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PIVOTING ALGORITHM IN CLASS OF ABS METHODS

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Summary: The paper deals with a pivoting modification of the algorithm in the class of ABS methods. Numerical experiments compare this pivoting modification with the fundamental version. A hybrid algorithm for the solution of the linear system with the Hankel matrix is introduced.

1. Introduction

In a recent monograph Abaffy and Spedicato [2] have introduced a class of direct methods for solution of the linear algebraic systems in the form:

$$Ax = b \tag{1}$$

where the matrix $A = (a_1, \ldots, a_m)^T$, the vectors $a_1, \ldots, a_m \in R^n, x \in R^n, b \in R^m$ and $m \le n$. The idea of the ABS methods consists in formation of the finite sequence of vectors $\{x_i\}_{i=1}^{m+1}$ with the property that the approximation x_{i+1} obtained at the i-th cycle is a solution of the first i equations of system (1). Then x_{m+1} solves the whole system (1). If $a_k^T \in R^n$ is the k-th row of the matrix A and b_k is the k-th component of the vector b, the system (1) is indicated componentwise

$$a_k^T x = b_k, \qquad k = 1, \ldots, m. \tag{2}$$

On the assumption that the vector $x_i \in R^n$ (the solution of the first $i-1$ equations of (2)) is known, it is possible to find the vector $x_{i+1} \in R^n$ so that it is the solution of the first i equations of (2).

Omitting the particular description of the theory about the ABS methods we introduce the general version of the ABS algorithm. While the ABS methods are able to solve underdetermined systems ($m < n$) we assume that $m = n$ and A is nonsingular. For more details see [1] and [2].

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ABS ALGORITHM

1) Let $x_1 \in R^n$ be an arbitrary vector. Let $H_1 \in R^{n,n}$ be an arbitrary nonsingular matrix.

2) Cycle for $i = 1, \dots, n$

a) Let $z_i \in R^n$ be a vector arbitrary save for the condition:

$$z_i^T H_i a_i \neq 0. \tag{3}$$

Compute search vector p_i :

$$p_i = H_i^T z_i. \tag{4}$$

b) Compute step size α_i :

$$\alpha_i = \frac{a_i^T x_i - b_i}{p_i^T a_i}, \tag{5}$$

which is well defined with regard to (3) and (4).

c) Compute the new approximation of the solution using

$$x_{i+1} = x_i - \alpha_i p_i. \tag{6}$$

If $i = n$ stop; x_{n+1} solves the system (1).

d) Let $w_i \in R^n$ be a vector arbitrary save for the condition:

$$w_i^T H_i a_i = 1, \tag{7}$$

and to update the matrix H_i:

$$H_{i+1} = H_i - H_i a_i w_i^T H_i. \tag{8}$$

There are three eligible parameters in the general version of the ABS algorithm: matrix H_1 and two systems of vectors z_i and w_i. The new algorithms or a new formulation of the classic algorithms can be created by a suitable choice of these parameters.

Abaffy et al. have studied the above system for a variety of choices of z_i and w_i, calculating the storage and arithmetic operations which are required to solve the systems with various kinds of matrices.

Many papers have dealt with the ABS modification of the LU decomposition. Numerical experiments with the ABS – LU algorithm have been made on special systems of linear equations by Bodon in [3], [4], [5]. Other experiments are available in a paper by Deng and Vespucci [7]. Bodon and Spedicato [6] have dealt with the LU, LQ and QU algorithms, and Phuan [9] has demonstrated that this method exploits sparsity in a natural way.

The purpose of this paper is to present one choice of the vectors z_i and w_i which corresponds to the choice of these vectors for the ABS – LU algorithm but does not require the strong nonsingularity of the matrix A. This choice of vectors leads to the method of behaviour similar to the known pivoting process in Gauss elimination. We will show one algorithm for solving the system (1) with Hankel or Toeplitz matrices which can be also used in a combination with the ABS algorithm.

In the section 2 we briefly describe some fundamental properties of the ABS methods and one simple choice of vectors z_i and w_i. Section 3 deals with the algorithm that Rissanen [10] has introduced for solving the linear systems of equations with Hankel (or Toeplitz) matrices, and the section 4 describes the relation of the Rissanen algorithm to the ABS algorithms. The substance of the paper resides in the section 5, where we deal with one modified parameter option in the ABS class. In the section 6 we evaluate the numerical pretension of the algorithms.

