to \gamma(t)$ of the system is determined by the restriction of the Lagrangian L to the subbundle $\mathcal{A}$. Then H is constant on the lift of γ to $\mathcal{A}^*$. We recover a result of [M].

b) Let P be a principal G-bundle. The pullback to TP of a Lagrangian L on the gauge Lie algebroid TP/G is a G-invariant function. The corresponding vector field on TP is projectable onto a vector field on TP/G which is the vector field obtained by the above procedure.

11. The case of the algebroid J_1TM

Consider a vector bundle $\pi : E \to M$ and the bundle $\pi^1 : J_1E \to M$. For any local section $s : U \subset M \to E$ (where $U \ni x$), the jet $j_x^1 s$ can be identified with $T_{s(x)}W$ where $W = s(U)$; it can be deduced that $(J_1E)_y$, set of all 1-jets of sections of E with target $y \in E$ can be identified with the set of all vector subspaces of T_yE which are complementary to the vertical subspace V_yE. Consequently $\beta : J_1E \to E$ is an affine bundle. A linear connection $\mathcal{C}$ on E i.e. a linear lifting $\mathcal{C} : E \to J_1E$ induces a splitting $TE = VE \oplus HE$, where $H_yE = \mathcal{C}(y)$.

Consider now the tangent bundle $p : TM \to M$. A hyperregular Lagrangian L on TM induces a Legendre transformation $\mathcal{L} : TM \to T^*M$ which can be prolongated to a diffeomorphism $\mathcal{L}^{(1)} : J_1TM \to J_1T^*M$. Indeed any local section $s : U \to TM$ is transformed into a section $\mathcal{L} \circ s$ of T^*M; passing to the jets we obtain $\mathcal{L}^{(1)}$. From what we have remarked just above, $\mathcal{L}^{(1)}$ is the restriction of $T\mathcal{L}$ to the vectors tangent to TM which are transverse to the vertical bundle VTM. By means of $\mathcal{L}^{(1)}$, the structure of Lie algebroid is carried on J_1T^*M with the anchor map $\beta \circ (\mathcal{L}^{(1)})^{-1}$.

On the other side, we have shown in [L.M] that J_1T^*M is endowed with a 2-form $\eta = \beta^* d\theta - \Theta$, where θ is the Liouville 1-form on T^*M, Θ the pullback of the Liouville 2-form on $\Lambda^2 T^*M$ by the projection $J_1T^*M \to \Lambda^2T^*M$. The characteristic property of η is the following; a local section $\sigma : U \to J_1T^*M$ is the jet $j^1\omega$ of a Pfaffian form ω on M if and only if $\sigma^*\eta = 0$. The form $(\mathcal{L}^{(1)})^*\eta$ has a similar property on J_1TM.

Now if we consider a linear connection $\mathcal{C} : T \to J_1T$, it induces a linear connection $\widetilde{\mathcal{C}} : T^* \to J_1T^*$ and splittings

$$TTM = VTM \oplus HTM\,, \qquad TT^*M = VT^*M \oplus HT^*M\,.$$

A hyperregular Lagrangian L on TM inducing a Legendre transformation $\mathcal{L} : TM \to T^*M$ could be called *adapted* to $\mathcal{C}$ if the image of HTM by $T\mathcal{L}$ is HT^*M. It would be interesting to study the following problems.

a) $\mathcal{C}$ being given, find conditions to the existence of an adapted L

b) $\mathcal{C}$ being given, find a hyperregular function L such that the geodesics of the connection are the projection on M of the integral curves of the lagrangian vector field $X_{\mathcal{E}}$. This problem is a particular case of one studied by I. Vaisman [V]: given a vector field X on TM defining a second order differential equation, study the existence and generality of the Poisson structures on TM for which X is a lagrangian vector field.

c) find relations between problem a) and problem b).

It is to notice that the connection $\mathcal{C} : T \to J_1 T$ induces a linear lifting $T \to TH|GL(\eta, \mathbb{R})$, hence a principal connection on H (see [E]).

12. The case of the Lie algebroid of a symplectic groupoid

A. Weinstein [W] has put the question whether the formulation due to J. Klein [Kl] (see also [G]) which produces the lagrangian vector field on the tangent bundle without using the cotangent bundle can be extended to general Lie algebroids.