2. Fundamental properties of ABS methods.

The vectors p_i, x_i and the matrix H_i generated by the ABS algorithm dispose of the interesting properties.

Theorem 1. *For $i = 2, \ldots, n+1$, $Null(H_i)$ is generated by the vectors $a_1, \ldots, a_{i-1}$ (the first $i-1$ rows of the matrix A),e.q.*

$$H_i a_j = 0, \qquad j = 1, \ldots, i-1. \tag{9}$$

If $m = n$, $Range(H_i)$ is generated by the vectors $H_i a_i, \ldots, H_i a_n$, e.q.

$$H_i a_j \neq 0, \qquad j = i, \ldots, n. \tag{10}$$

Proof: [2], p.23, Theorem 3.1

Theorem 1 implies the validity of the following theorem which plays a fundamental role in the analysis of the ABS class.

Theorem 2. *Let $p_1, \ldots, p_n$ be the search vectors generated by (4) and let $P = (p_1, \ldots, p_n)$. Then the matrix L, defined as*

$$L = AP, \tag{11}$$

is nonsingular and lower triangular.

Proof: [2], p.30, Theorem 3.12

Further we use the simplest and most obvious choice of z_i and w_i about which Abaffy, Broyden and Spedicato [1] had shown that it corresponds to an implicit LU factorization of A. It is known that such factorization of the square matrix A exists if and only if A is strongly nonsingular. If A is nonsingular but not strongly nonsingular, there exists a suitable permutation of the rows or columns of A after which the factorization $A = LU$ exists.

From (11) in Theorem 2 it follows that here is only one question: how to choose H_1, z_i and w_i to achieve that matrix P^{-1} (and also matrix P) is upper triangular.

Theorem 3. *The sufficient condition for matrix P to be upper triangular is the following choice of the parameters:*

a) *vectors $w_1, \dots, w_n$ arbitrary save for the condition (7),*
b) *matrix $H_1^T W$, where $W = (w_1, \dots, w_n)$ is upper triangular,*
c) *$z_i \simeq w_i$.*

Proof: [2], p.71, Theorem 6.1

Definition 1. The implicit LU factorization is defined by the following choice of the parameters:

$$H_1 = I, \qquad z_i = e_i, \qquad w_i = \frac{e_i}{e_i^T H_i a_i}. \tag{12}$$

Remark. This choice leads to the following formulas for the computation of the search vector, the new approximation of the solution and update of the matrix:

$$p_i = H_i^T e_i, \tag{13}$$

$$x_{i+1} = x_i - \frac{a_i^T x_i - b_i}{p_i^T a_i} p_i, \tag{14}$$

$$H_{i+1} = H_i - \frac{H_i a_i p_i^T}{p_i^T a_i}. \tag{15}$$

The parameter option (12) induces the structure of H_i, described by the following theorem.

Theorem 4. *Let $A \in R^{n,n}$ be a strongly nonsingular matrix and w_i be similar to (12). Let H_i^T and W be nonsingular upper triangular matrices. Consider the sequence of matrices H_i generated by (8). Then the following properties are true:*

a) *the first i rows of H_{i+1} are identically zero,*
b) *the last $n-i$ columns of H_{i+1} are equal to the last $n-i$ columns of H_1, i.e.,*

$$H_{i+1} = \begin{pmatrix} 0 & 0 \\ S_i & (H_1)_i \end{pmatrix}. \tag{16}$$

Proof: [2] , p.73, Theorem 6.3

The following theorem shows that a row pivoting strategy always exists so that the assumptions a), b) and c) of Theorem 3 are satisfied by some choices of H_1 and vectors w_i.

Theorem 5. *Let A be square nonsingular and let H_1^T and $\overline{W} = (\overline{w_1}, \dots, \overline{w_n})$, where $\overline{w_i} = H_i^T w_i$, be nonsingular upper triangular matrices. Then it is possible to choose for $i = 1, \dots, n$ an index j with $i \le j \le n$ and a scalar $\beta_i \ne 0$ such that $(\beta_i \overline{w_i})^T H_i \overline{a_j} = 1$ where $\overline{a_j}$ is the j-th row of the matrix $\overline{A}$ obtained after some permutation of the rows of A.*

Proof: [2], p.72, Theorem 6.2

3. Algorithm for systems with Hankel matrices.

The method for solving systems of linear equations described by Rissanen (1974) is based on the transformation A as follows:

$$SA = Q \tag{17}$$

where S is a lower triangular matrix with ones on the diagonal, and Q is

a) upper triangular when A is strongly nonsingular or

b) a matrix that can be made triangular by the row permutation.

Solving of the system (1) is equivalent after such a transformation to solving of the system:

$$Sb = Qx. \tag{18}$$

The author has described how such a transformation can be calculated in the case when A is a Hankel matrix:

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{pmatrix}. \tag{19}$$

As in [2] let $A(k), k = 1, \ldots, n$ denote the submatrices of the nonsingular Hankel matrix consisting of the first k rows of the matrix A. The rank of $A(k)$ is k. Define the set of natural numbers $E_k = \{i_1, \ldots, i_k\}, k = 1, \ldots, n$ as follows: i_1 is the index of the first non–zero element of row $A(1)$ and for $t = 2, \ldots, k$; i_t is the least natural number so that $i_t \neq i_1, \ldots, i_{t-1}$ and the i_t-th column of $A(t)$ is not in the linear span of the preceding columns. A simple example is the matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

with $E_1 = \{1\}, E_2 = \{1, 3\}, E_3 = \{1, 3, 2\}$.

The proof of the main theorem in [10] includes a constructive basis for creation of the algorithm for solution of the transformation (16) of the Hankel matrix.

Theorem 6. *Let $A \in R^{n,n}$ be a nonsingular Hankel matrix with the set of indices $E_n = \{i_1, \ldots, i_n\}$. Then lower triangular matrix $S \in R^{n,n}$ exists with ones on the diagonal that:*

$$SA = Q$$

where $Q \in R^{n,n}$ is a matrix with elements $q_{i,j}$ satisfying the conditions:

$$q_{k,j} = 0 \qquad j < i_k$$

$$q_{k,j} \neq 0 \qquad j = i_k$$

The algorithm described by Rissanen (1974) has two distinct stages which follow from the properties of the elements i_k of index set E_n:

a) if $i_k - 1 \in E_k$ or $i_k = 1$ then E_k is a permutation of the integers $1, \ldots, k$,

b) if $i_k > 1$ and $i_k - 1 \notin E_k$ then $i_{k+1} = i_k - 1$.

THE RISSANEN ALGORITHM

1) Initialization
$s_1 = (1, 0, \ldots, 0)$ (n components)
$q_1 = (a_1, \ldots, a_n)$
$E_1 = \{i_1\}$, where i_1 is the least number t, $1 \le t \le n$ such that $a_t \ne 0$
$U_1 = \{u_1\}$, where $u_1 = a_t$
2) For $k = 1, \ldots, n-1$
a) $s_{k+1} = (0, s_{k,1}, \ldots, s_{k,k-1}, 1, 0, \ldots, 0)$ (n components)
$q_{k+1} = (q_{k,2}, \ldots, q_{k,n}, q_{k+1,n})$
where $q_{k+1,n} = s_{k,1}a_{n+1} + \cdots + s_{k,k-1}a_{n+k-1} + a_{n+k}$
b) find the least number m, $1 \le m \le n$ such that $q_{k+1,m} \ne 0$
c) if $m = i_l$, $i_l \in E_k$ then
$d = q_{k+1,m}u_l^{-1}$
$s_{k+1} = s_{k+1} - ds_l$
$q_{k+1} = q_{k+1} - dq_l$
return to b)
d) if $m \notin E_k$ then
$i_{k+1} = m$ $u_{k+1} = q_{k+1,m}$
$E_{k+1} = E_k \cup \{i_{k+1}\}$ $U_{k+1} = U_k \cup \{u_{k+1}\}$

The output of this algorithm includes rows of S and Q, index set E_n and the set $U_n = (q_{1,i_1}, \ldots, q_{n,i_n})$ containing leading non–zero elements of the rows of Q.