We recall that J. Klein has introduced an endomorphism v on TTM, whose image is VTM and such that $v^2 = 0$. With v is associated a differential d_v such that $d_v f = \sum \frac{\partial f}{\partial \dot{x}^i} dx^i$, for any function for TM. This endomorphism v may be defined as $v = \varpi \circ J$ (see [L.M]) where ϖ is the projection $TTM \to TTM/VTM$ and J the isomorphism $TTM/VTM \to VTM$. Then given a Lagrangian L on TM, the 2-form $dd_v L$ is equal to the form $\mathcal{L}^* d\theta$ defined in section 7.

In general the isomorphism J does not exist for an algebroid $\mathcal{A}$; so these methods do no work for general Lie algebroids when the dimension of the fibers is different from the dimension of the base.

We shall investigate the case of the Lie algebroid $\mathcal{A}(\Gamma)$, where Γ is a *symplectic groupoid* which may be defined as follows when the α-fibers and β-fibers are connected (see [L_5]). A symplectic groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ is a groupoid endowed with a symplectic from such that 1°) the base Γ_0 is a lagrangian submanifold of Γ 2°) the α-foliation and the β-foliation are symplectically orthogonal.

Then the fibers and the base have the same dimension and the base Γ_0 is endowed with a Poisson structure.

It is remarked in [A.D] that the symplectic duality $\Omega^\#$ defines an isomorphism from $\mathcal{A}^*(\Gamma)$ onto $T\Gamma_0$ inducing a homogeneous Poisson structure on $T\Gamma_0$ and a Lie algebroid structure on $T^*\Gamma_0$. The authors have proved that this Lie algebroid structure on $T^*\Gamma_0$ coincides with that defined by the bracket $[df, dg] = d\{f, g\}$, where $\{\ \}$ is the Poisson bracket on Γ_0. The anchor map $\rho' : T^*\Gamma_0 \to T\Gamma_0$ is the map defined by the Poisson bivector field Λ on Γ_0. This map ρ' is also the map $\rho \circ \Omega^\#$ where ρ is the anchor map $\mathcal{A}(\Gamma) \to T\Gamma_0$.

Given a hyperregular Lagrangian L on $\mathcal{A}(\Gamma)$ (with Legendre transformation $\mathcal{L} : \mathcal{A}(\Gamma) \to \mathcal{A}^*(\Gamma)$, as in section 10) we obtain a lagrangian vector field $X_{\mathcal{E}}$ on $\mathcal{E}$ which is admissible with respect of ρ. By the isomorphism $\mathcal{I} : \mathcal{A}(\Gamma) \to T\Gamma_0$ (with $\mathcal{I} = \Omega^\# \circ \mathcal{L}$), we get a hyperregular Lagrangian $\widehat{L} = L \circ \mathcal{I}^{-1}$ on $T\Gamma_0$ and a lagrangian vector field $X_{\widehat{\mathcal{E}}}$ on $T\Gamma_0$, transform of $X_{\mathcal{E}}$ by $\mathcal{I}$, associated with the Poisson structure on $T\Gamma_0$.

On the other side the function $\widehat{L}$ defines a symplectic form $dd_v\widehat{L}$ on $T\Gamma_0$ and hence a vector field $Y_{\widehat{\mathcal{E}}}$ such that $i(Y_{\widehat{\mathcal{E}}})dd_v\widehat{L} = -d\widehat{\mathcal{E}}$.

In general the Poisson structure on $T\Gamma_0$ which was defined above and the symplectic structure do not coincide. So even in this case, it is not possible to use directly J. Klein's formalism. When $\Gamma = M \times M$ with M a symplectic manifold, then $\mathcal{A}(\Gamma)$ is the tangent bundle along the diagonal of $M \times M$ and it can be identified with TM.

The Lie algebroid of a symplectic groupoid is an example of *Lie bialgebroid*. We shall not tackle this theory which was initiated by K. Mackenzie and P. Xu [M.X]. See also the papers by Z. L. Liu, A. Weinstein, P. Xu [L.W.X] and by Y. Kosmann-Schwarzbach [K_1], [K_2].

A Lie bialgebroid $\mathcal{A}$ satisfying, among others, the property that both $\mathcal{A}$ and $\mathcal{A}^*$ have a Lie algebroid structure, the following question arises: under which conditions a Legendre transformation $\mathcal{L}$ on a Lie algebroid $\mathcal{A}$ induces a Lie bialgebroid structure on $\mathcal{A}$.

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