4. Relation between ABS–LU and Rissanen's algorithms.

In the case when A is a strongly nonsingular symmetric matrix, the equivalence of transformations (11) and (17) is obvious because the following well-known theorem is valid:

Theorem 7. *Matrix $A \in R^{n,n}$ has a LU factorization if $det(A^{k,k}) \ne 0$ for $k = 1, \ldots, n-1$. If the LU factorization exists and A is nonsingular then the LU factorization is unique.*

Proof: [8], p.96, Theorem 3.2.1

Corollary 1. *Let $A \in R^{n,n}$ be a strongly nonsingular Hankel matrix. Then for search vectors p_i generated by the ABS modification of the LU decomposition and vectors s_i containing the rows of matrix S generated by Rissanen's algorithm the following relation is true:*

$$\frac{p_i}{p_i^T a_i} = \frac{s_i}{s_i^T a_i}. \tag{20}$$

Proof: Because A is strongly nonsingular it follows that $P = (p_1, \ldots, p_n)$ is upper triangular in the decomposition (11). Lower triangular matrix L has the elements

$a_i^T p_i, i = 1, \ldots, n$ in the diagonal. Considering the following vector formulation of (11):

$$\begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{pmatrix} \begin{pmatrix} p_1 & p_2 & \ldots & p_n \end{pmatrix} = \begin{pmatrix} a_1^T p_1 & 0 & \ldots & 0 \\ a_2^T p_1 & a_2^T p_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n^T p_1 & a_n^T p_2 & \ldots & a_n^T p_n \end{pmatrix}$$

one can obtain:

$$\begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{pmatrix} \begin{pmatrix} \frac{p_1}{a_1^T p_1} & \frac{p_2}{a_2^T p_2} & \cdots & \frac{p_n}{a_n^T p_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ \frac{a_2^T p_1}{a_1^T p_1} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_n^T p_1}{a_1^T p_1} & \frac{a_n^T p_2}{a_2^T p_2} & \ldots & 1 \end{pmatrix}. \tag{21}$$

Because A is symmetric ($A = A^T$) we can transpose (17) into the form

$$AS^T = Q^T \tag{22}$$

where $S^T = (s_1, \ldots, s_n)$ is upper triangular and Q^T lower triangular. One can consider the vector formulation of (22) as

$$\begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{pmatrix} \begin{pmatrix} \frac{s_1}{a_1^T s_1} & \frac{s_2}{a_2^T s_2} & \cdots & \frac{s_n}{a_n^T s_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ \frac{a_2^T s_1}{a_1^T s_1} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_n^T s_1}{a_1^T s_1} & \frac{a_n^T s_2}{a_2^T s_2} & \ldots & 1 \end{pmatrix}. \tag{23}$$

which is analogical to the vector formulation (21).

It follows from the theorem 7 that the decompositions (21) and (23) are equivalent and relation (20) is true. □

Corollary 2. *The sequence $\{x_i\}_{i=1}^{n+1}$ generated from* (14) *in the ABS – LU algorithm is equivalent to the sequence of vectors which can be generated using the vectors s_i as the search vectors in the ABS algorithm.*

Proof: We can proceed by induction, using (14) and (20). □

It has been a trivial matter to show the equivalence of (11) and (17) in the case when A is a strongly nonsingular matrix. The application of the Rissanen algorithm in the case when A is not strongly nonsingular appears more interesting.

We explain our intention on the example of a matrix with $n = 3$ and index set $E_3 = (1, 3, 2)$. In the following vector formulation:

$$\begin{pmatrix} a_1^T \\ a_2^T \\ a_3^T \end{pmatrix} \begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix} = \begin{pmatrix} a_1^T s_1 & 0 & 0 \\ a_2^T s_1 & 0 & a_2^T s_3 \\ a_3^T s_1 & a_3^T s_2 & a_3^T s_3 \end{pmatrix}$$

matrix $S^T = (s_1, s_2, s_3)$ is upper triangular. After the column permutation of S^T given by E_3 we get:

$$\begin{pmatrix} a_1^T \\ a_2^T \\ a_3^T \end{pmatrix} \begin{pmatrix} s_1 & s_3 & s_2 \end{pmatrix} = \begin{pmatrix} a_1^T s_1 & 0 & 0 \\ a_2^T s_1 & a_2^T s_3 & 0 \\ a_3^T s_1 & a_3^T s_3 & a_3^T s_2 \end{pmatrix}$$

or:

$$\begin{pmatrix} a_1^T \\ a_2^T \\ a_3^T \end{pmatrix} \begin{pmatrix} \dfrac{s_1}{a_1^T s_1} & \dfrac{s_3}{a_2^T s_3} & \dfrac{s_2}{a_3^T s_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{a_2^T s_1}{a_1^T s_1} & 1 & 0 \\ \dfrac{a_3^T s_1}{a_1^T s_1} & \dfrac{a_3^T s_3}{a_2^T s_3} & 1 \end{pmatrix}$$

These formulations have the structure described in Theorem 2. Therefore, we can expect that it is possible to use vectors $s_i, i = 1, \ldots, n$ in the order given by E_n as search vectors p_i in the ABS algorithm, and to form a hybrid algorithm.

The values $a_i^T s_{k_i}$,which are in the denominator of (14), are contained in U_n generated using the Rissanen algorithm.

5. One parameter option in ABS algorithm.

We will discuss a simple modification of the parameter choise (12) in the ABS class. Rissanen's algorithm and Phuan (1988) inspire us to consider the next option of parameters.

Definition 2. The pivoting LU factorization is defined using the following choice of parameters:

$$(24) \qquad H_1 = I, \qquad z_i = e_j, \qquad e_j = \max_{1 \le k \le n} |e_k^T H_i a_i|, \qquad w_i = \frac{e_j}{e_j^T H_i a_i}.$$

Remark. The choice changes (13) into $p_i = H_i^T e_j$. The relations (14) and (15) are not changed.

We show that parameter z_i in (24) is well defined and it is determined by a permutation of columns of A. We also show how this permutation affects the structure of H_k.

Theorem 8. *Let $A \in R^{n,n}$ be a square nonsingular matrix. Then it is possible to choose for $i = 1, \dots, n$ an index j_i with $1 \leq j_i \leq n$ that:*

$$e_{j_i}^T H_i a_i \neq 0.$$

Index set $IS_n = \{j_1, \dots, j_n\}$ is a permutation of elements $1, \dots, n$.

Proof: We can proceed by induction. For $i = 1, v_1 = H_1 a_1 = a_1 \neq (0, \dots, 0)$ because of nonsingularity of A. Let j_1 be the index for which:

$$e_{j_1}^T H_1 a_1 = \max_{1 \leq l \leq n} |e_l^T H_i a_i|,$$

and $IS_1 = \{j_1\}$. Then $p_1 \equiv H_1^T e_{j_1} = e_{j_1}$, and:

$$H_2 \equiv H_1 - \frac{v_1 p_1^T}{p_1^T a_1} = I - \frac{a_1 e_{j_1}^T}{e_{j_1}^T a_1}$$

where $a_1 e_{j_1}^T$ is matrix with the elements $(a_{11}, \dots, a_{1n})$ in column j_1. Then the j_1-th row of new matrix H_2 is zero because $h_{j_1,j_1} \equiv 1 - a_{1,j_1}/a_{1,j_1} = 0$. The other elements of the j_1-th column are $h_{i,j_1} = -a_{i,j_1}$ and H_2 has the form:

$$H_2 = \begin{pmatrix} 1 & 0 & \dots & h_{1,j_1} & \dots & 0 \\ 0 & 1 & \dots & h_{2,j_1} & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & h_{n,j_1} & \dots & 0 \end{pmatrix}. \tag{25}$$

Then $v_2 = H_2 a_2$ is a vector with $v_{2,j_1} = 0$. It follows that index j_2 for which:

$$e_{j_2}^T H_2 a_2 = \max_{1 \leq k \leq n} |e_k^T H_2 a_2|,$$

fulfills the condition $j_2 \neq j_1$. Because of nonsingularity of A, element $v_{2,j_2} \neq 0$ exists in the vector v_2. The index set $IS_2 = \{j_1, j_2\}$ is created.

If it is assumed that the argument is true for $i = k$, index set $IS_k = \{j_1, \dots, j_k\}$ can also be created. Index j_{k+1}, for which:

$$e_{j_{k+1}}^T H_{k+1} a_{k+1} = \max_{1 \leq l \leq n} |e_l^T H_i a_i|, \tag{26}$$

may be found. Matrix H_{k+1} is given by:

$$H_{k+1} = H_k - \frac{v_k p_k^T}{p_k^T a_k}, \tag{27}$$

where H_k is a matrix with $j_1, \ldots, j_{k-1}$ zero rows and $v_k = H_k a_k$ is a vector with zero elements in the same positions. Search vector $p_k = H_k^T e_{j_k}$ is the j_k-th row of H_k and:

$$p_{k,j_k} = 1,$$

$$p_{ki} = 0 \qquad i \neq j_1, \ldots, j_k.$$

It follows that update (27) does not change the elements of H_{k+1} which lie in rows $j_1, \ldots, j_{k-1}$ or in columns $IS_n - \{j_1, \ldots, j_k\}$.

We take an interest in the change of the j_k-th row of H_{k+1}. Using (27) we obtain:

$$(H_{k+1})_{j_k} = p_k^T - \frac{p_k^T a_k p_k^T}{p_k^T a_k}.$$

One can see that the j_k-th row of H_{k+1} was zeroed. Vector $v_{k+1} = H_{k+1} a_{k+1}$ has zero elements in positions $j_1, \ldots, j_k$, and index j_{k+1} holds the condition:

$$j_{k+1} \neq j_i \qquad i = 1, \ldots, k.$$

From the nonsingularity of A it follows that there exists an element $v_{k+1,j_{k+1}} \neq 0$ in vector v_{k+1}. Then vector $e_{j_{k+1}}$ satisfying the condition (26) can be chosen.

Now we have $IS_{k+1} = \{j_1, \ldots, j_{k+1}\}$. After n steps of the process set of indices $IS_n = \{j_1, \ldots, j_n\}$ is obtained. It is a permutation of the elements $1, \ldots, n$. □

Corollary 3. *The structure of the matrices $H_{k+1}, k = 1, \ldots, n$ generated using the pivoting algorithm is, after interchanges of its rows and columns by the index set IS_k, the same as the structure described in the Theorem 4.*

Proof: We can proceed by induction again. For $i = 1$, as we can see from (25), the statement is true . If the validity of the statement is assumed up to the index $k - 1$ then the j_k-th row is zeroed and new elements arise in the j_k-th column except for the row positions $j_1, \ldots, j_k$ in H_{k+1}. If $j_k - th$ row and $j_k - th$ column move to the k-th position in H_{k+1}, respectively, the known structure is obtained.□

Remark. The numerical pretension of the pivoting algorithm is the same as in the ABS – LU algorithm. Authors in [1] and [2] and more exactly in [9] proved that $n^3/3 + O(n^2)$ multiplications and the same number of additions are needed to solve the system (1) by the ABS – LU algorithm, and that it is the same pretension as in the classic LU algorithm.

Remark. It is not very difficult to verify that the last $n-i+1$ rows of AH_i^T contain the same elements as the rows of A after $(i - 1)$ steps of the Gauss elimination with the column pivotization.

6. Numerical pretension of algorithms.

The Rissanen algorithm for the solution of transformation (17) in the case when A is the Hankel matrix in the section 3 has been introduced. Rissanen in [10] has proved that matrices S and Q can be determined by the order of n^2 arithmetic operations. His estimate of the operations is:

$$\sum_{k\in K_1}(2k-1)+2n\sum_{k\in K_2}(i_{k+1}-i_k+2),$$

where K_1 is a set of the row-indices k for which $i_{k+1}<i_k$, and K_2 is a set of the remaining ones. There are no more than $3n(n-1)$ multiplications and the same amount of additions.

The calculation of Sb in (18) takes $n(n+1)/2$ multiplications and $n(n-1)/2$ additions. The same amount of operations is needed for the solution of (18). Using the vectors $s_i, i=1,\ldots,n$ as search vectors in the ABS algorithm (as it has been described in the section 4) and the formula (14) solution of system (1) with the Hankel matrix takes n^2 multiplications, $n(n+1)/2$ additions and n divisions.

Table 1 compares the number of operations of the single algorithms with the hybrid algorithm described above. The numerical tests have been performed using the Hankel system with a strongly nonsingular matrix. It is the most difficult case for the Rissanen algorithm from the numerical point of view.

TABLE 1

	ABS – LU	Rissanen	Hybrid
additions	$n^3/3$	$4n^2$	$7/2n^2$
multiplications-divisions	$n^3/3$	$4n^2$	$4n^2$

In section 5 the modified choice of parameters has been introduced. This option has some pivoting quality and does not require the principal minors of A to be non-zero except $detA$.

Series of systems (1) with strongly nonsingular matrices have been solved via the ABS – LU algorithm [with the parameter option (12)] and via the pivoting algorithm [with the parameter option (24)] in order to obtain comparable results. The elements of A are randomly defined integers in the interval $[-100,100]$ and the elements of the exact solution x^+ are randomly defined integers in the interval $[-50,50]$. The right-hand side b is defined by computing Ax^+. Results in the next table are obtained testing 1000 systems of dimension between $n=10$ and $n=1000$. Minimum values of the relative error:

$$\sigma=\frac{\| x_{n+1}-x^+ \|}{\| x^+ \|}$$

contain the table:

TABLE 2

dimension	ABS – LU	pivoting	dimension	ABS – LU	pivoting
10	.1003E-14	.5310E-15	100	.4966E-13	.3457E-13
20	.4748E-14	.4442E-14	200	.9379E-13	.8862E-13
30	.1085E-13	.5886E-14	300	.1578E-12	.1295E-12
40	.1110E-13	.1175E-13	400	.1639E-12	.1919E-12
50	.1644E-13	.1626E-13	500	.2366E-12	.2217E-12
60	.2198E-13	.1866E-13	600	.2079E-12	.2550E-12
70	.2440E-13	.1790E-13	700	.3713E-12	.2800E-12
80	.2496E-13	.2958E-13	800	.4095E-12	.3341E-12
90	.3889E-13	.2138E-13	900	.4078E-12	.4339E-12
			1000	.4601E-12	.4404E-12

We solve series of the systems (1) with ill-conditioned randomly defined A (warning from DLSARG in fortran library IMSL, condition numbers 1.E+15 – 1.E+20). Minimum values of the relative error of test process are displayed in Table 3:

TABLE 3

dimension	ABS – LU	pivoting ABS – LU
10	.478E-11	.264E-11
20	.366E-09	.206E-11
30	.129E-09	.256E-11
40	.541E-09	.564E-11
50	.260E-09	.436E-11
60	.103E-08	.847E-11
70	.648E-08	.201E-10
80	.791E-09	.906E-11
90	.873E-09	.149E-10
100	.304E-08	.116E-10
120	.426E-08	.441E-10
140	.587E-08	.242E-10

Finally linear systems with matrices $A_n \in R^{n,n}$ of the form

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

have been solved. The growth factor is value which considerable affects on precision of the result of (1) with such matrix.

TABLE 4

dimension	ABS – LU	pivoting ABS – LU
50	0	0
55	.1348E+00	.4334E-15
60	.2480E+00	.2237E-15
70	.4396E+00	.3278E-15
80	.5477E+00	.3696E-15
90	.6021E+00	.4412E-15
100	.6472E+00	.4537E-15
200	.8388E+00	.9909E-15

7. Conclusion.

The hybrid algorithm does not afford more accurate results than the ABS – LU or the Rissanen algorithms. However, this algorithm has two advantages in comparison with these two algorithms: i) it uses smaller amount of numerical operations and ii) it does not require the strong nonsingularity of A.

From the results in Table 3 and Table 4 we can conclude that the pivoting algorithm demonstrates an improvement of the accuracy of the solution. There is the next possibility to modify this algorithm. Its combination with a row pivoting process can lead to the complete pivotization which is well-known in the Gauss elimination.

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